L33InverseLaplaceHeavyside

Note Title 8/15/2020

Inverse Laplace Transforms involving $u_c(t)$

Table Entry*

use for inverse $\qquad u_c(t)\,y(t-c) \qquad e^{-cs}\,Y(s)$

or

use for Fwd $\qquad \mathcal{L}\{u_c(t)\,y(t)\} = e^{-cs}\,\mathcal{L}\{y(t+c)\}$

Problem - Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-2s}\,\frac{1}{s^2}\right\} = u_2(t)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Big|_{t\mapsto t-2}\right]$$

$$= u_2(t)\left(t\,\Big|_{t\mapsto t-2}\right)$$

$$= u_2(t)\,(t-2)$$

* Full table on last page

Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\Bigg|_{\text{“}t \mapsto t-2\text{”}}$$

<u>Step 1</u> Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}$

$$\frac{1}{s(s+4)} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\} = 1/4 \cdot 1 - 1/4\, e^{-4t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\Bigg|_{\text{“}t \mapsto t-2\text{”}}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\left(\frac{1}{4} - \frac{e^{-4t}}{4}\right)\Bigg|_{\text{“}t \mapsto t-2\text{”}}$$

$$= u_2(t)\left(\frac{1}{4} - \frac{e^{-4(t-2)}}{4}\right)$$

Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}\Bigg|_{t\mapsto t-2}$$

Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1 = A(s^2+9) + (Bs+C)s$$

$$1 = (A+B)s^2 + Cs + 9A$$

so $\quad 9A = 1 \qquad C = 0 \qquad A+B=0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s} \quad \frac{-1/9\,s}{s^2+9}\right\}$$

$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= 1/9 - \frac{1}{9}\cos 3t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\} = u_2(t)\left(\frac{1}{9} - \frac{1}{9}\cos 3t\right)\Bigg|_{t\mapsto t-2}$$

$$= \frac{u_2(t)}{9}\left(1 - \cos 3(t-2)\right)$$

Problem 2 Solve the IVP

$$y'' + y = g(t) \qquad g(t) = \begin{cases} e^{-t} & t<2 \\ 0 & 2<t \end{cases}$$

$$y(0)=0 \qquad y'(0)=0$$

Laplace Transforms

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1}\mathcal{L}\{g\}$$

Compute $\mathcal{L}\{g\}$

$$g(t) = e^{-t}(1-u_2(t))$$

$$\mathcal{L}\{g\} = \mathcal{L}\{e^{-t}\} - \mathcal{L}\{u_2(t)\, e^{-t}\}$$

---

version of Table Entry for Fwd Transform

$$\mathcal{L}\{u_c(t)\, y(t)\} = e^{-cs}\mathcal{L}\{y(t+c)\}$$

$$= \frac{1}{s+1} - e^{-2s}\mathcal{L}\{e^{-(t+2)}\}$$

$$= \frac{1}{s+1} - e^{-2s}\mathcal{L}\{e^{-2}e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s}e^{-2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s-2}\frac{1}{s+1}$$

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1}\mathcal{L}\{g\}$$

$$\mathcal{L}\{g\} = \frac{1}{s+1} - e^{-2s-2}\frac{1}{s+1}$$

so

$$Y = \frac{1}{(s^2+1)(s+1)} - e^{-2}e^{-2s}\frac{1}{(s^2+1)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s^2+1)(s+1)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}\Bigg|_{t=t-2}$$

Calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}$

$$\frac{1}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = (s^2+1)A + (Bs+C)(s+1)$$

$$= (A+B)s^2 + (B+C)s + (A+C)$$

$A+B=0$ and $B+C=0$ and $A+C=1$

so $A=-B=C$ and $2A=1$

$$\frac{1}{(s^2+1)(s+1)} = \frac{1/2}{s+1} + \frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

$$\frac{1}{(s^2+1)(s+1)} = \frac{1/2}{s+1} + \frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}u_2(t)\left.\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}\right|_{t=t-2}$$

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right) - e^{-2}u_2(t)\frac{1}{2}\left(e^{-(t-2)} - \cos(t-2) + \sin(t-2)\right)$$

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$
$\delta_a(t)$	e^{-as}
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - y(0)s - y'(0)$
$e^{at}y(t)$	$Y(s-a)$
$ty(t)$	$-\frac{d}{ds}Y(s)$
$u_a(t)y(t-a)$	$e^{-as}Y(s)$
$u_a(t)y(t)$	$e^{-as}\mathcal{L}\{y(t+a)\}$
$y(at)$	$\frac{1}{a}Y\left(\frac{s}{a}\right)$