Resonance and Beats $\ddot{y} + y = \cos \omega t$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Two Cases $\omega = 1$ $y(t) = \frac{t \sin t}{2}$

(see lecture L22 Undetermined Coefficients3)

$\omega \neq 1$ $y_H = C_1 \cos t + C_2 \sin t$

Seek $y_p = A \cos \omega t + B \sin \omega t$

$1 \times y_p = A \cos \omega t + B \sin \omega t$

$0 \times \dot{y}_p = -\omega A \sin \omega t + \omega B \cos \omega t$

$+ \quad 1 \times \ddot{y}_p = -A \omega^2 \cos \omega t - \omega^2 B \sin \omega t$

$$\cos t = (1 - \omega^2) A \cos \omega t + (1 - \omega^2) B \sin \omega t$$

so $A = \frac{1}{(1 - \omega^2)}$ $B = 0$ and $y_p = \frac{\cos \omega t}{(1 - \omega^2)}$

$$y(t) = \frac{\cos \omega t}{(1 - \omega^2)} + C_1 \cos t + C_2 \sin t$$

Setting $0 = y(0)$ yields $C_1 = \frac{-1}{(1 - \omega^2)}$

Setting $0 = \dot{y}(0)$ yields $C_2 = 0$

$$\boxed{y(t) = \frac{\cos(\omega t) - \cos t}{1 - \omega^2}}$$

* It would not be unreasonable to ask you to do
a calculation like this on an exam.

$$\ddot{y} + y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

$$\omega \neq 1 \qquad y(t) = \frac{\cos \omega t - \cos t}{1 - \omega^2}$$

$$\omega = 1 \qquad y(t) = \frac{t \sin t}{2}$$

Slightly more general

$$\ddot{y} + \omega_0^2 \, y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

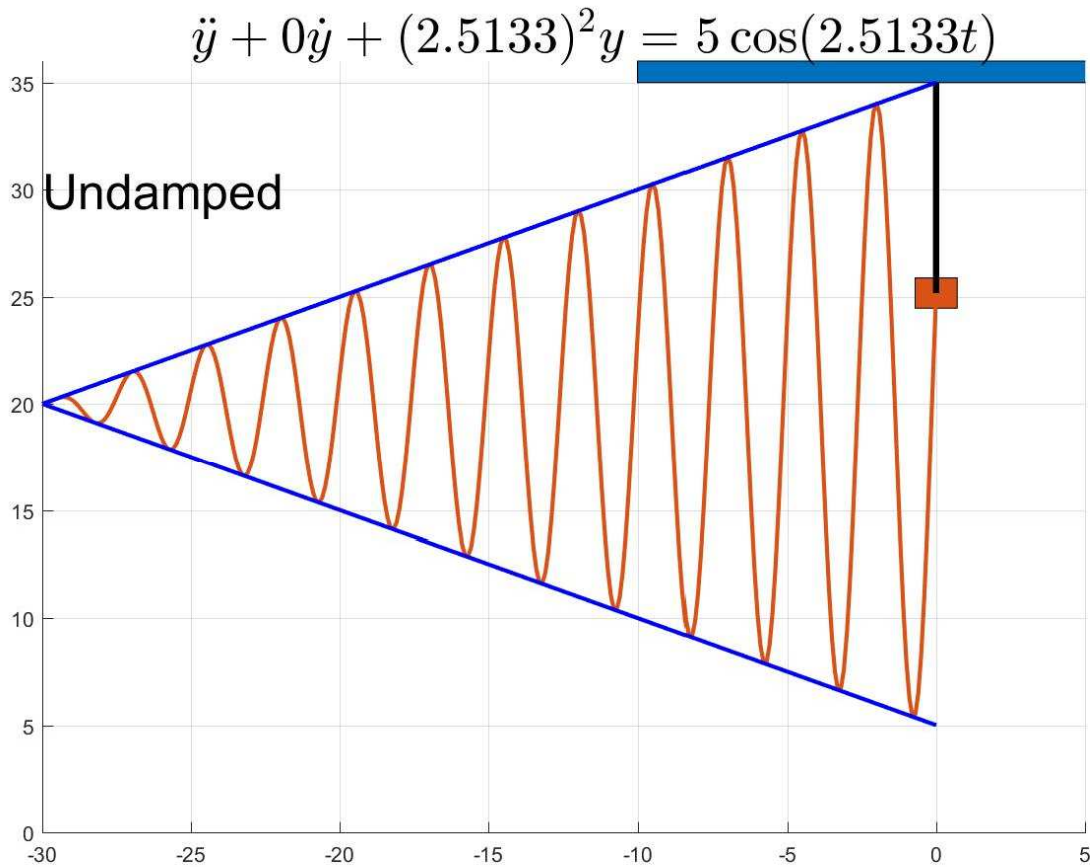$$\omega \neq \omega_0 \qquad y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

