L33InverseLaplaceHeavyside
Inverse Laplace Transforms involving $u_{e}(t)$
Table Entry*
use for inverse $\quad u_{c}(t) y(t-c) \quad e^{-c s} Y(s)$
or
use for fud $\quad \mathcal{L}\left\{u_{c}(t) y(t)\right\}=e^{-c s} \mathcal{L}\{y(t+c)\}$

$$
\begin{aligned}
& \text { Problem -find } \begin{aligned}
& \mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s^{2}}\right\} \\
&\left.\left.\begin{array}{rl}
\mathcal{L}^{-1}\left\{e^{-2 s} \frac{1}{s^{2}}\right\} & =u_{2}(-1)
\end{array} \mathcal{L}^{-1}\left\{\frac{1}{s^{2}}\right\}\right|_{t \mapsto t-2}\right] \\
&\left.=\left.u_{2}(t)(t)\right|_{t-1-2}\right) \\
&=u_{2}(t)(t-2)
\end{aligned}
\end{aligned}
$$

* Full table on last page

$$
\begin{aligned}
& \text { Find } \mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s(s+4)}\right\} \\
& \mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s(s+4)}\right\}=\left.u_{2}(t) \mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\right|_{t \mapsto t-2}
\end{aligned}
$$

Step 1 Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}$

$$
\begin{gathered}
\frac{1}{s(s+4)}=\frac{1 / 4}{s}-\frac{1 / 4}{s+4} \\
\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}=1_{4} \cdot 1-1 / 4 e^{-4 t} \\
\mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s(s+4)}\right\}=\left.u_{2}(t) \mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\right|_{t \mapsto t-2^{\prime \prime}} \\
\mathcal{L}_{t}^{-1}\left\{\frac{e^{-2 s}}{s(s+4)}\right\}=\left.u_{2}(t)\left(\frac{1}{4}-\frac{e^{-4 t}}{4}\right)\right|_{t \mapsto t-2^{\prime \prime}} \\
=u_{2}(t)\left(\frac{1}{4}-\frac{e^{-4(t-2)}}{4}\right)
\end{gathered}
$$

$$
\begin{aligned}
& \text { Find } \mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s\left(s^{2}+q\right)}\right\}^{-1} \\
& \mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s\left(s^{2}+9\right)}\right\}=\left.u_{2}(t) \mathcal{L}^{-1}\left\{\frac{1}{s\left(s^{2}+9\right)}\right\}\right|_{t \mapsto t-2}
\end{aligned}
$$

Compute $\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^{2}+9\right)}\right\}$

$$
\begin{aligned}
\frac{1}{s\left(s^{2}+9\right)} & =\frac{A}{s}+\frac{B s+C}{s^{2}+9} \\
1 & =A\left(s^{2}+9\right)+(B S+C) s \\
1 & =(A+B) s^{2}+C s+9 A
\end{aligned}
$$

So $\quad a A=1 \quad C=0 \quad A+B=0$

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^{2}+9\right)}\right\} & =\mathcal{L}^{-1}\left\{\frac{1 / 9}{s}-\frac{1 / q s}{s^{2}+9}\right\} \\
& =\frac{1}{q} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{q} \mathcal{L}^{-1}\left\{\frac{s}{s^{2}+q}\right\} \\
& =1 / 9-\frac{1}{q} \cos 3 t \\
\mathcal{L}^{-1}\left\{\frac{e^{-2 s}}{s\left(s^{2}+9\right)}\right\} & =\left.u_{2}(t)\left(\frac{1}{q}-\frac{1}{q} \cos 3 t\right)\right|_{t \mapsto t-2} \\
& =\frac{u_{2}(t)}{q}(1-\cos 3(t-2))
\end{aligned}
$$

Problem 2 Solve the IVP

$$
\begin{aligned}
& y^{\prime \prime}+y=g(t) \\
& y(0)=0 \quad y^{\prime}(0)=0
\end{aligned} \quad g(t)=\left\{\begin{array}{cc}
e^{-t} & t<2 \\
0 & 2<t
\end{array}\right.
$$

Laplace Transforms

$$
\begin{array}{r}
s^{2} y+y=\mathcal{L}\{g\} \\
y=\frac{1}{s^{2}+1} \mathcal{L}\{g\}
\end{array}
$$

