Steady State and Transients

$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

① $r^2 + 0.2r + 1 = 0$

$$(r+0.1)^2 = -0.99$$

$$r = -0.1 \pm i\sqrt{0.99}$$

$$y_H = c_1 e^{-0.1t}\cos(0.995t) + c_2 e^{-0.1t}\sin(0.995t)$$

② $y_p = A\cos 2t + B\sin 2t$

$$\dot{y}_p = -2A\sin 2t + 2B\cos 2t$$

$$\ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\ddot{y}_p + 0.2\dot{y}_p + y_p = (-3A + 0.4B)\cos 2t$$

$$+ (-3B - 0.4A)\sin 2t$$

$$\ddot{y}_p + 0.2\dot{y}_p + y_p = (-3A + 0.4B)\cos 2t + (-3B - 0.4A)\sin 2t$$

$$\cos 2t = (-3A + 0.4B)\cos 2t + (-3B - 0.4A)\sin 2t$$

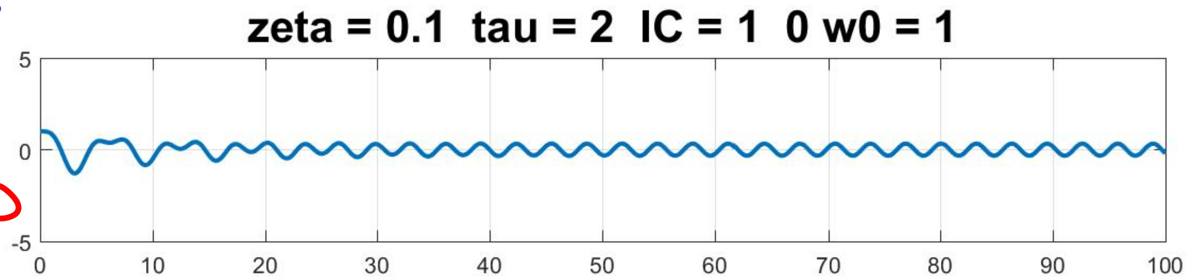
$$\left.\begin{aligned} 1 &= -3A + 0.4B \\ 0 &= -0.4A - 3B \end{aligned}\right\} \quad \begin{aligned} A &= -0.3275 \\ B &= 0.0437 \end{aligned}$$

$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

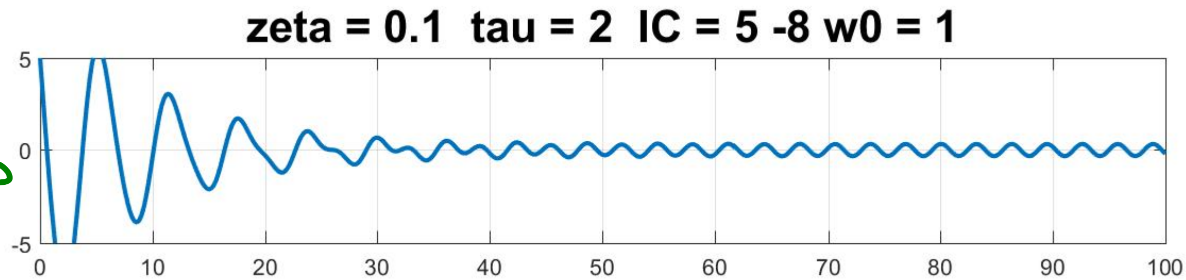
$$y = -0.375\cos 2t + 0.0437\sin 2t$$
$$+ \, C_1 e^{-0.1t}\cos(0.995t) + C_2 e^{-0.1t}\sin(0.995t)$$

