L28LaplaceTransformTable

Note Title 8/13/2020

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st} y(t)\, dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	?.
$\cos \omega t$	?.
1	?.
t	?.
$\frac{t^n}{n!}$	?.

We will fill in the rest of the table using "general rules" rather than by computing integrals.

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$

Alternative Notation

Rule 1 $\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$ $\left[= s\mathcal{L}\{y\} - y(0) \right]$

Rule 2 $\mathcal{L}\{e^{at} y(t)\} = Y(s-a)$ $\left[= \mathcal{L}\{y\}\big|_{s \to s-a} \right]$

Rule 3 $\mathcal{L}\{t\, y(t)\} = -\frac{d}{ds} Y(s)$ $\left[= -\frac{d}{ds} \mathcal{L}\{y\} \right]$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$ $\left[= \frac{1}{a} \mathcal{L}\{y\}\big|_{s \to \frac{s}{a}} \right]$

<u>General Rule</u> $\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$

<u>Proof</u>

$$\int_0^\infty \dot{y}\, e^{-st}\, dt = y e^{-st}\Big|_0^\infty + s\int_0^\infty y(t)\, e^{-st}\, dt$$

$$= -y(0) + sY(s)$$

<u>Problem</u>

Combine $\mathcal{L}\{0\} = 0$ and $\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$

to calculate $\mathcal{L}\{1\}$, using the fact that $\frac{d}{dt}1 = 0$.

<u>Answer:</u>

$$0 = \mathcal{L}\{0\} = \mathcal{L}\{\tfrac{d}{dt}1\} = s\,\mathcal{L}\{1\} - 1$$

$$0 = s\,\mathcal{L}\{1\} - 1$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

<u>Problem</u> Compute $\mathcal{L}\{t\}$ using $\dot{t} = 1$ and $\mathcal{L}\{1\} = \frac{1}{s}$

<u>Answer</u>

General Rule $\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$

$$\mathcal{L}\{\dot{t}\} = s\,\mathcal{L}\{t\} - 0$$

$$\frac{1}{s} = \mathcal{L}\{1\} = s\,\mathcal{L}\{t\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Use the general rule to:

1) Compute $\mathcal{L}\{\cos t\}$ in terms of $\mathcal{L}\{\sin t\}$

$$\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{\cos t\} = \mathcal{L}\left\{\frac{d\sin t}{dt}\right\} = s\,\mathcal{L}\{\sin t\} - \sin(0)$$

$$\mathcal{L}\{\cos t\} = s\,\mathcal{L}\{\sin t\}$$

2) Compute $\mathcal{L}\{\sin t\}$ in terms of $\mathcal{L}\{\cos t\}$

$$\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{\sin t\} = -\mathcal{L}\left\{\frac{d\cos t}{dt}\right\} = -\left(s\,\mathcal{L}\{\cos t\} - \cos(0)\right)$$

$$\mathcal{L}\{\sin t\} = -s\,\mathcal{L}\{\cos t\} + 1$$

3) Combine 1) and 2) to compute $\mathcal{L}\{\cos t\}$ and $\mathcal{L}\{\sin t\}$

$$\mathcal{L}\{\sin t\} = -s\,\mathcal{L}\{\cos t\} \qquad +1$$

$$\mathcal{L}\{\sin t\} = -s\cdot s\,\mathcal{L}\{\sin t\} \qquad +1$$

$$(1+s^2)\,\mathcal{L}\{\sin t\} = 1$$

$$\mathcal{L}\{\sin t\} = \frac{1}{1+s^2}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

Notation: where $Y(s) = \mathcal{L}\{y(t)\}$

General Rule 2

$$\mathcal{L}\{e^{at} y(t)\} = \mathcal{L}\{y(t)\}\Big|_{s \to s-a} \quad \left[= Y(s-a)\right]$$

Proof

$$\mathcal{L}\{e^{at} y(t)\} = \int_0^\infty e^{-st} e^{at} y(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} y(t)\,dt$$

Notation

$$= Y(s-a) \quad \left[= \mathcal{L}\{y(t)\}\Big|_{s-a}\right]$$

Use Rule 2 to compute $\mathcal{L}\{e^{at}\cos t\}$

$$\mathcal{L}\{e^{at}\cos t\} = \mathcal{L}\{\cos t\}\Big|_{s \to s-a}$$

$$= \frac{s}{s^2+1}\Big|_{s \to s-a}$$

$$= \frac{(s-a)}{(s-a)^2+1}$$

Similarly,

$$\mathcal{L}\{e^{at}\sin t\} = \mathcal{L}\{\sin t\}\Big|_{s \to s-a}$$

$$= \frac{1}{(s-a)^2+1}$$

Rule 3 $\mathcal{L}\{t\,y(t)\} = -\frac{d}{ds} Y(s)$ $\left[= -\frac{d}{ds}\mathcal{L}\{y(t)\}\right]$

Proof

$$\frac{d}{ds} Y(s) = \frac{d}{ds}\int_0^\infty e^{-st}\, y(t)\, dt$$

$$= \int_0^\infty \left(\frac{d}{ds} e^{-st}\right) y(t)\, dt$$

$$= \int_0^\infty \left(-t e^{-st}\right) y(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, t\, y(t)\, dt$$

$$= -\mathcal{L}\{t\,y(t)\}$$

Compute $\mathcal{L}\{t e^{at}\}$ using Rule 3

$$\mathcal{L}\{t\,y(t)\} = -\frac{d}{ds}\mathcal{L}\{y(t)\}$$

$$\mathcal{L}\{t e^{at}\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\} = -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$\boxed{\mathcal{L}\{t e^{at}\} = \frac{1}{(s-a)^2}}$$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$

Use Rule 4 to calculate $\mathcal{L}\{\sin \omega t\}$

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{\omega} \mathcal{L}\{\sin t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{1}{1+s^2}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{1}{1+\left(\frac{s}{\omega}\right)^2}$$

simplify to match standard table entry

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{1}{\omega} \mathcal{L}\{\cos t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{s}{s^2+1}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{s/\omega}{\left(\frac{s}{\omega}\right)^2+1}$$

simplify

$$= \frac{1}{\omega} \frac{\omega s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$

Proof of Rule 4

$$\mathcal{L}\{y(at)\} = \int_0^\infty e^{-st}\, y(at)\, dt$$

$$= \int_0^\infty e^{-\frac{s}{a}(at)}\, y(at)\, dt$$

Let $\tau = at$

$$= \int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, d\tau$$

$$= \frac{1}{a}\, \mathcal{L}\{y(t)\}\Big|_{s \mapsto \frac{s}{a}} = \frac{1}{a}\, Y\!\left(\frac{s}{a}\right)$$

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st}\, y(t)\, dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
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$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$

Rule 1 $\mathcal{L}\{\dot{y}\} = sY(s) - y(0) \quad \left[= s\,\mathcal{L}\{y\} - y(0)\right]$

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