L26SecondOrderSummary

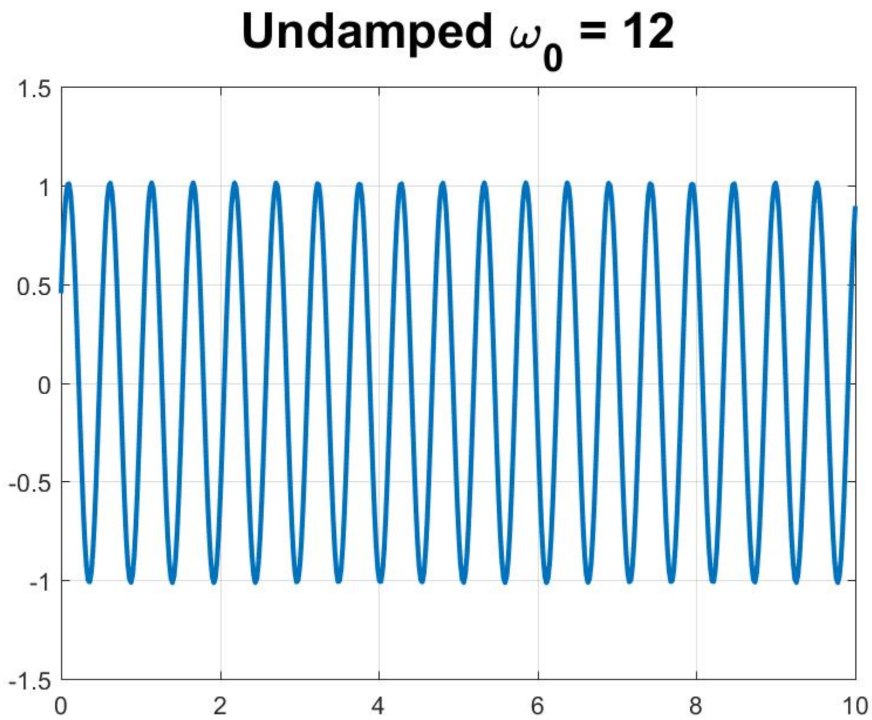
Note Title 8/9/2020

Summary -

Second Order Constant Coefficient ODE

Harmonic Motion

Unforced Motion Undamped



$$\ddot{y} + \frac{k}{m} y = 0$$ mass-spring

$$\ddot{y} + \omega_0^2 y = 0$$ harmonic oscillator

ω_0 = natural frequency

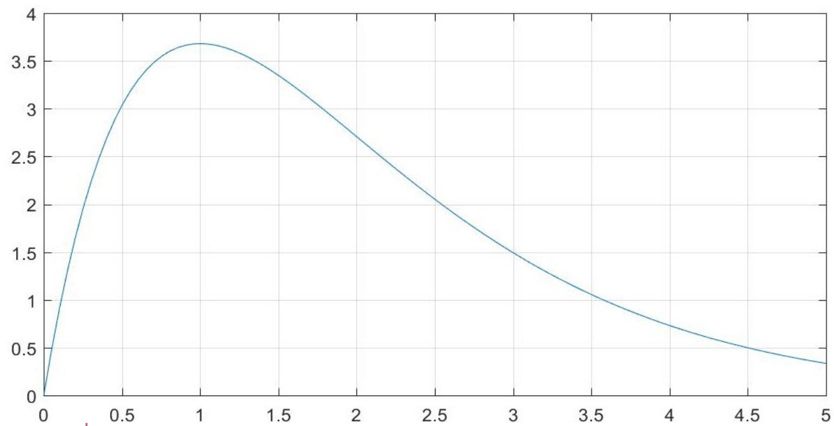
$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$= A \cos(\omega_0 t - \varphi)$$