$$\omega = \omega_0 \qquad y(t) = \frac{t \sin \omega_0 t}{2 \omega_0}$$

$$\omega_0 = \text{Natural frequency}$$

$$\omega = \text{Forcing frequency}$$

What do the graphs of these solutions look like?

$\omega_0 = \omega$ Resonance

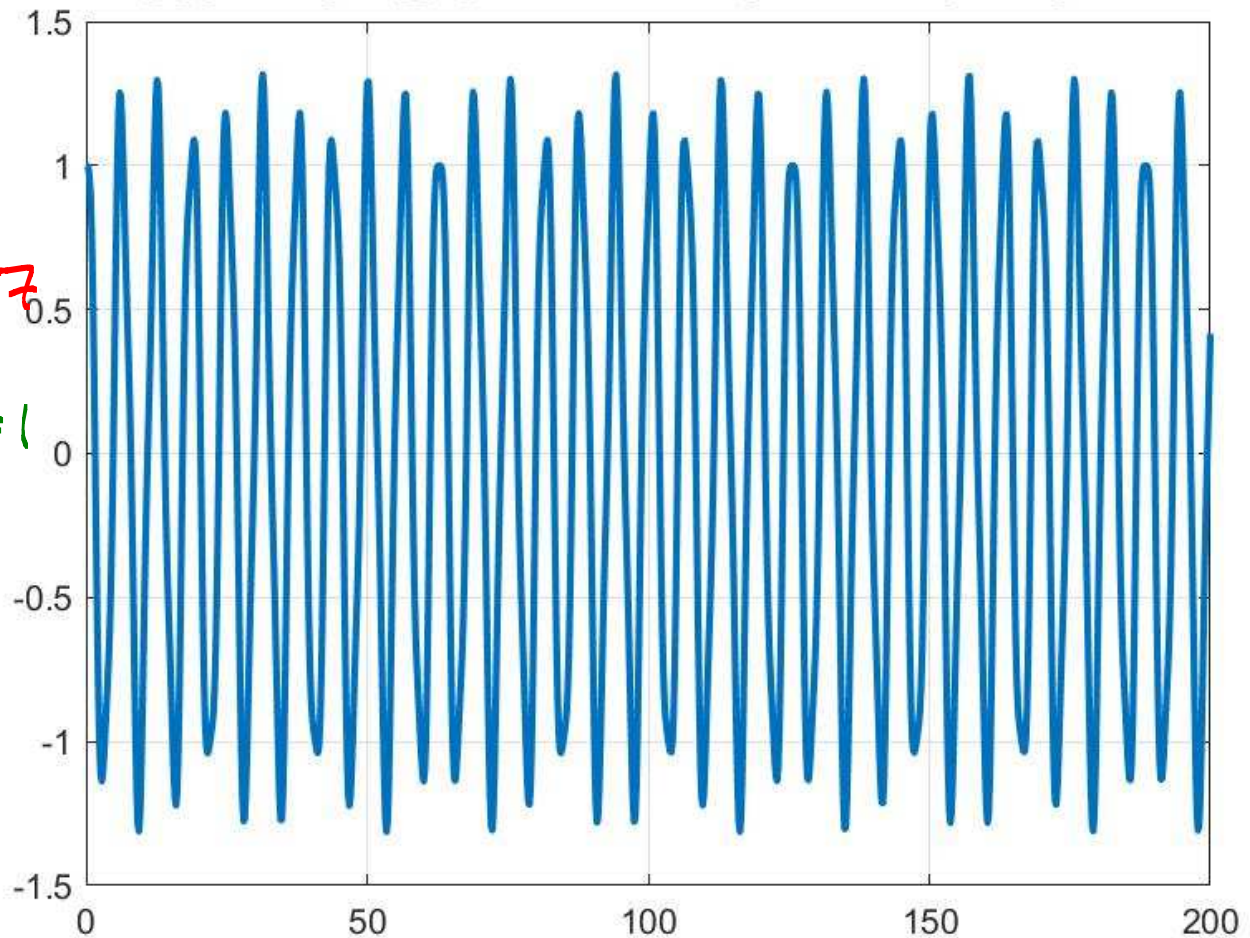$$\ddot{y} + 0\dot{y} + (2.5133)^2 y = 5\cos(2.5133t)$$

