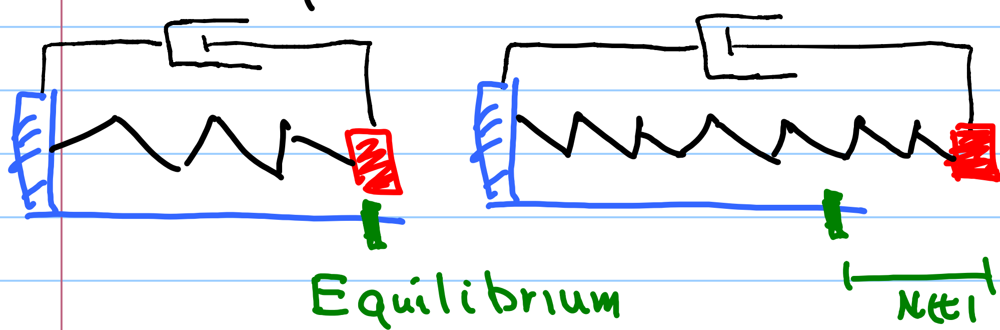
L18DampedHarmonicOscillator

Note Title 8/4/2020

Damped Harmonic Oscillator 2



Suppose the spring constant is 145 kg sec^{-2}

Damping coeff. = 2 kg sec^{-1}. $m = 1$ kg

Initial displacement = 1 meters

Initial velocity = 2 meters/sec

① Write the Initial Value Problem

② Find a formula for $x(t)$

Answer

$$m\ddot{x} = -r\dot{x} - kx$$

$$1\ddot{x} = -2\dot{x} - 145x$$

$$\ddot{x} + 2\dot{x} + 145x = 0 \quad \text{(IVP)}$$

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

$$\ddot{x} + 2\dot{x} + 145x = 0 \qquad \text{(IVP)}$$

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

Seek $\quad x(t) = e^{rt}$

$$r^2 e^{rt} + 2r e^{rt} + 145 e^{rt} = 0$$

$$r^2 + 2r + 145 = 0$$

$$r^2 + 2r + 1 = -144$$

$$(r+1)^2 = -144$$

$$r + 1 = \pm 12i$$

$$r = -1 \pm 12i$$

We could write

$$x(t) = c_1 e^{(-1+12i)t} + c_2 e^{(-1-12i)t}$$

But, instead we write

$$x(t) = D_1 e^{-t} \cos 12t + D_2 e^{-t} \sin 12t$$

$$x(t) = D_1 e^{-t} \cos 12t + D_2 e^{-t} \sin 12t$$

and require $\quad x(0) = 1 \quad \dot{x}(0) = 2$

$$1 = x(0) = D_1 + \cancel{D_2 \cdot 0}$$

$$2 = \dot{x}(0) = D_1 (-e^{-t} \cos 12t - 12 e^{-t} \sin 12t) \Big|_{t=0}$$
$$+ D_2 (-e^{-t} \sin 12t + 12 e^{-t} \cos 12t) \Big|_{t=0}$$