Damped

$$\ddot{y} + 2\zeta\omega_0 y + \omega_0^2 y = 0$$ harmonic oscillator

$$\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m} y = 0$$ mass-spring



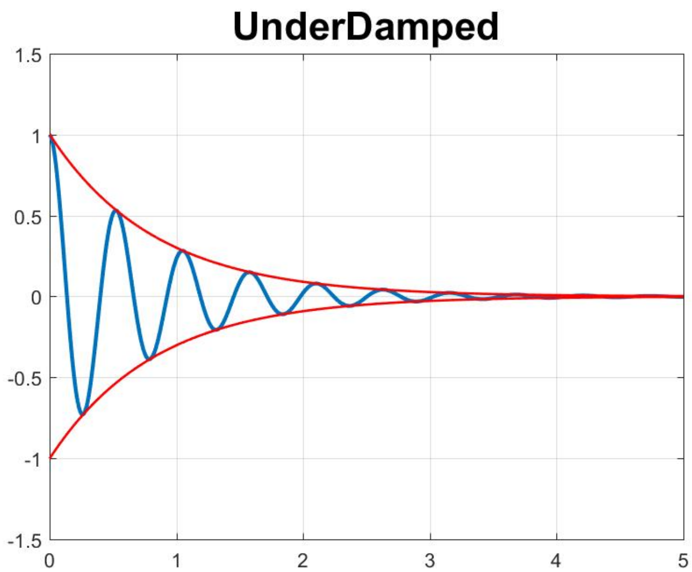
overdamped

$$y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$$

$$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$$

has 2 real roots

[This is equivalent to $\zeta > 1$]



$$y = c_1 e^{-\frac{\gamma}{2m}t}\cos\omega_d t + c_2 e^{-\frac{\gamma}{2m}t}\sin\omega_d t$$

$$y = A e^{-\frac{\gamma}{2m}t}\cos(\omega_d t - \phi)$$

$$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$$

has complex roots

[$\zeta < 1$]

Critically Damped

$$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$$

has 1 (real) root

$$y = c_1 e^{-rt} + c_2 t e^{-rt}$$

[This is equivalent to $\zeta = 1$]

