Note Title 8/9/2020

Damped Force Oscillations (Filters)

Example $\ddot{y} + 2\zeta\dot{y} + y = \cos(\omega t)$

① Find the steady state solution

② Find the amplitude and phase as functions of ζ and ω

③ At what frequency ω is the amplitude largest?

The calculation we are about to do involves a lot of letters. It takes time to do it accurately. However, the steps are just what we use when zeta and omega are specific numbers: 1) Undetermined coefficients and 2) conversion of the solution to polar (frequency,amplitude, phase) form.

Answer

Seek $y = A\cos\omega t + B\sin\omega t$

$$2\zeta\dot{y} = -2\zeta A\omega\sin\omega t \quad 2\zeta B\omega\cos\omega t$$

$$+ \quad \ddot{y} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$\ddot{y} + 2\zeta\dot{y} + y = [(1-\omega^2)A + 2\zeta\omega B]\cos\omega t + [(1-\omega^2)B - 2\zeta\omega A]\sin\omega t$$

$$\cos\omega t = [(1-\omega^2)A + 2\zeta\omega B]\cos\omega t + [(1-\omega^2)B - 2\zeta\omega A]\sin\omega t$$

so

$$[(1-\omega^2)A + 2\zeta\omega B] = 1$$

$$[(1-\omega^2)B - 2\zeta\omega A] = 0$$

$$(1-\omega^2)A + 2\zeta\omega B = 1$$

$$(1-\omega^2)B - 2\zeta\omega A = 0 \implies A = \frac{(1-\omega^2)}{2\zeta\omega} B$$

$$\frac{(1-\omega^2)^2}{2\zeta\omega} B + 2\zeta\omega B = 1 \implies \left[(1-\omega^2)^2 + (2\zeta\omega)^2\right] B = 2\zeta\omega$$

$$B = \frac{2\zeta\omega}{(1-\omega^2)^2 + (2\zeta\omega)^2} \qquad A = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (2\zeta\omega)^2}$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (2\zeta\omega)^2} \cos\omega t + \frac{2\zeta\omega}{(1-\omega^2)^2 \, (2\zeta\omega)^2} \sin\omega t$$

We want to know how big the steady state solution will become, so we write y_{ss} in amplitude phase form

$$y_{ss} = A\cos(\omega t - \varphi)$$

Amplitude Phase Form

$$A\cos(\omega t - \varphi) = A\cos\varphi\cos t + A\sin\varphi\sin\omega t$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2 \; (2\zeta\omega)^2}\cos\omega t + \frac{2\zeta\omega}{(1-\omega^2)^2+(2\zeta\omega)^2}\sin\omega t$$

$$A\cos\varphi = \frac{(1-\omega^2)}{(1-\omega^2)^2+(2\zeta\omega)^2} \qquad A\sin\varphi = \frac{2\zeta\omega}{(1-\omega^2)^2+(2\zeta\omega)^2}$$

$$A^2 = \frac{(1-\omega^2)^2+(2\zeta\omega)^2}{\left[(1-\omega^2)^2+(2\zeta\omega)^2\right]^2} = \boxed{\frac{1}{(1-\omega^2)^2+(2\zeta\omega)^2}}$$

$$\boxed{A = \frac{1}{\left[(1-\omega^2)^2+(2\zeta\omega)^2\right]^{1/2}}} \qquad \tan\varphi = \frac{2\zeta\omega}{(1-\omega^2)}$$

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{2\zeta\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2+(2\zeta\omega)^2\right]^{1/2}}}$$

I won't ask you to derive or memorize this formula, but I might give you the formula below and expect you to use it to answer questions like (3) and (4) on the last page.

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{2\zeta\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2 + (2\zeta\omega)^2\right]^{1/2}}}$$

This formula answers many important questions.

$$A(\omega,\zeta) = \frac{1}{\left[(1-\omega^2)^2 + (2\zeta\omega)^2\right]^{1/2}}$$

The amplitude tells us how much the forcing amplitude is magnified or damped.