Undamped

$$y(t) = \frac{t\sin(2.5133t)}{2 \cdot 2.5133}$$

Resonance – displacement $y(t)$ grows larger and larger – this can lead to disaster (e.g. Tacoma Narrows Bridge Collapse)

ω_0 far from ω — Nothing Special

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega = 2.7$

$\omega_0 = 1$

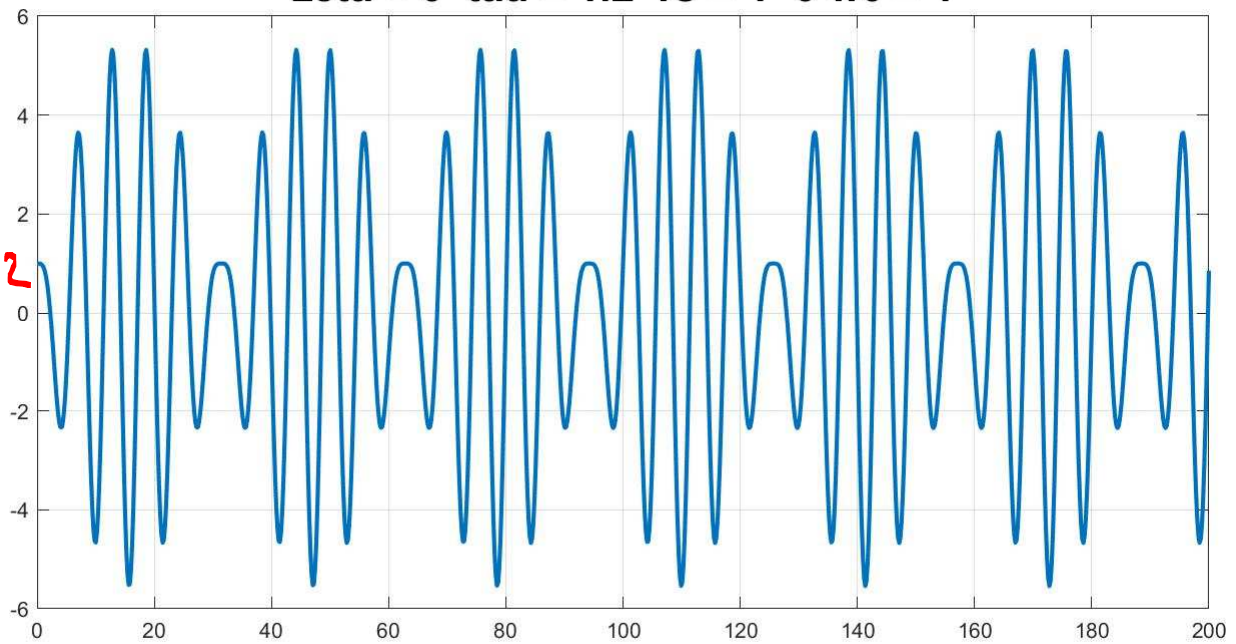**zeta = 0 tau = 2.7 IC = 1 0 w0 = 1**

ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

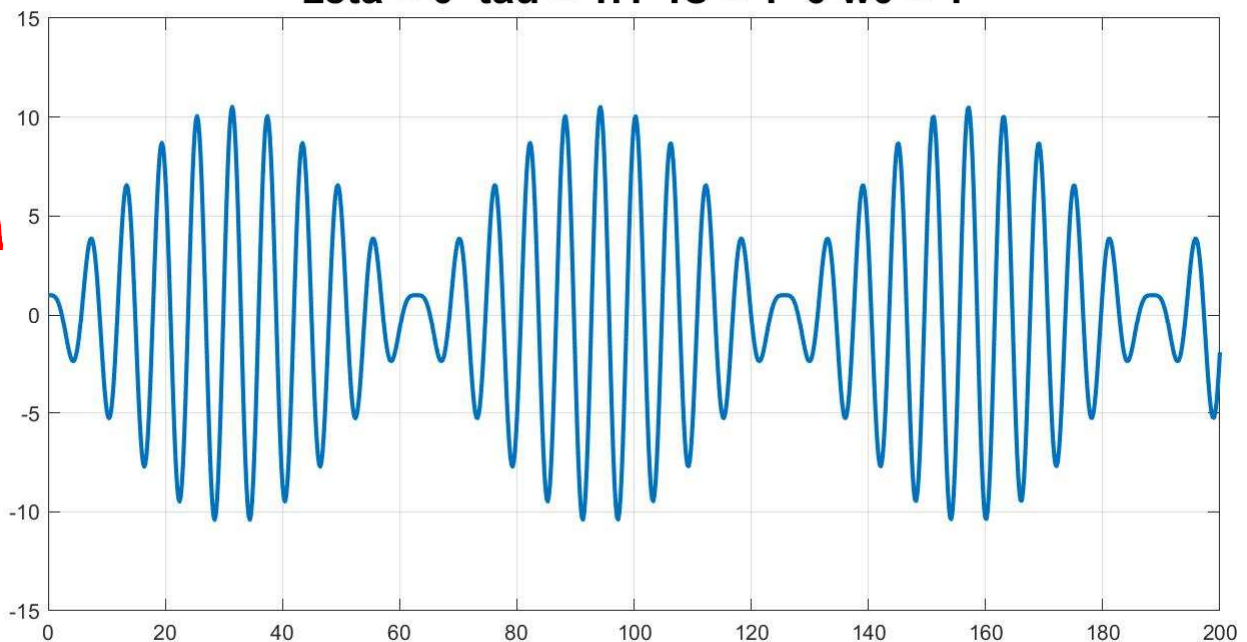**zeta = 0 tau = 1.2 IC = 1 0 w0 = 1**

$\omega_0 = 1$

$\omega = 1.2$

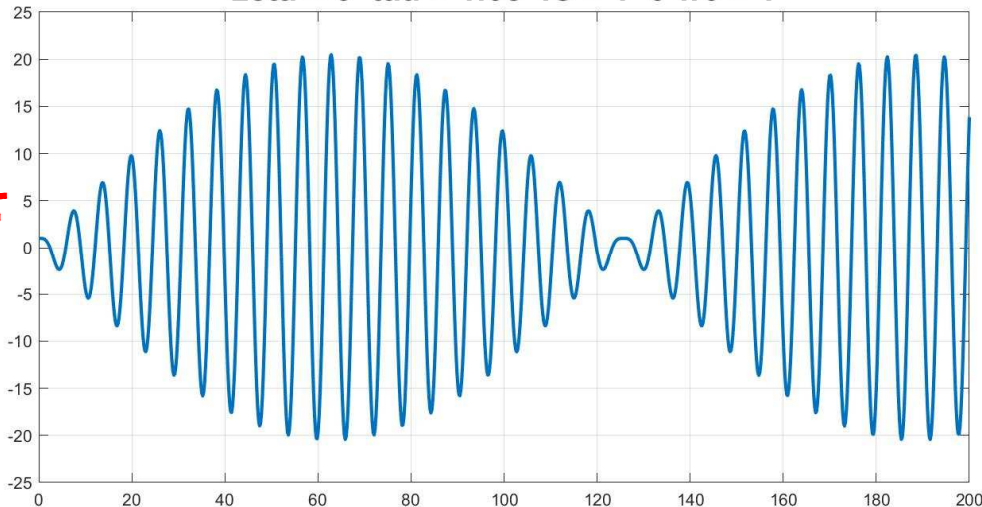**zeta = 0 tau = 1.1 IC = 1 0 w0 = 1**

