Forced Harmonic Oscillator

$m\ddot{y} + \gamma\dot{y} + ky = F(t)$ = external force

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = e^{-4t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Technique

① Fi general solution to

$\ddot{y}_H + 3\dot{y}_H + y_H = 0$ called "homogeneous solution"

② Find any solution to

$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$ called "particular solution", or "steady state solution"

Set $y = y_p + y_H$

③ Choose the constants from ① to satisfy IC's.

Example

$$\ddot{y} + 3\dot{y} + 2y = e^{-4t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Seek $y(t) = y_H + y_P$

where

$$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$$

and

$$\ddot{y}_P + 3\dot{y}_P + 2y_P = e^{-4t}$$

① Find general solution to homogeneous equation.

Seek $y_H = e^{rt}$

$$r^2 + 3r + 2 = 0$$
$$(r+2)(r+1) = 0$$

$$y_H = c_1 e^{-2t} + C_2 e^{-t}$$

② Find any solution to

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$$

Rule:

The form of the particular solution must contain constants times

① The forcing term [e^{-4t} in this example]

② All derivatives of the forcing term

In this example, derivatives of e^{-4t} are just constants times e^{-4t}, so we don't need any other terms.

In this case, we set $y_p = Ae^{-4t}$

Substitute y_p into (DE) and solve for A

$$(-4)^2 A e^{-4t} + 3(-4) A e^{-4t} + 2Ae^{-4t} = e^{-4t}$$

$$(16 - 12 + 2) A = 1$$

$$6A = 1$$

$$A = \frac{1}{6}$$

$$\boxed{y_p = \frac{1}{6} e^{-4t}}$$

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t} \quad \text{(DE)}$$ 4

$$y_p = \frac{1}{6}e^{-4t}$$

Recall $y_H = C_1 e^{-2t} + C_2 e^{-t}$

$$y(t) = y_p + y_H$$

SO $y = \frac{1}{6}e^{-4t} + C_1 e^{-2t} + C_2 e^{-t}$

Use initial conditions to find C_1 and C_2

(IC)'s $\quad y(0) = 0 \quad \dot{y}(0) = 0$

$$0 = y(0) = \frac{1}{6} + C_1 + C_2$$

$$0 = \dot{y}(0) = -\frac{4}{6} - 2C_1 - C_2$$

$$\begin{array}{rl} C_1 + C_2 & = -1/6 \\ + \quad -2C_1 - C_2 & = 4/6 \\ \hline -C_1 & = 1/2 \end{array}$$

$$C_1 = -1/2$$

$$C_2 = 1/3$$

$$\boxed{y = \frac{1}{6}e^{-4t} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-t}}$$

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

① Find general solution to

$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$ called "homogeneous solution"

② Find any solution to

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$$

called "particular solution", or "steady state solution"

Set $\quad y = y_p + y_H$

③ Choose the constants from ① to satisfy IC's.

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

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① $\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$

seek $y = e^{rt}$ $\quad r^2 + 3r + 2 = 0$

$$\boxed{y_H(t) = C_1 e^{-t} + C_2 e^{-2t}}$$

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② $\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

Seek $y_p = A\cos 2t$ ← Not good enough

Seek $y_p = A\cos 2t + B\sin 2t$ ← This one works

Rule: The particular solution must contain constants times

① The forcing term

② All terms that are derivatives of forcing terms

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$$

This doesn't work

Try $y_p = A\cos 2t$

$$2y_p = 2A\cos 2t$$

$$3\dot{y}_p = -2A\sin 2t \cdot 3$$

$$+\ \ddot{y}_p = -4A\cos 2t$$

$$\cos 2t = \underbrace{-2A}_{=1}\cos 2t \underbrace{-6A}_{=0}\sin 2t$$

Can't solve this

The forcing term $\cos 2t$ $\quad y_p = A\cos 2t + ??$

$$\downarrow \tfrac{d}{dt}$$

All derivatives of the forcing term $\quad -2\sin 2t \quad y_p = A\cos 2t + B\sin 2t + ??$

$$\downarrow \tfrac{d}{dt}$$

* $-4\cos 2t \quad y_p = A\cos 2t + B\sin 2t + C\cancel{\cos 2t}$

I already have a $\cos 2t$ term. I can stop now

$$y_p = A\cos 2t + B\sin 2t$$

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* Purple means "not important"

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$$

$$2\,y_p = 2A\cos 2t + 2B\sin 2t$$

$$3\cdot\dot{y}_p = 3\cdot(-2)A\sin 2t + 3\cdot 2B\cos 2t$$

$$+\quad \ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\cos 2t = (2A + 6B - 4A)\cos 2t + (2B - 6A - 4B)\sin 2t$$

$$0\sin 2t + 1\cos 2t = (-2B - 6A)\sin 2t + (6B - 2A)\cos 2t$$

$$0 = (-2B - 6A) \implies B = -3A$$

$$1 = (6B - 2A) \Rightarrow 1 = -20A$$

$$A = -\frac{1}{20} \qquad B = \frac{3}{20}$$

$$y_p = -\frac{1}{20}\cos 2t + \frac{3}{20}\sin 2t$$

Now go to step 3

$$y(t) = y_P + y_H$$

$$= \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + C_1 e^{-t} + C_2 e^{-2t}$$

$$0 = y(0) = \frac{-1}{20}\cdot 1 + \frac{3}{20}\cdot 0 + C_1 \cdot 1 + C_2 \cdot 1$$

$$1 = \dot{y}(0) = \frac{2}{20}\cdot 0 + \frac{6}{20}\cdot 1 - C_1 - 2C_2$$

$$\left.\begin{aligned} C_1 + C_2 &= \frac{1}{20} \\ -C_1 - 2C_2 &= \frac{14}{20} \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \frac{16}{20} \\ C_2 &= \frac{-15}{20} \end{aligned}$$

$$\boxed{y(t) = \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + \frac{16}{20} e^{-t} - \frac{15}{20} e^{-2t}}$$