L30IVPandPartialFractions1

Note Title 8/13/2020

Solve the IVP using Laplace Transforms

$$y'' + 2y' + 10y = 0$$

$$y(0) = 1 \qquad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$* \; [s^2Y(s) - s\,y(0) - y'(0)] + 2 * [sY(s) - y(0)] + 10[Y(s)] = 0$$

$$[s^2Y(s) - s\cdot 1 - 2] + 2[sY(s) - 1] + 10[Y(s)] = 0$$

$$[s^2 + 2s + 10]\,Y(s) - s - 4 = 0$$

$$Y(s) = \frac{s+4}{s^2+2s+10}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

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* using the table on the last page

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = e^{-t}\cos 3t + e^{-t}\sin 3t}$$

Inhomogeneous DE's via Laplace Transform

Solve using Laplace Transform

$$\ddot{y} + y = e^{-2t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

<u>Solution</u>

$$\mathcal{L}\{\ddot{y} + y\} = \mathcal{L}\{e^{-2t}\}$$

$$\mathcal{L}\{\ddot{y}\} + \mathcal{L}\{y\} = \frac{1}{s+2}$$

$$s^2 Y - s\,y(0) - \dot{y}(0) + Y = 1/(s+2)$$

$$s^2 Y - 0 - 0 + Y = \frac{1}{s+2}$$

$$(s^2+1)\,Y = \frac{1}{s+2}$$

$$Y = \frac{1}{(s+2)(s^2+1)}$$

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

General Partial Fractions

$$\frac{Q(s)}{P(s)}$$

degree Q < degree P

$P(s)$ factors $P(s) = P_1(s)\,P_2(s)$

In General

$$\frac{Q(s)}{P_1(s)\,P_2(s)} = \frac{Q_1}{P_1(s)} + \frac{Q_2}{P_2(s)}$$

$\deg Q_1 = \deg P_1 - 1$ $\deg Q_2 = \deg P_2 - 1$

IF P_1 and P_2 have no common factor.

Example:

degree < 3

degree 0

degree 1

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

degree 3

degree 1 polynomial

degree 2 polynomial

Example

degree < 3

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

degree 3

Bad Example

degree < 3

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs}{s^2+1}$$

degree 3

Not good enough!!
Need most general degree 1 polynomial
$Bs+C$

Examples

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$ ✓

$$\frac{6s^3+4s^2+3s+6}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$ ✓

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{As+B}{(s+3)^2} + \frac{Cs+D}{(s+2)^2}$$

Correct, but not as useful as the one below

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{A}{s+3} + \frac{\tilde{B}}{(s+3)^2} + \frac{C}{s+2} + \frac{\tilde{D}}{(s+2)^2}$$

If you check $\tilde{B} = B-3A$

$\tilde{D} = D-2C$

We use this one because $\frac{1}{s+3}$, $\frac{1}{(s+3)^2}$, $\frac{1}{s+2}$, $\frac{1}{(s+2)^2}$

are easier to find in the transform table.

Example

$$\frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

Back to Initial Value Problem on page 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

Clear denominators and write as polynomial equality *

$$1 = A(s^2+1) + (Bs+C)(s+2)$$

$$1 = As^2 + A + Bs^2 + Cs + 2Bs + 2C$$

$$1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$$0s^2 + 0s + 1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$A + 2C = 1 \qquad A + B = 0 \qquad C + 2B = 0$

Eliminate A

$$\left.\begin{array}{r} A + 2C = 1 \\ -\quad A + B = 0 \\ \hline 2C - B = 1 \end{array}\right\}$$

Combine with $C + 2B = 0 \Rightarrow C = -2B$

$$2C - B = 1 \Rightarrow -4B - B = 1 \Rightarrow B = -\frac{1}{5}$$

$$A = -B = \frac{1}{5} \text{ and } C = -2B = \frac{2}{5}$$

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* We could set $s = -2$ and find $A = 1/5$, before expanding in powers to find B and C

$$\frac{1}{(s+2)(s^2+1)} = \frac{1/5}{s+2} + \frac{-\frac{1}{5}s + 2/5}{s^2+1}$$

$$= 1/5\,\frac{1}{s+2} - 1/5\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\} = 1/5\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 1/5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = 1/5\, e^{-2t} - 1/5 \cos t + \frac{2}{5} \sin t}$$

Table of Laplace Transforms

General	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
	$y(at)$	$\frac{1}{a}\,Y\left(\frac{s}{a}\right)$
Specific	e^{at}	$1/(s-a)$
	$\sin\omega t$	$\omega/(s^2+\omega^2)$
	$\cos\omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$
	1	$\frac{1}{s}$