$$2 = D_1 (-1 - 0) + D_2 (0 + 12)$$

so $\quad 1 = D_1$

and $\quad 2 = -D_1 + 12 D_2$

so $\quad D_2 = \frac{3}{12} = \frac{1}{4}$

$$\boxed{x(t) = 1 e^{-t} \cos 12t + \frac{1}{4} e^{-t} \sin 12t}$$

$$x(t) = 1\, e^{-t} \cos 12t + \frac{1}{4} e^{-t} \sin 12t$$

$$x(t) = e^{-t}\left(1 \cos 12t + \frac{1}{4} \sin 12t\right)$$

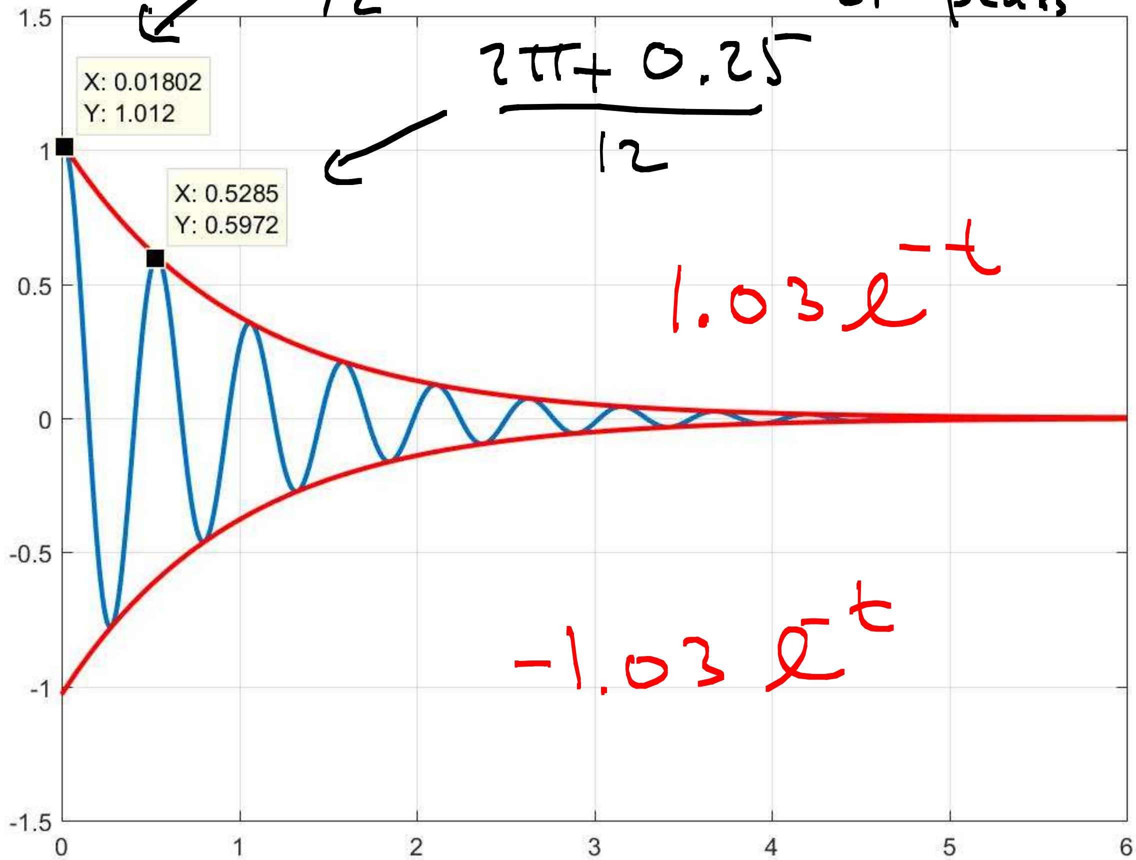
$$A = \sqrt{1 + \left(\frac{1}{4}\right)^2} \qquad \phi = \operatorname{atan}\left(\frac{1/4}{1}\right)$$

