L31PartialFractions2

Note Title 8/13/2020

General Partial Fractions

$$\frac{Q(s)}{P(s)} \qquad \text{degree } Q < \text{degree } P$$

$P(s)$ factors $\quad P(s) = P_1(s)\,P_2(s)$

In General

$$\frac{Q(s)}{P_1(s)\,P_2(s)} = \frac{Q_1}{P_1(s)} + \frac{Q_2}{P_2(s)}$$

$$\deg Q_1 = \deg P_1 - 1 \qquad \deg Q_2 = \deg P_2 - 1$$

IF P_1 and P_2 have no common factor.

Example:

degree < 3 $\quad$ degree 0 $\quad$ degree 1

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

degree 3 $\quad$ degree 1 polynomial $\quad$ degree 2 polynomial

degree < 3 Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

degree 3

degree < 3 Bad Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs}{s^2+1}$$

Not good enough!!
Need most general degree 1 polynomial
$Bs+C$

degree 3

Examples

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} \quad \checkmark$$

$$\frac{6s^3+4s^2+3s+6}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} \quad \checkmark$$

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{As+B}{(s+3)^2} + \frac{Cs+D}{(s+2)^2}$$

Correct, but not as useful as the one below

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{A}{s+3} + \frac{\tilde{B}}{(s+3)^2} + \frac{C}{s+2} + \frac{\tilde{D}}{(s+2)^2}$$

If you check $\tilde{B} = B-3A$

$\tilde{D} = D-2C$

We use this one because $\frac{1}{s+3}$, $\frac{1}{(s+3)^2}$, $\frac{1}{s+2}$, $\frac{1}{(s+2)^2}$

are easier to find in the transform table.

Example

$$\frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

How to find the constants?

Example

$$\frac{1}{(S-1)(S-2)(S-3)} = \frac{A_1}{(S-1)} + \frac{A_2}{S-2} + \frac{A_3}{(S-3)}$$

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Always works, but lots of calculating

$$1 = (A_1+A_2+A_3)s^2 + (-5A_1-4A_2-3A_3)s + (6A_1+4A_2+2A_3)$$

$$A_1 + A_2 + A_3 = 0$$
$$-5A_1 - 4A_2 - 3A_3 = 0$$
$$6A_1 + 4A_2 + 7A_3 = 1$$

Three equations in three unknowns
Solve them

Faster, especially for simple roots

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Set $S=1$

$$1 = A_1(1-2)(1-3) + 0 + 0$$

$$\boxed{\frac{1}{2} = A_1}$$

Set $S=2$

$$1 = 0 + A_2(2-1)(2-3)$$

$$\boxed{-1 = A_2}$$

Set $s=3$

$$1 = 0 + 0 + A_3(3-1)(3-2)$$

$$\boxed{\frac{1}{2} = A_3}$$

Example

$$\frac{s+2}{(s-1)(s-2)^2} = \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{B_2}{(s-2)^2}$$

$$s+2 = A_1(s-2)^2 + A_2(s-2)(s-1) + B_2(s-1)$$

set $s = 1$

$$1+2 = A_1(1-2)^2 + 0 + 0$$

$$\boxed{3 = A_1}$$

set $s = 2$

$$2+2 = 0 + 0 + B_2(2-1)$$

$$\boxed{4 = B_2}$$

set $s = 0$ (or anything else

$$0+2 = 3(0-2)^2 + A_2(0-2)(0-1) + 4(0-1)$$

$$2 - 12 + 4 = 2A_2$$

$$\boxed{-3 = A_2}$$

Example

$$\frac{1}{(s-1)(s^2+4)} = \frac{A_1}{s-1} + \frac{A_2 s + B_2}{s^2+4}$$

$$1 = A_1(s^2+4) + (A_2 s + B_2)(s-1)$$

set $s=1$ $\boxed{A_1 = \frac{1}{5}}$

Now multiply it out

$$1 = (A_1 + A_2)s^2 + (B_2 - A_2)s + (4A_1 - B_2)$$

$A_1 + A_2 = 0$ so $\boxed{A_2 = -\frac{1}{5}}$

$B_2 - A_2 = 0$ so $\boxed{B_2 = -1/5}$

$4\left(\frac{1}{5}\right) - \left(\frac{-1}{5}\right) = 1$ ✓

$$1 = A_1(s^2+4) + (A_2 s + B_2)(s-1)$$

set $s=1$ $A_1 = \frac{1}{5}$

To eliminate A_2 term

set $s=0$ $1 = \frac{1}{5}(0+4) + (A_2 \cdot 0 + B_2(0-1))$

$B_2 = -1/5$

set $s=2$ Just choosing some number other than 0 and 1

$1 = \frac{1}{5}(2^2+4) + (2A_2 - \frac{1}{5})(2-1)$

$1 = \frac{8}{5} + 2A_2 - \frac{1}{5}$

$1 = \frac{7}{5} + 2A_2$

$-\frac{2}{5} = 2A_2$ $A_2 = -\frac{1}{5}$

One More Example

$$y'' + 9y = \cos 2t \qquad y(0) = 0 \quad y'(0) = 0$$

Laplace Transform

$$s^2 Y + 9Y = \frac{s}{s^2+4}$$

$$(s^2+1)\,Y = \frac{s}{s^2+4}$$

$$Y = \frac{s}{(s^2+9)(s^2+4)}$$

Inverse Laplace Transform
Partial Fractions

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{(s^2+9)} + \frac{Cs+D}{(s^2+4)}$$

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

$$= As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 9Cs + 9D$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$0\,s^3 + 0\,s^2 + 1\,s + 0 =$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$A+C=0 \Rightarrow A=-C$$

$$B+D=0 \Rightarrow B=-D$$

$$4B+9D=0 \Rightarrow -4D+9D=0 \Rightarrow D=0 \Rightarrow B=0$$

$$4A+9C=1 \Rightarrow -4C+9C=1 \Rightarrow C=\frac{1}{5} \Rightarrow A=-\frac{1}{5}$$

$$Y(s) = \frac{-1}{5}\,\frac{s}{s^2+9} + \frac{1}{5}\,\frac{s}{s^2+4}$$

$$y(t) = \frac{-1}{5}\cos 3t + \frac{1}{5}\cos 2t$$