L21UndeterminedCoefficients

Note Title 8/10/2020

Slightly More Complicated Example

$$\ddot{y} + 3\dot{y} + 2y = t^2 e^{-4t}$$

Homogeneous Solution is

$$y_H = c_1 e^{-2t} + c_2 e^{-t}$$

What is the form of y_P ?

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y_P = the most general expression you can get by repeatedly differentiating the forcing term.

Start with $y_P = A t^2 e^{-4t} + ??$

Differentiate $\dot{y}_P = 2t e^{-4t} - 8 t^2 e^{-4t}$

I could add the whole derivative but its simpler to just add the new term

$$\dot{y}_P = \underbrace{2t e^{-4t}}_{\text{new term}} \underbrace{-8t^2 e^{-4t}}_{\text{old term}}$$

$$y_P = A t^2 e^{-4t} + B t e^{-4t} + ??$$

* purple means "not important"

$$y_p = At^2e^{-4t} + Bte^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term.

$$(te^{-4t})' = \underbrace{e^{-4t}}_{\text{new term}} - 4\underbrace{te^{-4t}}_{\text{old term}}$$

Add the new term

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term

$$(e^{-4t})' = -4\underbrace{e^{-4t}}_{\text{old term}}$$

No new term - I can stop!!

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

The calculation below does exactly the same thing as we did on the previous page. Its just organized as a diagram and there are fewer words.

$t^2 e^{-4t}$ ① Forcing Function = first term

$\downarrow \dfrac{d}{dt}$

$2t e^{-4t}$ ② second term $\quad -4 t^2 e^{-4t}$ old term - drop it

STOP

$\downarrow \dfrac{d}{dt}$

$2 e^{-4t}$ ③ third term $\quad -4 t e^{-4t}$ old term - drop it

STOP

$\downarrow$

$-8 e^{-4t}$ old term - drop it

STOP

$$y_p(t) = A t^2 e^{-4t} + B t e^{-4t} + C e^{-4t}$$

or

$$(A t^2 + B t + C) e^{-4t}$$

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

Now substitute y_p into

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = t^2e^{-4t}$$

and solve for A, B, C

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

$$\dot{y}_p = -4At^2e^{-4t} + (2A-4B)te^{-4t} + (B-4C)e^{-4t}$$

$$\ddot{y}_p = 16At^2e^{-4t} + (-16A+16B)te^{-4t} + (2A-8B+16C)e^{-4t}$$

Notice the way the terms are organized after each differentiation. Here is how I calculated $\dot{y}_p$.

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

$$\dot{y}_p = -4At^2e^{-4t} + 2Ate^{-4t} + -4Bte^{-4t} + Be^{-4t} + -4Ce^{-4t}$$

$$\dot{y}_p = -4At^2e^{-4t} + (2A-4B)te^{-4t} + (B-4C)e^{-4t}$$

We keep each of the terms, t^2e^{-4t}, te^{-4t}, e^{-4t} together.

Now use the (DE)

$$\ddot{y}_p = 16At^2e^{-4t} + (-16A+16B)te^{-4t} + (2A-8B+16C)e^{-4t}$$

$$2y_p = 2At^2e^{-4t} + 2Bte^{-4t} + 2Ce^{-4t}$$

$$+\ 3\dot{y}_p = -12At^2e^{-4t} + (6A-12B)te^{-4t} + (3B-12C)e^{-4t}$$

$$t^2e^{-4t} = 6At^2e^{-4t} + (-10A-6B)te^{-4t} + (2A-5B+6C)e^{-4t}$$

$$\left.\begin{array}{l} 6A = 1 \\ -10A-6B=0 \\ 2A-5B+6C=0 \end{array}\right\} \quad \begin{array}{l} A = \frac{1}{6} \\ B = \frac{-10}{36} = \frac{-5}{18} \\ C = \frac{-31}{18 \cdot 6} \end{array}$$