$$A = 1.03 \qquad \phi = 0.25$$

$$x(t) = 1.03\, e^{-t} \cos(12t - 0.25)$$

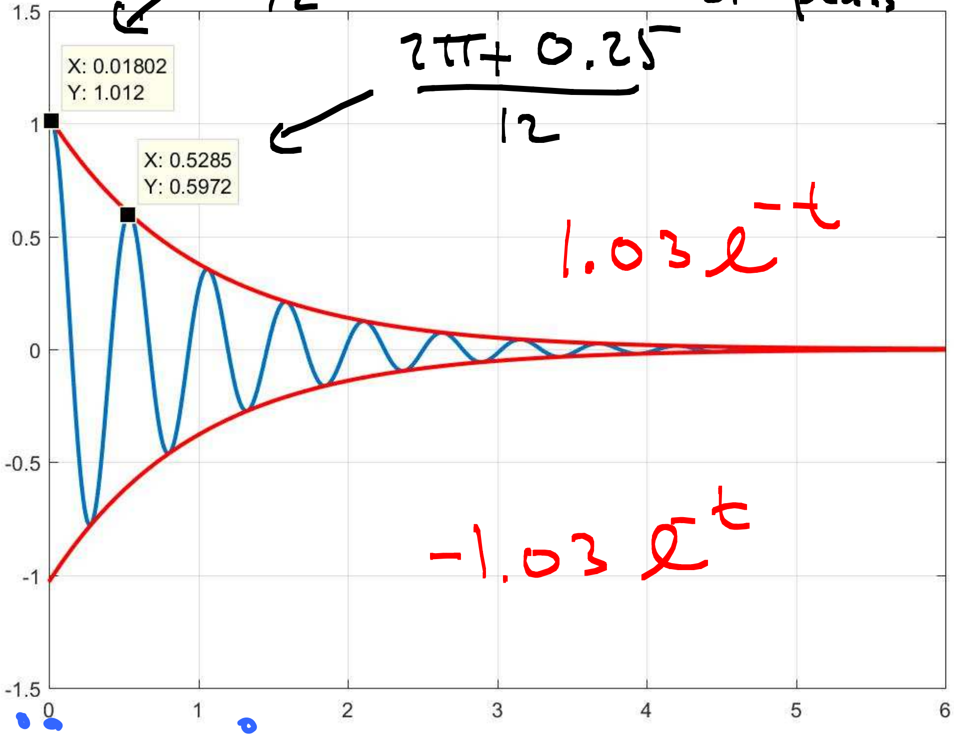
$\frac{0.25}{12}$

Approximate locations of peaks



$\frac{0.25}{12}$ Approximate locations of peaks

$\frac{2\pi + 0.25}{12}$



$$\ddot{x} + 2\dot{x} + 145x = 0$$

$$x(t) = 1.03\, e^{-t} \cos(12t - 0.25)$$

Some Facts

1. The peaks occur approximately, but not exactly, at the same times as the peaks of $\cos(12t - 0.25)$.

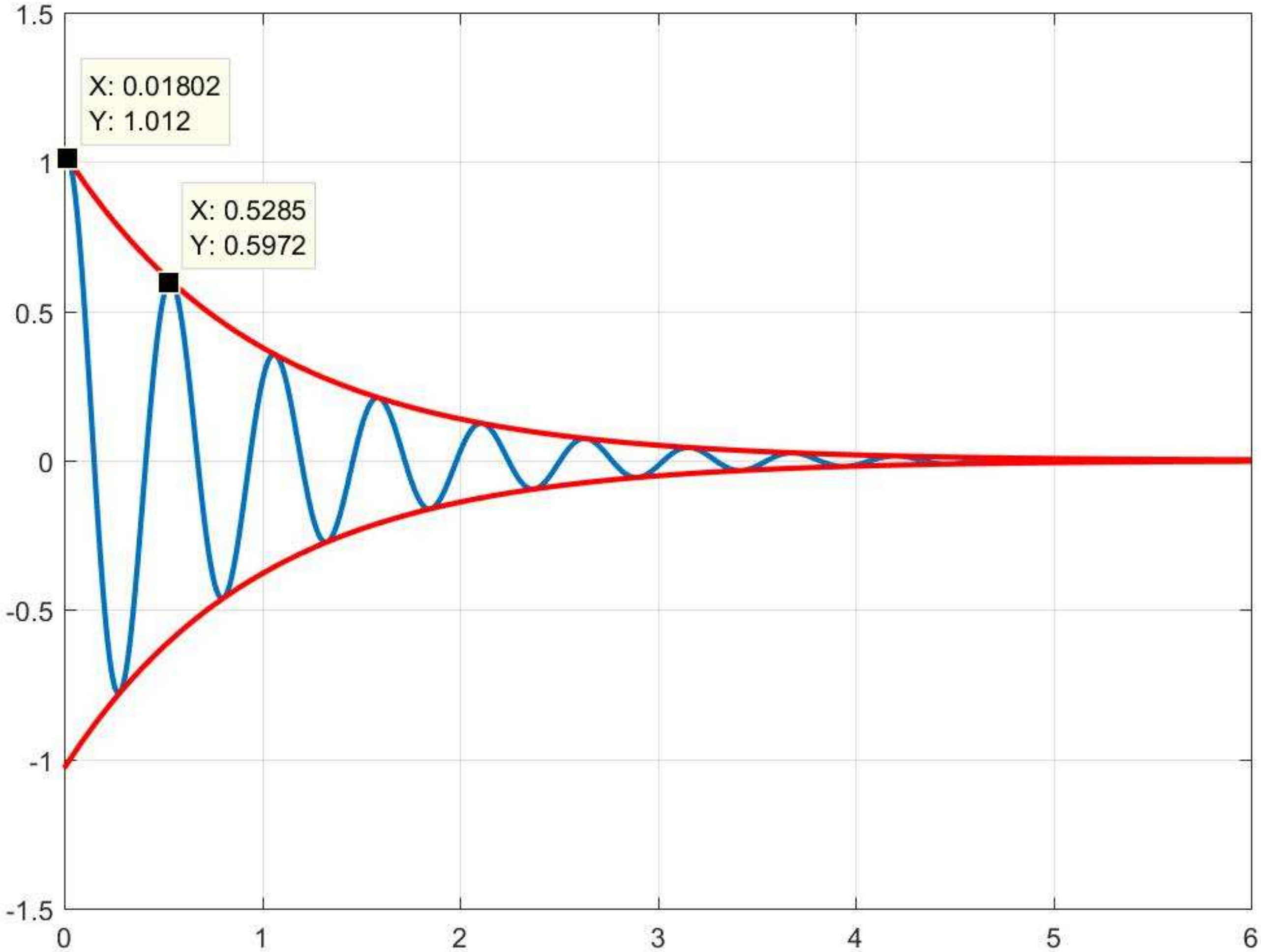
2. The time between the peaks is exactly the same as the time between the peaks of $\cos(12t - 0.25)$.

