L19CriticallyDamped

Note Title 8/6/2020

Critically Damped

$$\ddot{x} + 2\dot{x} + x = 0 \quad \text{(DE)}$$

$$x(0) = 0 \quad \dot{x}(0) = 10 \quad \text{(IC)}$$

① Solve the IVP ② Is the displacement ever $= 0$?

③ What is the maximum displacement?

Seek $x(t) = e^{rt}$ $\quad r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$ so $r = -1$

When there is only one value of r that solves the indicial equation, the general solution is $c_1 e^{rt} + c_2 t e^{rt}$.

explanation after we finish the problem.

General Solution $c_1 e^{-t} + c_2 t e^{-t}$

Impose IC's

$0 = x(0) = c_1 + 0c_2$ so $c_1 = 0$

$10 = \dot{x}(0) = -c_1 + c_2$ so $c_2 = 10$

① $x(t) = 0e^{-t} + 10te^{-t} = 10te^{-t}$

$$x(t) = 10t\,e^{-t}$$

② where is displacement $= 0$

$$0 \overset{?}{=} 10t\,e^{-t}$$

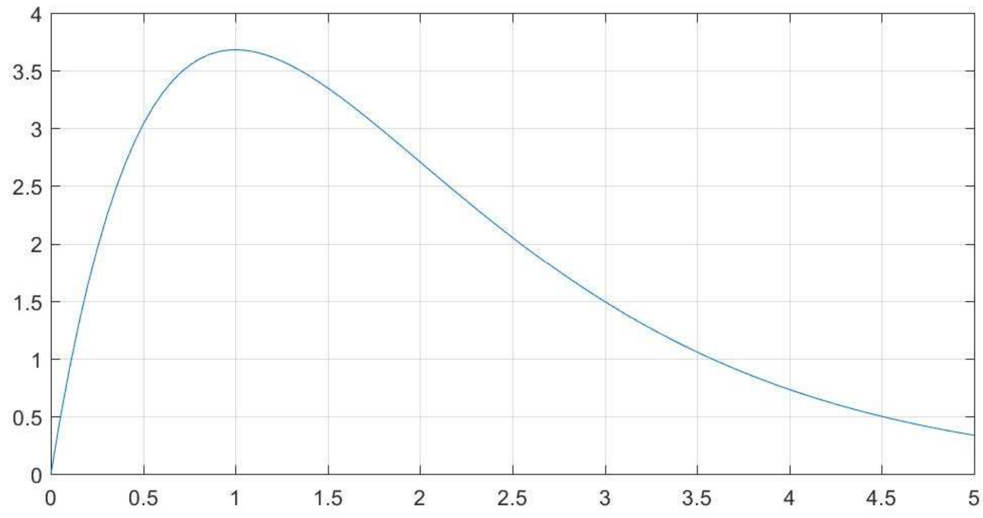
Only zero at $t = 0$

③ What is the max displacement?

Max displacement occurs when $\dot{x} = 0$

$$0 = \dot{x}(t) = 10(e^{-t} - t e^{-t}) = 10 e^{-t}(1-t)$$

$\dot{x}$ vanishes at $t=1$. Max displacement $= 10e^{-1}$



Looks just like overdamped

Where did the "t" come from:

I'll show you two examples that illustrate how you can find these solutions, without someone else telling you what they are. The technique involves:

Find solutions that you don't know by taking limits of solutions that you do know!!

It can be applied to many different problems in math and science.

The first example is the simplest

$$\ddot{x} - \omega^2 x = 0 \qquad x(0) = 0 \qquad \dot{x}(0) = 1$$

For $\omega \neq 0$, we set $x = e^{rt}$ and find two roots $r = \pm\omega$.

$$\ddot{x} - \omega^2 x = 0 \qquad x(0) = 0 \qquad \dot{x}(0) = 1$$

For $\omega \neq 0$, we set $x = e^{rt}$. If $\omega \neq 0$, we find two roots $r = \pm\omega$. So

$$x(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

We use the initial conditions to find c_1 and c_2. It turns out that $c_1 = -c_2 = \frac{1}{2\omega}$ so

$$x(t) = \frac{e^{\omega t} - e^{-\omega t}}{2\omega}$$

If $\omega = 0$, the DE is

$$\ddot{x} = 0$$

When we set $x(t) = e^{rt}$, the indicial equation is

$$r^2 = 0$$

and we find only one root $r = 0$, so we have only one solution $x(t) = e^{0t} = \underline{1}$

We can find the other solution by going back to the case $\omega \neq 0$ and taking the limit as $\omega \to 0$.

$$\lim_{\omega \to 0} \frac{e^{\omega t} - e^{-\omega t}}{2\omega} \text{"} = \frac{e^{0t} - e^{-0t}}{2 \cdot 0} = \frac{0}{0} \text{"}$$

So we need to use L'Hôpitals rule

$$\lim_{\omega\to 0} \frac{e^{\omega t} - e^{-\omega t}}{2\omega} \text{"} = \frac{e^{0t} - e^{-0t}}{2\cdot 0} = \frac{0}{0} \text{"}$$

So we need to use L'Hôpitals rule

$$\lim_{\omega\to 0} \frac{e^{\omega t} - e^{-\omega t}}{2\omega} = \lim_{\omega\to 0} \frac{\frac{d}{d\omega}\left(e^{\omega t} - e^{-\omega t}\right)}{\frac{d}{d\omega}(2\omega)}$$

$$= \lim_{\omega\to 0} \frac{te^{\omega t} + te^{-\omega t}}{2}$$

$$= \lim_{\omega\to 0} \frac{t\cdot 1 + t\cdot 1}{2} = t$$

We have found a second solution $x(t) = t$

so the general solution is

$$x(t) = c_1 + c_2 t$$

This is the simplest example. You might have found this general solution on your own, by just integrating $\ddot{x} = 0$ twice. The example on the next page is a little more complicated.

Example 2 $\quad \ddot{x} + 2\dot{x} + x = 0$

This is the example from the first page. We looked for $x = e^{rt}$ and only found one root, $r = -1$, so we had only one solution $x(t) = e^{-t}$.

To find the second solution, we modify the DE to
$$\ddot{x} + 2\dot{x} + (1-\omega^2)x = 0 \text{ and,}$$
keep the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 10$

The indicial equation for r becomes
$$r^2 + 2r + (1-\omega^2) = 0$$
and now there are two roots $r = -1 \pm \omega$.

The general solution is
$$x(t) = c_1 e^{(-1+\omega)t} + c_2 e^{(-1-\omega)t}$$
and use the initial conditions $x(0) = 0$ and $\dot{x}(0) = 10$
$$x(t) = 10\left(\frac{e^{(-1+\omega)t} - e^{(-1-\omega)t}}{2\omega}\right) = 10e^{-t}\left(\frac{e^{\omega t} - e^{-\omega t}}{2\omega}\right)$$

$$x(t) = e^{-t}\left(\frac{e^{\omega t} - e^{-\omega t}}{2\omega}\right)$$

Now, let $\omega \to 0$,

$$\lim_{\omega \to 0} x(t) = e^{-t} \lim_{\omega \to 0} \left(\frac{e^{\omega t} - e^{-\omega t}}{2\omega}\right)$$

We again need to apply L'Hôpitals rule exactly as before and find

$$x(t) = b\, t\, e^{-t}$$

We know have two independent solution to the DE $\ddot{x} + 2\dot{x} + x = 0$

The general solution is

$$x(t) = c_1 e^{-t} + c_2\, t e^{-t}$$