$\omega_0 = 1$

$\omega = 1.1$

ω_0 close to ω — Beats

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$
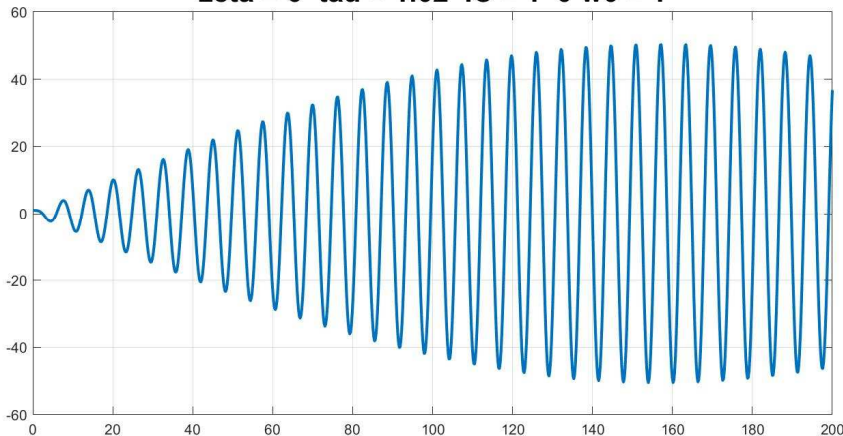
zeta = 0 tau = 1.05 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.05$

zeta = 0 tau = 1.02 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.02$

zeta = 0 tau = 1.01 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.01$

ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

zeta = 0 tau = 1.2 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.2$

20π

Beat period = 20π
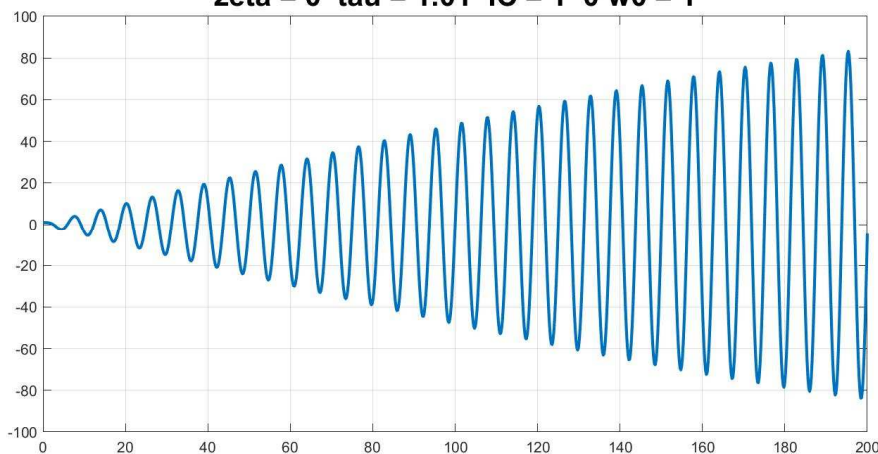
$\omega_{Beat} = \frac{2\pi}{20\pi} = 0.1$