3. $\cos(12t - 0.25)$ has the same value at all peaks

12 is called the quasi-frequency and $\frac{2\pi}{12}$ is called the quasi-period.

Find the solution from the graph

$$x(t) \doteq e^{-rt}\cos(\omega t - \varphi)$$



To Find ω

② The time between the peaks is exactly the same as the time between the peaks of $\cos(\omega t - \varphi)$.

so time between peaks = period of $\cos(\omega t)$

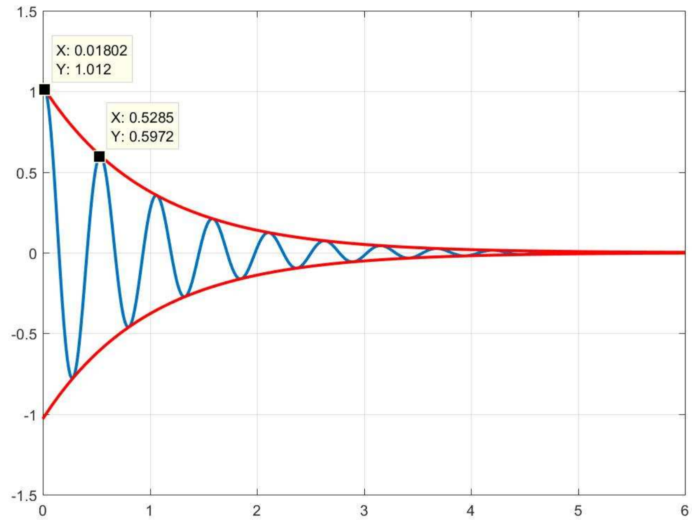
so $0.5285 - 0.0182 = 0.51$

$$\omega = \frac{2\pi}{0.51} \approx 4\pi \; \boxed{\approx 12}$$

$$\boxed{\omega = 4\pi}$$

$$x(t) = e^{-rt}\cos(\omega t - \varphi)$$

Find y(t) from the graph



To Find r

(3) $\cos(\omega t - \varphi)$ has the same value at all peaks

At first peak,

$$e^{-0.018 r}\cos(\omega \cdot 0.018 - \varphi) = 1.012$$

At second peak

$$e^{-0.5285 r}\cos(\omega \cdot 0.5285 - \varphi) = 0.5972$$

Dividing,
$$\frac{e^{-0.018 r}\cos(\omega \cdot 0.018 - \varphi)}{e^{-0.5285 r}\cos(\omega \cdot 0.5285 - \varphi)} = \frac{1.012}{0.5972}$$