Compute $\mathcal{L}\{g\}$

$$
\begin{aligned}
& g(t)=e^{-t}\left(1-u_{2}(t)\right) \\
& \mathcal{L}\{g\}=\mathcal{L}\left\{e^{-t}\right\}-\mathcal{L}\left\{u_{2}(t) \quad e^{-t}\right\} \\
& \begin{array}{ll}
\text { Version of Table } & \mathcal{L}\left\{u_{c}(t) y(t)\right\}=e^{-\operatorname{cs}}\{\{y(t+c)\} \\
\text { Entry for Fud }
\end{array} \\
& \begin{array}{l}
\text { Entry for Fud } \\
\text { Trans form }
\end{array} \\
& =\frac{1}{s+1}-e^{-2 s} \mathcal{L}\left\{e^{-(t+2)}\right\} \\
& =\frac{1}{s+1}-e^{-2 s} \mathcal{L}\left\{e^{-2} e^{-t}\right\} \\
& =\frac{1}{s+1}-e^{-2 s} e^{-2} \mathcal{L}\left\{e^{-t}\right\} \\
& =\frac{1}{s+1}-e^{-2 s-2} \frac{1}{s+1}
\end{aligned}
$$

$$
\begin{gathered}
s^{2} y+y=\mathcal{L}\{g\} \\
y=\frac{1}{s^{2}+1} \mathcal{L}\{g\} \\
\mathcal{L}\{g\}=\frac{1}{s+1}-e^{-2 s-2} \frac{1}{s+1}
\end{gathered}
$$

So

$$
\begin{aligned}
& \text { So } \\
& y=\frac{1}{\left(s^{2}+1\right)(s+1)}-e^{-2} e^{-2 s} \frac{1}{\left(s^{2}+1\right)(s+1)} \\
& y(t)=\mathcal{L}^{-1}\left\{\frac{1}{\left.\left(s^{2}+1\right)(s+1)\right)}\right\}-e^{-2} \alpha^{-1}\left\{e^{-2 s} \frac{1}{\left.s^{2}+1\right)(s+1)}\right\} \\
& y(t)=\mathcal{L}^{-1}\left\{\left.\frac{1}{\left.\left(s^{2}+1\right)(s+1)\right\}-e^{-2} u_{2}(t) \mathcal{L}^{-1}\left\{\frac{1}{\left(s^{2}+1 s+1\right.}\right\}}\right|_{t=t-2}\right.
\end{aligned}
$$

Calculate $\mathcal{L}^{-1}\left\{\frac{1}{\left(s^{2}+1\right)(s+1)}\right\}$

$$
\begin{aligned}
& \frac{1}{\left(s^{2}+1\right)(S+1)}=\frac{A}{S+1}+\frac{B s+C}{S^{2}+1} \\
& 1=\left(s^{2}+1\right) A+(B s+C)(S+1) \\
& =(A+B) s^{2}+(B+C) s+(A+C)
\end{aligned}
$$

$A+B=0$ and $B+C=0$ and $A+C=1$
so $A=-B=C$ and $2 A=1$

$$
\frac{1}{\left(s^{2}+1\right)(s+1)}=\frac{1 / 2}{s+1}+\frac{-1 / 2 s+1 / 2}{s^{2}+1}
$$

$$
\begin{aligned}
& \frac{1}{\left(s^{2}+1\right)(s+1)}=\frac{\frac{1 / 2}{s+1}+\frac{-1 / 2 s+1 / 2}{s^{2}+1}}{\mathcal{L}^{-1}\left\{\frac{1}{\left(s^{2}+1\right)(s+1)}\right\}} \begin{aligned}
& =\frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-1 / 2 \rho^{-1}\left\{\frac{s}{s^{2}+1}\right\}+\frac{1}{2} d^{-1}\left\{\frac{1}{s^{2}+1}\right\} \\
& =\frac{1}{2} e^{-t}-1 / 2 \cos t+1 / 2 \sin t \\
& =\frac{1}{2}\left(e^{-t}-\cos t+\sin t\right)
\end{aligned} \\
& \begin{aligned}
y(t) & =\mathcal{L}^{-1}\left\{\left.\frac{1}{\left.\left(s^{2}+1\right)(s+1)\right\}-e^{-2} u_{2}(t)} \alpha^{-1}\left\{\frac{1}{\left(s^{2}+1\right)(s+1)}\right\}\right|_{t=t-2}\right. \\
y(t) & =\frac{1}{2}\left(e^{-t}-\cos t+s \sin t\right)-e^{-2} u_{2}(t) \frac{1}{2}\left(e^{-(t-2)}-\cos (t-2)+\sin (t-2)\right.
\end{aligned}
\end{aligned}
$$

$$
\begin{array}{lc}
1 & \frac{1}{s} \\
e^{a t} & \frac{1}{s-a} \\
t^{n} & \frac{n!}{s^{n+1}} \\
\sin (\omega t) & \frac{\omega}{s^{2}+\omega^{2}} \\
\cos (\omega t) & \frac{s}{s^{2}+\omega^{2}} \\
u_{a}(t) & \frac{e^{-a s}}{s} \\
\delta_{a}(t) & e^{-a s} \\
y^{\prime}(t) & s Y(s)-y(0) \\
y^{\prime \prime}(t) & s^{2} Y(s)-y(0) s-y^{\prime}(0) \\
e^{a t} y(t) & Y(s-a) \\
t y(t) & -\frac{d}{d s} Y(s) \\
u_{a}(t) y(t-a) & e^{-a s} Y(s) \\
u_{a}(t) y(t) \\
y(a t) & e^{-a s} \mathcal{L}\{y(t+a)\} \\
t^{2}(t) \\
a
\end{array}
$$