zeta = 0 tau = 1.1 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.1$

40π

Beat period = 40π

$\omega_{Beat} = \frac{2\pi}{40\pi} = 0.05$

ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega_0 = 1$

$\omega = 1.05$

zeta = 0 tau = 1.05 IC = 1 0 w0 = 1

Beat period = 80π 40π $\omega_{Beat} = \frac{2\pi}{80\pi} = 0.025$

$\omega_0 = 1$

$\omega = 1.02$

zeta = 0 tau = 1.02 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.01$
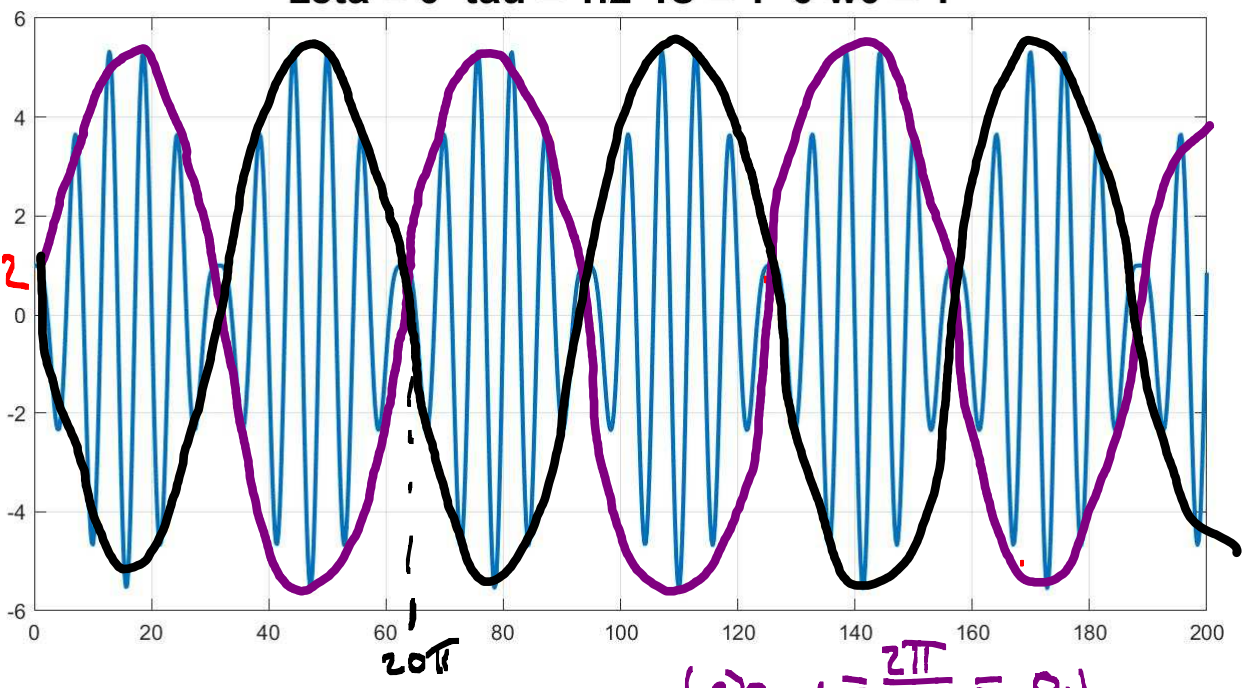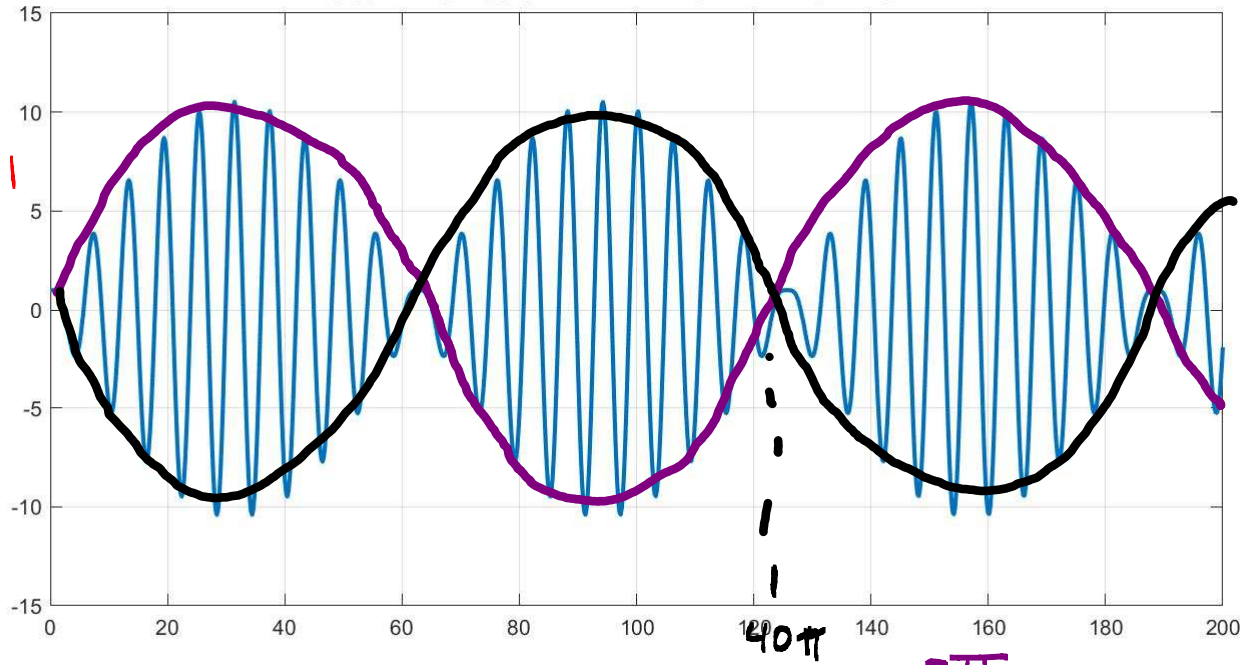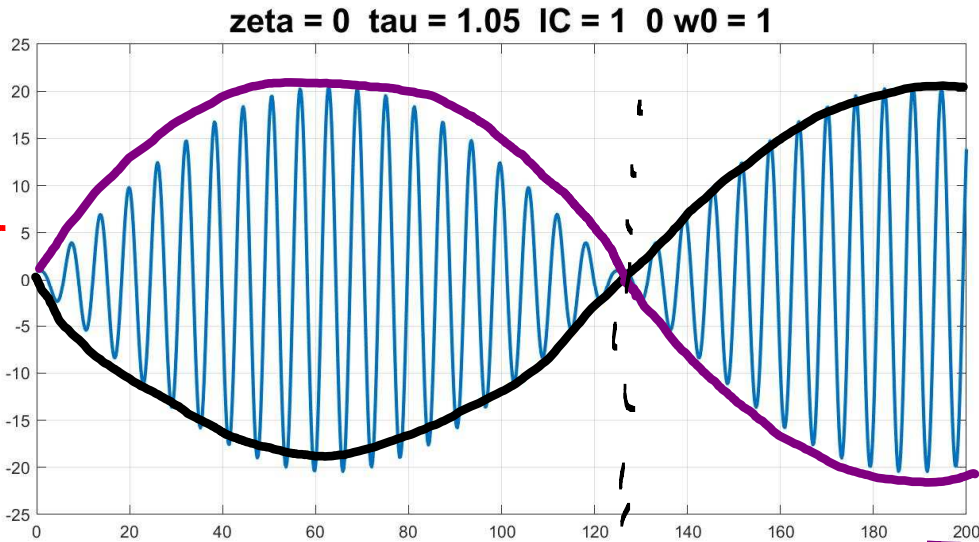
zeta = 0 tau = 1.01 IC = 1 0 w0 = 1