using fact (3), we can cancel

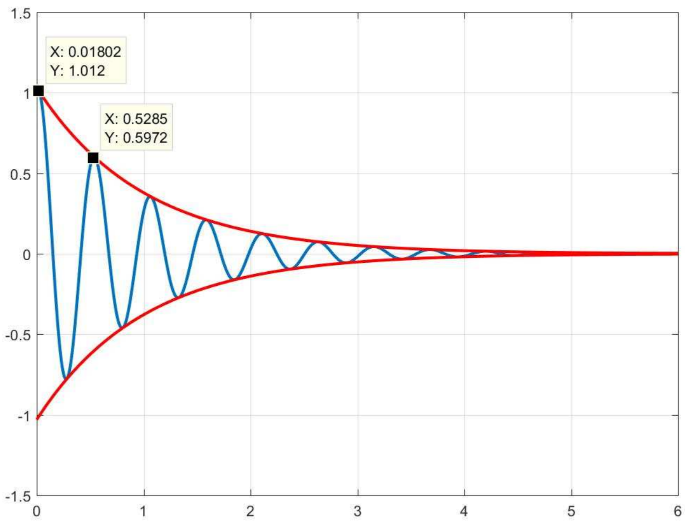
$$\frac{e^{-0.018 r}\cancel{\cos(\omega \cdot 0.018 - \varphi)}}{e^{-0.5285 r}\cancel{\cos(\omega \cdot 0.5285 - \varphi)}} = \frac{1.012}{0.5972}$$

so

$$e^{0.51 r} = 1.7$$

$$r = \frac{\ln(1.7)}{0.51} = 1.1 \approx 1$$

$$\boxed{r = 1.1}$$



Find φ

(1) The peaks occur approximately, but not exactly, at the same times as the peaks of $\cos(\omega t - \varphi)$.

so $\cos(\omega \cdot 0.0182 - \varphi) = 1$

$$\omega \cdot 0.0182 - \varphi = 0$$

$$\varphi = 0.0182 \cdot \omega = 0.0182 \cdot 4\pi$$

$$\boxed{\varphi = 0.23}$$

Justification of Facts

peak = local maximum

(3) $\cos(\omega t - \phi)$ has the same value at all peaks

Peak of $y(t) = e^{-rt}\cos(\omega t - \phi)$

occurs at

$$0 = \dot{y}(t) = -r\cancel{e^{-rt}}\cos(\omega t - \phi) - \omega\cancel{e^{-rt}}\sin(\omega t - \phi)$$

$$0 = r\cos(\omega t - \phi) - \omega\sin(\omega t - \phi)$$

$$r\cos(\omega t - \phi) = \omega\sin(\omega t - \phi)$$

$$\frac{r}{\omega} = \tan(\omega t - \phi)$$

So, at all t's that are maxima or minima,

$\tan(\omega t - \phi)$ has the same value. If the

tangent is the same at t_1 and t_2

$$\cos(\omega t_1 - \phi) = \pm\cos(\omega t_2 - \phi)$$

At maxima, cosine is positive so

$$\cos(\omega t_1 - \phi) = \cos(\omega t_2 - \phi)$$

Also, if $\tan\theta_1 = \tan\theta_2$,

$$\theta_1 - \theta_2 = N\pi, \quad N = \text{integer}$$

If both cosines are positive, $\Theta_1 - \Theta_2 = 2N\pi$

so the distance between successive peaks satisfy

$$(\omega t_2 - \varphi) - (\omega t_1 - \varphi) = 2\pi$$

$$t_2 - t_1 = \frac{2\pi}{\omega}$$

which justifies Fact ②

② The time between the peaks is exactly the same as the time between the peaks of $\cos(\omega t - \varphi)$.