Laplace Transform - a method for solving constant coefficient DE's/IVP's

Definition:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Notation: The Laplace transform of $f(t)$ is $F(s)$. Also written as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

These two statements mean the same thing:

$\mathcal{L}\{e^{-t}\} = \dfrac{1}{s+1}$ $\qquad f(t) = e^{-t} \quad F(s) = \dfrac{1}{s+1}$

The Laplace transform changes a function of t into a function of s.

Problem - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t}\,dt = \int_0^\infty e^{-(s+1)t}\,dt$$

$$= \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^\infty = \frac{0}{-(s+1)} - \frac{1}{-(s+1)} = \frac{1}{s+1}$$

Problem - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t}\, dt = \int_0^\infty e^{-(s+1)t}\, dt$$

Some technical details

$$\int_0^\infty e^{-(s+1)t}\, dt = \lim_{M\to\infty} \int_0^M e^{-(s+1)t}\, dt$$

$$= \lim_{M\to\infty} \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^M$$

$$= \lim_{M\to\infty} \left(\frac{1}{s+1} - \frac{e^{-M(s+1)}}{s+1} \right)$$

we always assume s is big enough so that this limit = 0

$$= \frac{1}{s+1} - 0 \qquad [\text{for } s > -1]$$

Calculate $\mathcal{L}\{e^{rt}\}$ where r is a constant.

$$\mathcal{L}\{e^{rt}\} = \int_0^\infty e^{-st} e^{rt}\, dt = \int_0^\infty e^{-(s-r)t}\, dt$$

$$= \left. \frac{e^{-(s-r)t}}{-(s-r)} \right|_0^\infty$$

$$= 0 + \frac{1}{(s-r)}$$

$$\boxed{\mathcal{L}\{e^{rt}\} = \frac{1}{(s-r)}}$$

Why we use Laplace Transform

The Laplace Transform of the derivative

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,\mathcal{L}\{y\} - y(0)$$

Proof

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \int_0^\infty e^{-ts}\frac{dy}{dt}\,dt$$

$$= e^{-ts}y(t)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}e^{-ts}\,y(t)\,dt$$

$$= 0 - y(0) - \int_0^\infty (-s)\,e^{-ts}\,y(t)\,dt$$

$$= -y(0) + s\int_0^\infty e^{-ts}\,y(t)\,dt$$

$$= s\,\mathcal{L}\{y\} - y(0)$$

$\blacksquare$

Laplace Transform Table

	$y(t)$	$Y(s) = \mathcal{L}\{y(t)\}$
①	e^{rt}	$\dfrac{1}{s-r}$
②	$\dfrac{dy}{dt}$	$sY(s) - y(0)$
③	$f(t) + g(t)$	$F(s) + G(s)$

$Y(s)$ is notation for $\mathcal{L}\{y(t)\}$

Table summarizes these facts

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

Laplace transform is linear

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \qquad \mathcal{L}\{0\} = 0$$

One Additional Fact

Laplace transform is one to one.

If $\mathcal{L}\{y(t)\} = 0$ then $y(t) = 0$.

Solve IVP $\begin{cases} \frac{dy}{dt} + 4y = 0 & \text{(DE)} \\ y(0) = 1 & \text{(IC)} \end{cases}$ using Laplace transforms

$$\frac{dy}{dt} + 4y = 0$$

$$\mathcal{L}\left\{\frac{dy}{dt} + 4y\right\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y\} = 0$$ uses 3 from table

$$sY(s) - y(0) + 4Y(s) = 0$$ uses 2 from table

$$(s+4)Y(s) - 1 = 0$$

$$Y(s) = \frac{1}{s+4}$$

$$Y(s) = \mathcal{L}\{e^{-4t}\}$$ uses 1 from table

$$\boxed{y(t) = e^{-4t}}$$

We have solved (IVP) without integrating or differentiating. We only used algebra.

Laplace Transform Approach to IVP

$$y'' + 2y' + y = \cos\omega t$$
$$y(0) = 0$$
$$y'(0) = 0$$

$\mathcal{L}$ Laplace Transform

$$s^2 Y + 2sY + Y = \frac{s}{s^2+\omega^2}$$

where

$$Y(s) = \mathcal{L}\{y\}$$
$$\frac{s}{s^2+\omega^2} = \mathcal{L}\{\cos\omega t\}$$

$$Y(s) = \frac{s}{(s^2+2s+1)(s^2+\omega^2)}$$

$\mathcal{L}^{-1}$ "Inverse" Laplace Transform

I haven't explained this step yet

$$\frac{2\omega \sin\omega t - (\omega^2-1)\cos\omega t - e^{-t} - (2\omega^2+1)\, t e^{-t}}{(\omega^2+1)^2}$$

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st} y(t)\,dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\frac{t^n}{n!}$	$\frac{1}{(s-a)^{n+1}}$
$\dot{y}(t)$	$sY(s) - y(0)$
$\ddot{y}(t)$	$s^2Y(s) - sy(0) - \dot{y}(0)$

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$