<u>Example</u> $\zeta = 0.1$

$$\ddot{y} + 0.1\dot{y} + y = \cos(1.1t) + \cos(3t)$$

Because the equation is linear, y_{ss} can be written as a sum of two solutions:

$$\ddot{y}_1 + 0.1\dot{y}_1 + y_1 = \cos(1.1t)$$

$$\ddot{y}_2 + 0.1\dot{y}_2 + y_2 = \cos(3t)$$

$$y_1 = \frac{1}{\left[(1-1.1^2)^2 + (0.11)^2\right]^{1/2}} \cos\left(1.1t - \operatorname{atan}\left(\frac{0.11}{1-1.1^2}\right)\right)$$

$$y_2 = \frac{1}{\left[(1-3^2)^2 + (0.33)^2\right]^{1/2}} \cos\left(3t - \operatorname{atan}\left(\frac{0.33}{1-3^2}\right)\right)$$

$$y_1 = 4.2\cos(1.1t - 2.7)$$

$$y_2 = 0.125\cos(3t - 3.10)$$

$$y_{ss} = y_1 + y_2 = 4.2\cos(1.1t - 2.7) + 0.125\cos(3t - 3.10)$$

Important Feature

$$\frac{4.2}{0.125} = 33.7$$

The response to $\cos(1.1t)$ is 34 times bigger than the response to $\cos(3t)$.

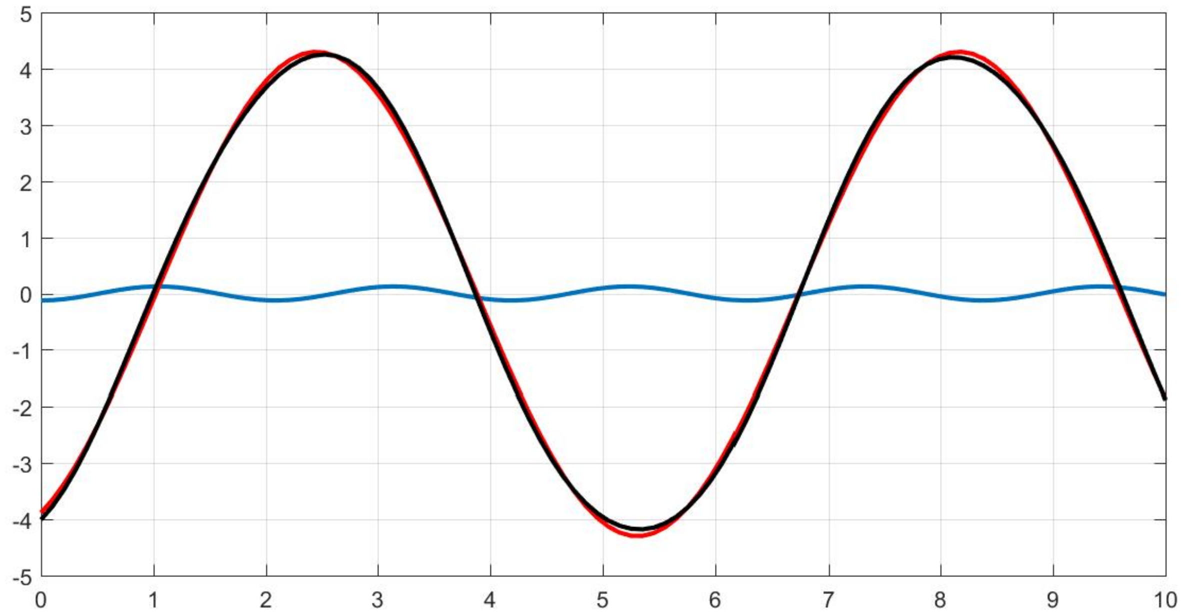
Summary

$$\ddot{y} + 0.1\dot{y} + y = \cos(1.1t) + \cos(3t)$$

$$y_{ss} = 4.2\cos(1.1t + 2.7) + 0.125\cos(3t + 3.10)$$

$4.2\cos(1.1t + 2.7)$ $\quad$ $4.2\cos(1.1t + 2.7) + 0.125\cos(3t + 3.10)$

$0.125\cos(3t + 3.10)$



The steady state solution magnifies frequencies near $\omega = 1$ and suppresses frequencies far from $\omega = 1$. It "filters out" frequencies away from $\omega = 1$.