Graph looks like overdamped

Forced Motion

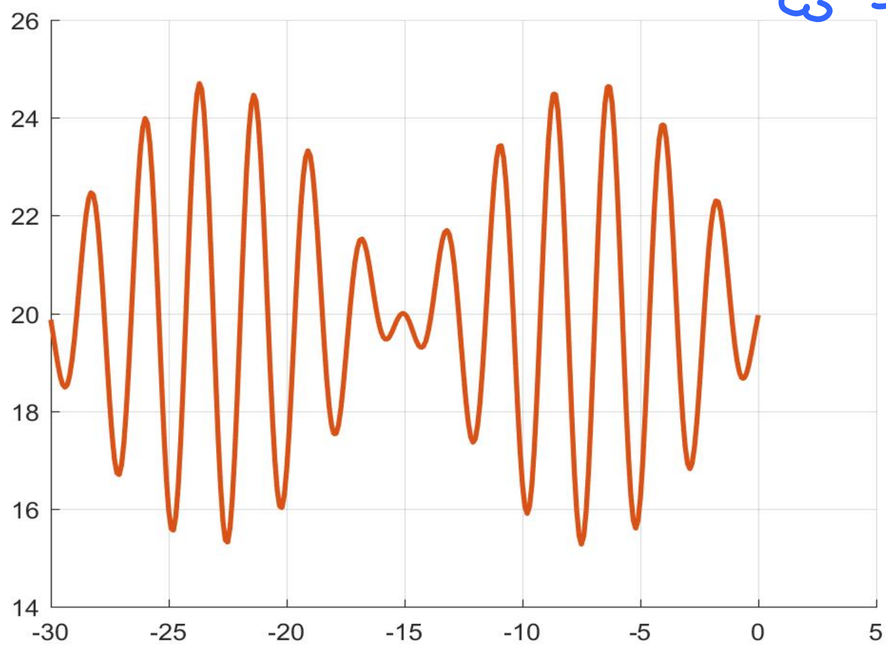
Undamped

Beats $\ddot{y} + \omega_0^2 y = \cos\omega t$

$y(0) = 0 \quad \dot{y}(0) = 0$

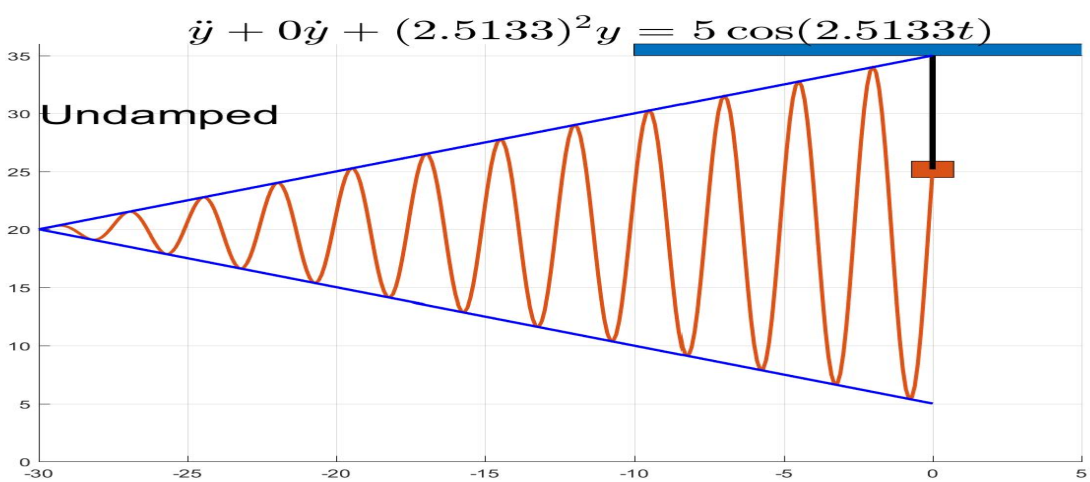
average freq.
half difference

$$y = \frac{2\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)}{\omega^2-\omega_0^2}$$



Resonance $\ddot{y} + \omega_0^2 y = \cos\omega_0 t \quad y(0) = 0 \quad \dot{y}(0) = \overset{0}{\cancel{1/2}}$

$$y = \frac{t\sin\omega_0 t}{2\omega_0}$$



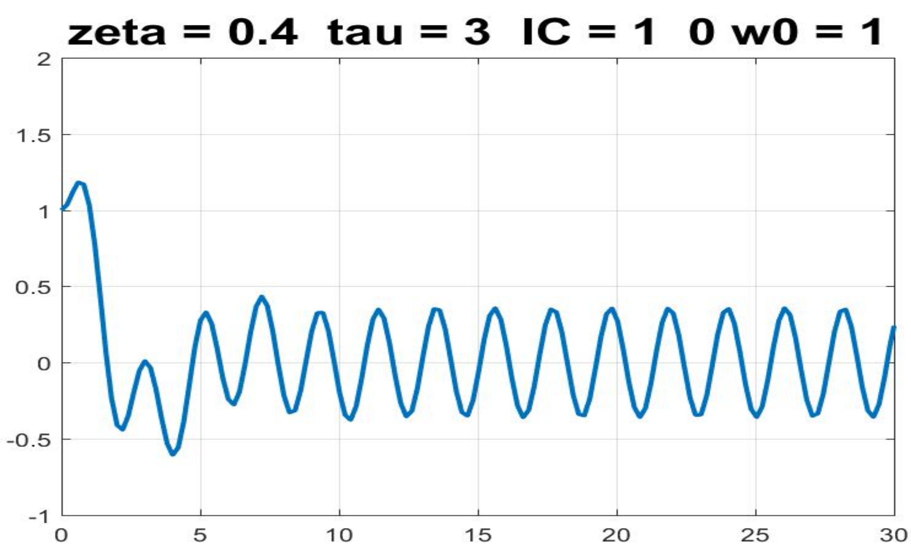
Damped Forced $\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m}y = \cos\omega t$

$$\left[\text{or } \ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = \cos\omega t\right]$$

$$y_{ss} = A\cos(\omega t - \varphi)$$

we can calculate A and φ in terms of ζ, ω_0, ω

$$A = \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$



I don't expect you to memorize any of these formulas. I expect you to be able to work them out in specific cases.

Mathematical Methods for Second Order Constant Coefficient ODE

① Find homogeneous solutions as linear combinations of functions e^{rt} or $t e^{rt}$.

② Particular Solutions are linear combinations of forcing functions and their derivatives.

If forcing terms satisfy homogeneous equation, particular solution must be multiplied by t. Repeat until no term satisfies homogeneous equation.