Observation

$$\omega_{Beat} = \frac{\omega_0 - \omega}{2}$$

We can't easily see this from the formula:

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

but this formula can be rewritten in a way that makes the beats phenemonon visible

$$y(t) = -2 \frac{\sin \left(\frac{\omega_0 - \omega}{2} t\right)}{\omega_0 - \omega} \cdot \frac{\sin \left(\frac{\omega_0 + \omega}{2} t\right)}{\omega_0 + \omega}$$

Product of Sines with

$\frac{\omega_0 + \omega}{2}$ average frequency $\frac{\omega_0 - \omega}{2}$ half difference

$$\omega = \frac{\omega + \omega_0}{2} + \frac{\omega - \omega_0}{2}$$

$$\omega_0 = \frac{\omega + \omega_0}{2} - \frac{\omega - \omega_0}{2}$$

$\frac{\omega + \omega_0}{2}$ Average

$\frac{\omega - \omega_0}{2}$ Half Difference

Sum of angles formula for cosine

$$\cos \omega t = \cos\left(\frac{\omega + \omega_0}{2}t + \frac{\omega - \omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega + \omega_0}{2}t\right)\cos\left(\frac{\omega - \omega_0}{2}t\right) - \sin\left(\frac{\omega + \omega_0}{2}t\right)\sin\left(\frac{\omega - \omega_0}{2}t\right) \quad (1)$$

$$\cos \omega_0 t = \cos\left(\frac{\omega + \omega_0}{2}t - \frac{\omega - \omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega + \omega_0}{2}t\right)\cos\left(\frac{\omega - \omega_0}{2}t\right) + \sin\left(\frac{\omega + \omega_0}{2}t\right)\sin\left(\frac{\omega - \omega_0}{2}t\right) \quad (2)$$

subtract (2) from (1)

$$\cos \omega t - \cos \omega_0 t = -2 \sin\left(\frac{\omega + \omega_0}{2}t\right)\sin\left(\frac{\omega - \omega_0}{2}t\right)$$

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

$$= \frac{-2 \sin\left(\frac{\omega + \omega_0}{2} t\right) \sin\left(\frac{\omega - \omega_0}{2} t\right)}{(\omega_0 - \omega)(\omega_0 + \omega)}$$

$$y(t) = \frac{2 \sin\left(\frac{\omega - \omega_0}{2} t\right)}{\omega - \omega_0} \cdot \frac{\sin\left(\frac{\omega + \omega_0}{2} t\right)}{\omega + \omega_0}$$

Digression

$$\lim_{\omega \to \omega_0} \frac{2 \sin\left(\frac{\omega - \omega_0}{2} t\right)}{\omega - \omega_0} = t$$

so $\quad \displaystyle\lim_{\omega \to \omega_0} y(t) = \frac{t \sin(\omega_0 t)}{2 \omega_0}$

so we see the "resonance solution" as a limit of "beats solutions".

