L22OptionalWhyTheT

Note Title 8/10/2020

Where does the "t" come from?

$$\ddot{y} + y = \cos(1.0001t)$$

$$y_p = A\cos(1.0001t) + B\sin(1.0001t)$$

But

$$\ddot{y} + y = \cos t$$

$$y_p = A\,t\cos t + B\,t\sin t$$

To see why we solve

$$\ddot{y} + y = \cos\omega t$$

with initial conditions

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

(These initial conditions make the calculation simpler, but any initial conditions will work)

We will write down a formula for the solution for $\omega \neq 1$, and then take the limit as $\omega \to 1$.

$$\ddot{y} + y = \cos \omega t \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(IC)}$$

Homogeneous Solution

$$\ddot{y}_H + y_H = 0 \qquad y_H = c_1 \cos t + c_2 \sin t$$

Seek particular solution

$$y_p = A \cos \omega t + B \sin \omega t$$

$$\ddot{y}_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

Insert y_p in (DE)

$$(1-\omega^2) A \cos \omega t + (1-\omega^2) B \sin \omega t = \ddot{y}_p + y_p = \cos \omega t$$

Conclude: $\quad A = \dfrac{1}{1-\omega^2} \qquad B = 0$

so $\quad y(t) = \dfrac{\cos \omega t}{1-\omega^2} + c_1 \cos t + c_2 \sin t$

Now use (IC)

$$0 = y(0) = \frac{1}{1-\omega^2} + c_1 \Longrightarrow c_1 = \frac{-1}{1-\omega^2}$$

$$0 = \dot{y}(0) = c_2 \Longrightarrow c_2 = 0$$

$$y(t) = \frac{\cos \omega t - \cos t}{\omega^2 - 1}$$

The solution to

$$\ddot{y} + y = \cos\omega t \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(IC)}$$

is $y(t) = \dfrac{\cos\omega t - \cos t}{\omega^2 - 1}$

Now we can let $\omega \to 1$

$$\lim_{\omega\to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \frac{\cos t - \cos t}{1 - 1} = \text{"}\frac{0}{0}\text{"}$$

so we use L'Hôpital's rule

$$\lim_{\omega\to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \lim_{\omega\to 1} \frac{\frac{d}{d\omega}(\cos\omega t - \cos t)}{\frac{d}{d\omega}(\omega^2 - 1)}$$

$$= \lim_{\omega\to 1} \frac{t\omega \sin\omega t}{2\omega} = \frac{t \sin t}{2}$$

So we see that the forcing function $\cos t$ results in a solution $y(t)$ of the form $A\underbrace{t}\sin t$. This is one way to see why we multiply the forcing term by t when the forcing term matches a term from the homogeneous solution.