Initial Conditions

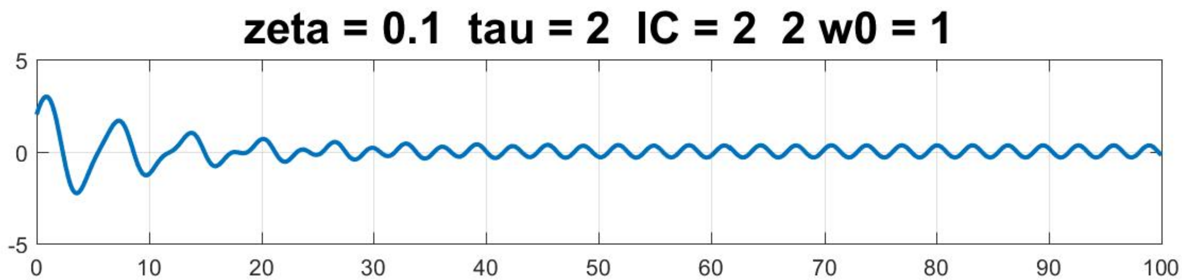
$y(0) = 1$
$\dot{y}(0) = 0$



$y(0) = 5$
$\dot{y}(0) = -8$



$y(0) = 2$
$\dot{y}(0) = 2$



All solutions are essentially the same for $t > 50$. Why?

All solutions are essentially the same for $t > 50$. Why?

We want to compare the sizes of y_P and y_H so we write them in amplitude-phase form.

$$y_P = -0.3275 \cos 2t + 0.0437 \sin 2t$$

$$= 0.3304 \cos(2t - 0.116)$$

$$y_H = c_1 e^{-0.1t} \cos(0.995t) + c_2 e^{-0.1t} \sin(0.995t)$$

$$= e^{-0.1t} (c_1^2 + c_2^2)^{1/2} \cos(0.995t + \text{atan2}(c_2, c_1))$$

$$e^{-0.1t}\Big|_{t=50} \qquad e^{-0.1 \cdot 50} = e^{-5} = 0.067$$

For $t \geq 50$ the amplitude of y_P is 0.3304, while the amplitude of y_H is less than 0.067 times the initial amplitude. y_H is so small that we can't see it on the graph. So small that it's irrelevant in many applications.

$$y_p = 0.3304 \cos(2t - 0.116)$$

constant (steady) amplitude 0.3775

$$y_H = \underbrace{e^{-0.1t} (c_1^2 + c_2^2)^{1/2}}_{\text{decaying amplitude}} \cos(0.995t + \text{atan2}(c_2, c_1))$$

Something which decays with time is called transient. If you wait long enough, its gone.

When the homogeneous solution decays, we call y_p the steady state solution, and y_H the transient.

$$y = 0.3304 \cos(2t - 0.116) + e^{-0.1t} (c_1^2 + c_2^2)^{1/2} \cos(0.995t + \text{atan2}(c_2, c_1))$$

$$y = y_{ss} + y_{tr}$$

$$y = 0.3304\cos(2t - 0.116)$$

$$+ \; e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \text{atan2}(c_2, c_1))$$

$$y = y_{ss} + y_{tr}$$

The steady state does not depend on the initial conditions. It only depends on the forcing term.

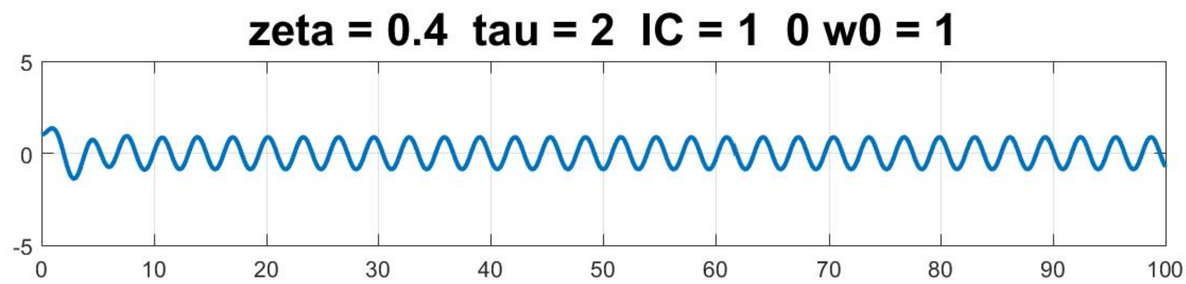
Only the transient part depends on the initial conditions.