Example

$$\ddot{y} + 4^2 y = \cos 5t$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Solution (after some calculations)

$$y = \frac{\cos 4t - \cos 5t}{9}$$

We write y as a product to visualize the "beats".

$$\omega_{AV} = \frac{4+5}{2} \qquad \omega_{HD} = \frac{5-4}{2} \qquad \begin{bmatrix} \text{use positive} \\ \text{difference} \\ \text{for convenience} \end{bmatrix}$$

$$\cos 4t = \cos\left(\tfrac{9}{2}t - \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$
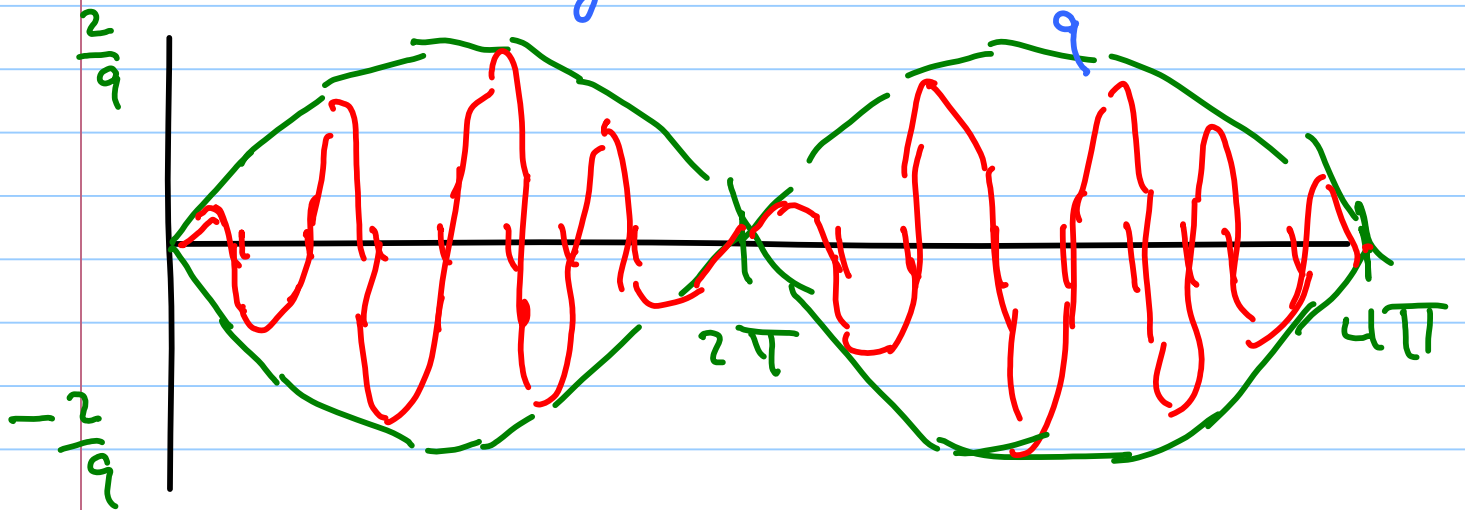
$$\cos 5t = \cos\left(\tfrac{9}{2}t + \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) - \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$

$$\cos 4t - \cos 5t = -2 \sin\tfrac{9}{2}t \, \sin\tfrac{t}{2}$$

$$y(t) = \frac{-2 \sin\tfrac{9}{2}t \, \sin\tfrac{t}{2}}{9}$$

Sketch

$$y(t) = 2 \sin \frac{9}{2} t \, \sin \frac{t}{2}$$

$\frac{2}{9}$

$-\frac{2}{9}$

2π 4π

$\sin \frac{t}{2}$ $P = \frac{2\pi}{\frac{1}{2}} = 4\pi \approx 12$

$\sin \frac{9}{2} t$ period $= \frac{2\pi}{9/2} = \frac{4\pi}{9} \approx \frac{12}{9}$

oscillates 9 times faster than $\sin \frac{t}{2}$

$\sin \frac{9}{2} t$ has 9 zeroes for every zero of $\sin \frac{t}{2}$