L29 Inverse Laplace Transform

Note Title 8/13/2020

There is a formula ($Y(s) = \int_0^\infty e^{-st} y(t)\,dt$) to compute the Laplace Transform, but there is no comparable formula for the Inverse Laplace Transform. We compute the inverse Laplace transform using the table below.*

<u>General</u>	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
Specific	e^{at}	$1/(s-a)$
	$\sin\omega t$	$\omega/(s^2+\omega^2)$
	$\cos\omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$

* Actually, there are formulas for the Inverse Laplace Transform, but they are weird, and typically not so useful

https://en.wikipedia.org/wiki/Post's_inversion_formula

https://www.rose-hulman.edu/~bryan/invlap.pdf

Table of Laplace Transforms

General	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
	$y(at)$	$\frac{1}{a}Y\left(\frac{s}{a}\right)$
Specific	e^{at}	$1/(s-a)$
	$\sin\omega t$	$\omega/(s^2+\omega^2)$
	$\cos\omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$
	1	$\frac{1}{s}$

Examples

Compute $\mathcal{L}^{-1}\{y(t)\}$ for each of the following

$$Y(s) = \frac{1}{s-4}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$Y(s) = \frac{3s+4}{s^2+9}$$

$$Y(s) = \frac{1}{s^2+2s+5}$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

$$Y(s) = \frac{1}{s-4}$$

This is $\frac{1}{s-a}$

with $a=4$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$$

Table

$y(t)$	$Y(s)$
e^{at}	$1/(s-a)$

Different Ways I could label the table

Standard label	$y(t)$	$Y(s)$
Alternative 1	$y(t)$	$\mathcal{L}\{y(t)\}$
Alternative 2	$\mathcal{L}^{-1}\{Y(s)\}$	$Y(s)$
typical entry	e^{at}	$\frac{1}{s-a}$

expresed like Alternative 1 $\quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

expresed like Alternative 2 $\quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$\frac{1}{(s-4)^2} = \text{" } \frac{1}{s^2}\Big|_{s \mapsto s-4} \text{ "}$$

" $= \frac{1}{s^2}$ with s replaced by $(s-4)$ "

Table

$y(t)$	$Y(s)$
$e^{at}y(t)$	$Y(s-a)$
t^n	$\frac{n!}{s^{n+1}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{4t}\cdot t$$

Alternatively,

$$Y(s) = \frac{1}{(s-4)^2}$$

This is minus the derivative of $\frac{1}{s-4}$

$$\frac{d}{ds}\frac{1}{s-4} = \frac{-1}{(s-4)^2}$$

so

$$\frac{1}{(s-4)^2} = -\frac{d}{ds}\frac{1}{(s-4)}$$

Table Entries

$t\,y(t)$	$-\frac{d}{ds}Y(s)$
$e^{at}y(t)$	$Y(s-a)$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = te^{4t}$$

$$Y(s) = \frac{5s+4}{s^2+9}$$

Two table entries

$\cos 3t$	$\frac{s}{s^2+9}$
$\sin 3t$	$\frac{3}{s^2+9}$

$$Y(s) = 5 \cdot \frac{s}{s^2+9} + \frac{4}{3} \cdot \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y\} = 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{4}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = 5\cos 3t + \frac{4}{3}\sin 3t}$$

$$Y(s) = \frac{1}{s^2+2s+5}$$

what are the roots?

$s^2+2s+5=0$

$s^2+2s+1=-4$

$(s+1)^2=-4$

$s=-1\pm \cancel{4}\,2i$

This should be 2, NOT 4

table entries

$\cos 2t$	$\frac{s}{s^2+4}$
$\sin 2t$	$\frac{2}{s^2+4}$
$e^{-1t}y(t)$	$Y(s-(-1))$

Complex Roots - "complete the square"

$$Y(s) = \frac{1}{(s+1)^2+4} = \left.\frac{1}{s^2+4}\right|_{s\to s+1}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{e^{-t}}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{e^{-t}}{2}\sin 2t$$

$$\boxed{y(t) = \frac{e^{-t}}{2}\sin 2t}$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$= \frac{3s+4}{(s+1)^2+4}$$

I want to recognize $Y(s)$ as something with s replaced by s+1

$$= \frac{3(s+1)-3+4}{(s+1)^2+4}$$

$$= \frac{3(s+1)+1}{(s+1)^2+4} \overset{\text{"}}{=} \left.\frac{3s+1}{s^2+4}\right|_{s \mapsto s+1}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{3\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\right\}$$

$$= e^{-t}\left[3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$\boxed{y(t) = 3e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t}$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

Roots of Denominator

$$s^2+2s+1 = 9 \implies (s+1)^2 = 9$$

$$s = -1 \pm 3 = 2, -4$$

$$Y(s) = \frac{1}{(s-2)(s+4)}$$

$\frac{1}{s-2}$ and $\frac{1}{s+4}$ are in the table

Partial Fractions

$$\frac{1}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s-4}$$

$$1 = A(s+4) + B(s-2)$$

Set $s = 2$ $\quad 1 = A(2+4) \quad A = 1/6$

set $s = -4$ $\quad 1 = B(-4-2) \quad B = -1/6$

$$Y(s) = \frac{1}{(s-2)(s+4)} = \frac{1/6}{s-2} - \frac{1/6}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1/6}{s-2} - \frac{1/6}{s-4}\right\} = 1/6\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 1/6\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}$$