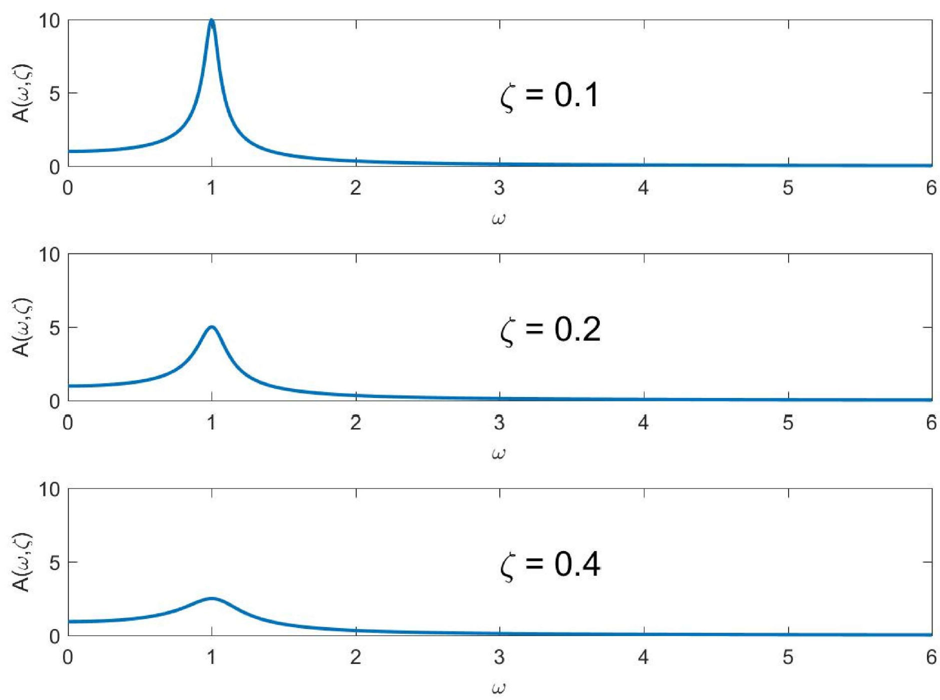
This is what makes (over the air) radios work. The antenna receives many transmissions from many different stations, each with its own distinct frequency. By turning a dial (nowadays its a digitally assisted dial), you adjust the natural frequency (in our example the natural frequency is omega=1). The response to the signal from the station nearest the natural frequency is magnified, and the reponse to the stations with frequencies far from the natural frequency is

$$\ddot{y} + 2\zeta\dot{y} + y = \cos\omega t$$

$$y_{ss} = A(\omega,\zeta)\cos(\omega t - \varphi(\omega,\zeta))$$

$$A(\omega,\zeta) = \frac{1}{\left[(1-\omega^2)^2 + (2\zeta\omega)^2\right]^{1/2}}$$

The smaller the value of ζ, the higher and narrower the peak.



If we replace the DE with

$$\ddot{y} + 2\omega_0\zeta\dot{y} + \omega_0^2 y = \cos\omega t$$

then the frequency peak moves to ω_0

③ At what frequency ω is the amplitude largest?

$$A(\omega,\zeta) = \frac{1}{\left[(1-\omega^2)^2+(2\zeta\omega)^2\right]^{1/2}}$$

A is largest when $(1-\omega^2)^2+(2\zeta\omega)^2$ is smallest

$$0 = \frac{d}{d\omega}\left[(1-\omega^2)^2+(2\zeta\omega)^2\right] = -\cancel{2}\omega\cancel{2}(1-\omega^2)+2\cancel{(}2\zeta\cancel{\omega})\cdot\cancel{2}\zeta$$

$$0 = -1+\omega^2+2\zeta^2$$

$$\omega_{max} = \sqrt{1-2\zeta^2}$$

④ What is the maximum amplitude?

$$A(\omega_{max},\zeta) = \frac{1}{\sqrt{(1-(1-2\zeta^2))^2+4\zeta^2(1-2\zeta^2)}}$$

$$= \frac{1}{\sqrt{4\zeta^4+4\zeta^2-8\zeta^4}}$$

$$= \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$