The more damping, the faster the transient part decays

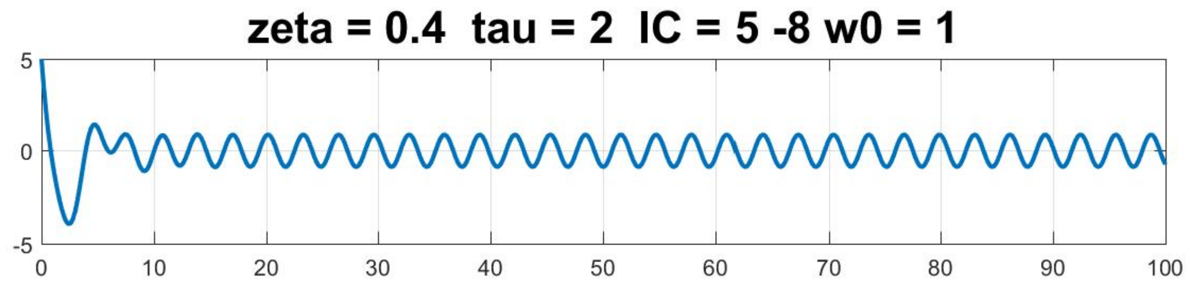
An Example with more damping

$$\ddot{y} + 0.8\dot{y} + y = 3\cos 2t$$

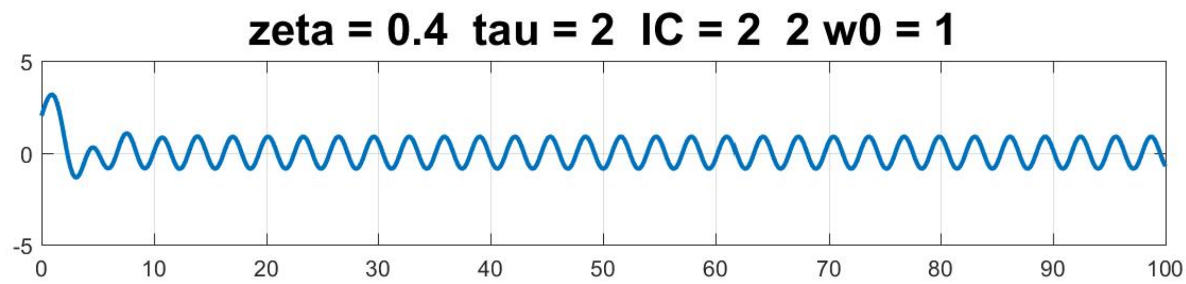
$y(0) = 1$
$\dot{y}(0) = 0$



$y(0) = 5$
$\dot{y}(0) = -8$



$y(0) = 2$
$\dot{y}(0) = 2$



$$y_H = y_{tr} = e^{-0.4t} A\cos(0.6t - \phi_1)$$

These depend on IC's (A and ϕ_1)

$$y_p = y_{ss} = 0.8824\cos(2t - 2.65)$$

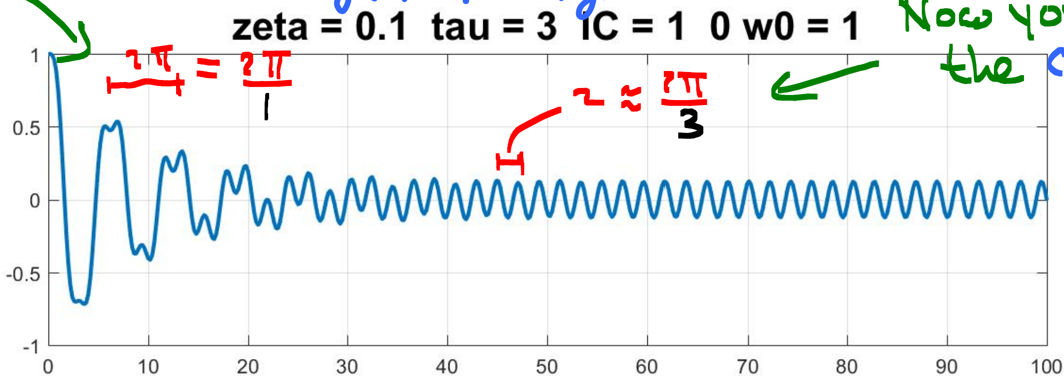
Now only takes 10 seconds for transient to decay enough that its invisible.

