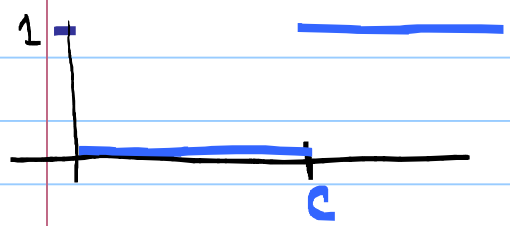
L32Heavyside

Note Title 8/13/2020

The Heaviside Function and Time Delay

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



$u_c(t)$ is called the Heavyside function or the step function.

https://en.wikipedia.org/wiki/Heaviside_step_functio

Multiplying a function by $u_c(t)$ turns it on at time $t=c$. Multiplying by $(1-u_c(t))$ turns it off at $t=c$.

Example

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ e^{(t-1)} & 1 < t \end{cases}$$ *

f equals t until time 1, then it becomes e^{t-1}

Problem: Write $f(t)$ using the Heavyside function.

$$f(t) = \underbrace{t\cdot(1-u_1(t))}_{\text{turn off } t} + \underbrace{e^{(t-1)}\,u_1(t)}_{\text{turn on } e^{(t-1)}}$$

* In problems involving Laplace transforms, one usually assumes that all functions $f(t)$ are only defined for $t \geq 0$.

Example 2 Write $f(t)$ using the Heavyside function.

$$f(t) = \begin{cases} 6 & 0 \le t \le 1 \\ e^t & 1 < t \le 2 \\ \frac{t-1}{2} & 2 < t \le 3 \\ 4 & 3 < t \end{cases}$$

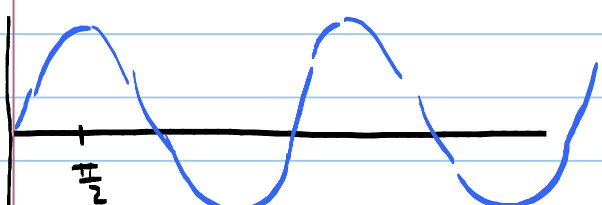
$$f(t) = 6(1 - u_1(t)) + e^t(u_1(t) - u_2(t)) + \left(\tfrac{t-1}{2}\right)(u_2(t) - u_3(t)) + 4u_3(t)$$

Turn on the 6 term | Turn it off | Turn on the e^t term | Turn off the e^t term | Turn on | Turn off | Turn on

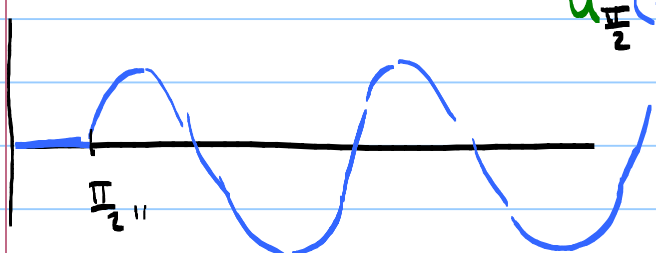
$$1 - u_1(t) = \begin{cases} 1 & t < 1 & \text{on} \\ 0 & t > 1 & \text{off} \end{cases}$$

$$(u_1(t) - u_2(t)) = \begin{cases} 0 & t < 1 & \text{off} \\ 1 & 1 < t < 2 & \text{on} \\ 0 & 2 < t & \text{off} \end{cases}$$

Interpretation of u_c as a delay



$\sin t$



$u_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right)$ = "$\sin t$ delayed by $\frac{\pi}{2}$"

$$= \begin{cases} 0 & t < \frac{\pi}{2} \\ \sin\left(t - \frac{\pi}{2}\right) & \frac{\pi}{2} < t \end{cases}$$

Laplace Transform of u_c

$$\mathcal{L}\{u_c(t)\, y(t-c)\} = e^{-cs} Y(s)$$ ⟵ most useful for calculating inverse transform

Alternatively,

$$\mathcal{L}\{u_c(t) y(t)\} = e^{-cs} \mathcal{L}\{y(t+c)\}$$ ⟵ most useful for calculating transform

Calculating $\mathcal{L}\{u_c(t)\}$ Derivation of Formulas

$$\mathcal{L}\{u_c(t) y(t)\} = \int_0^\infty e^{-st} u_c(t)\, y(t)\, dt$$

$$= \int_c^\infty e^{-st} y(t)\, dt$$

Change variables $t = \tau + c$

$$= \int_0^\infty e^{-s(\tau+c)} y(\tau+c)\, d\tau$$

$$= e^{-sc} \int_0^\infty e^{-s\tau} y(\tau+c)\, d\tau$$

(1) $$\mathcal{L}\{u_c(t) y(t)\} = e^{-sc} \mathcal{L}\{y(t+c)\}$$

To see the other formulation $\left(\mathcal{L}\{u_c(t) y(t-c)\} = e^{-cs} Y(s)\right)$

Define a new function $h(t) = y(t+c)$

then $y(t) = h(t-c)$ and (1) becomes

$$\mathcal{L}\{u_c(t) h(t-c)\} = e^{-sc} \mathcal{L}\{h(t)\} = e^{-cs} H(s)$$

Example 1 - Find the Laplace transform of $f(t)$

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{(t-1)} & 1 < t \end{cases} \qquad f(t) = u_1(t)\, e^{(t-1)}$$

$$F(s) = \mathcal{L}\{u_1(t)\, e^{(t-1)}\} = e^{-1s}\, \mathcal{L}\{e^{t}\} = e^{-s} \cdot \frac{1}{s-1}$$

Example 2 - Find the Laplace transform of $f(t)$

$$f(t) = \begin{cases} 0 & 0 \le t \le 2 \\ e^{t} & 2 < t \end{cases} \qquad f(t) = u_2(t)\, e^{t}$$

$$F(s) = \mathcal{L}\{u_2(t)\, e^{t}\} = e^{-2s}\, \mathcal{L}\{e^{t+2}\}$$

$$= e^{-2s}\, \mathcal{L}\{e^{2} \cdot e^{t}\}$$

$$= e^{-2s} \cdot e^{2}\, \mathcal{L}\{e^{t}\}$$

$$= e^{-2s+2} \cdot \frac{1}{s-1}$$

Example 3 - Find the Laplace transform of $f(t)$

$$f(t) = \begin{cases} \sin t & 0 < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t \end{cases}$$

$$f(t) = \sin t - u_{\pi/2}(t)\sin t$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{\pi/2}(t)\sin t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\sin(t+\tfrac{\pi}{2})\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$$