$$\ddot{y} + 0.2\dot{y} + 1^2 y = \cos 3t$$

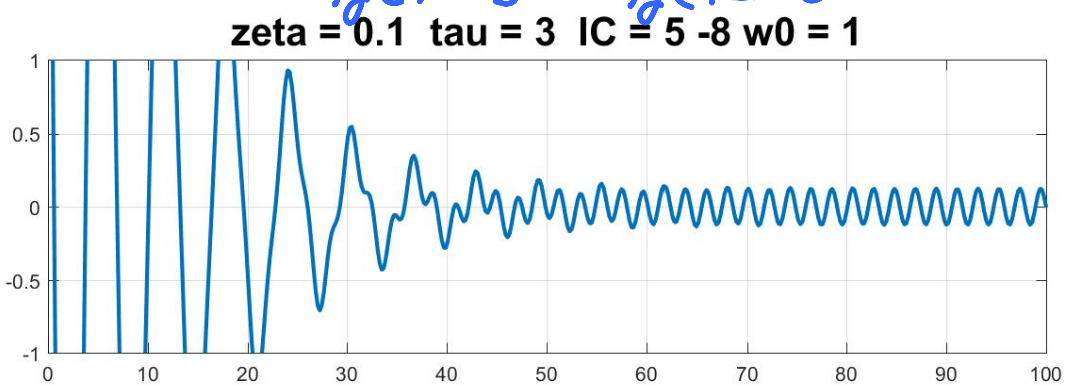
You can see the $\cos(1t)$ term

$y(0) = 1 \quad \dot{y}(0) = 0$

Now you only see the $\cos(3t)$ term

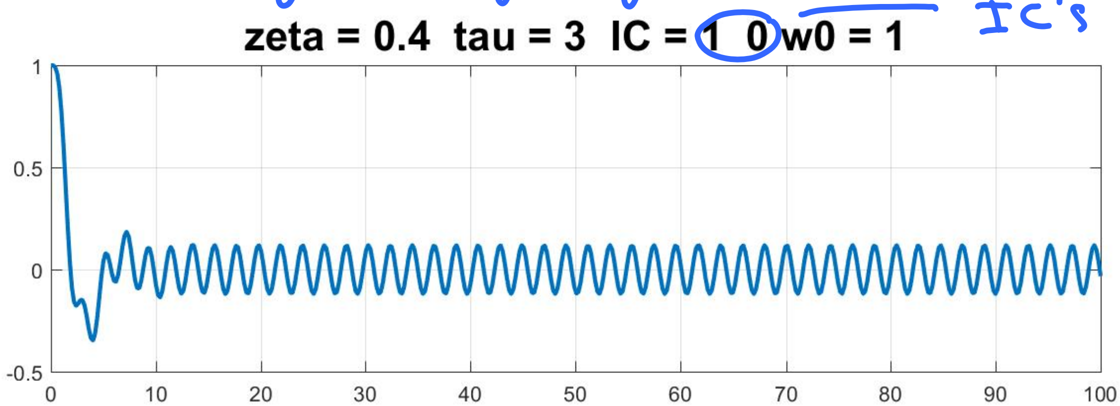


$y(0) = 5 \quad \dot{y}(0) = -8$



$$\ddot{y} + 0.4\dot{y} + y = \cos 3t$$

IC's



IC's