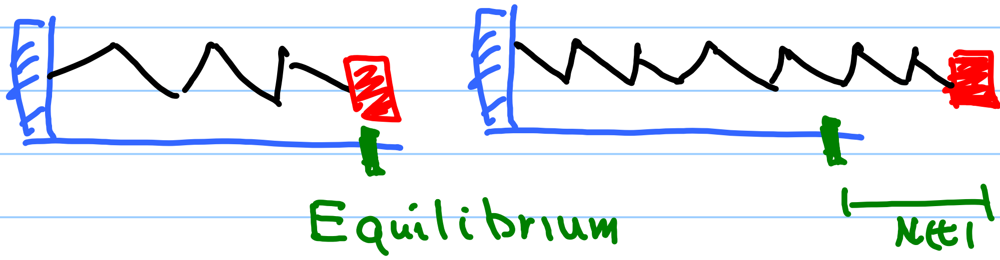
L17UndampedHarmonicOscillator

Note Title 8/5/2020

Undamped Harmonic Oscillator



Suppose the spring constant is 50 kg sec^{-2}

No damping $m = 2$ kg

Initial displacement = 5 meters

Initial velocity = 20 meters/sec

① Write the Initial Value Problem

② Find a formula for $x(t)$

Answer mass · acceleration = Σ Forces

$$m\ddot{x} = -kx$$

$$2\ddot{x} = -50x$$

or $$\ddot{x} + 25x = 0$$

$$x(0) = 5 \qquad \dot{x}(0) = 20$$

IVP $\ddot{x} + 25x = 0$

$x(0) = 5 \qquad \dot{x}(0) = 20$

Seek $x(t) = e^{rt}$

$$\begin{aligned} \ddot{x} &= r^2 e^{rt} \\ + \quad 25x &= 25 e^{rt} \\ \hline \ddot{x} + 25x &= (r^2+25)e^{rt} = 0 \end{aligned}$$

so $r^2 + 25 = 0$

or $r = \pm 5i$

Two solutions:

$x_1(t) = e^{5ti}$ and $x_2(t) = e^{-5ti}$

Because this is a Linear DE

$x(t) = c_1 e^{5ti} + c_2 e^{-5ti}$

is a solution for any constants c_1 and c_2

$$x(t) = c_1 e^{5ti} + c_2 e^{-5ti}$$

We could use the initial conditions

$$x(0) = 5 \qquad \dot{x}(0) = 20$$

to solve for the (complex) constants c_1 and c_2, but there is an easier way to do this.

Euler's Formula

$$x(t) = c_1(\cos 5t + i\sin 5t) + c_2(\cos 5t - i\sin 5t)$$

$$= \underbrace{(c_1 + c_2)}_{\text{Let's call this } D_1}\cos 5t + \underbrace{i(c_1 - c_2)}_{\text{Lets call this } D_2}\sin 5t$$

$$= D_1 \cos 5t + D_2 \sin 5t$$

$$x(t) = D_1 \cos 5t + D_2 \sin 5t$$

We solve for D_1 and D_2.

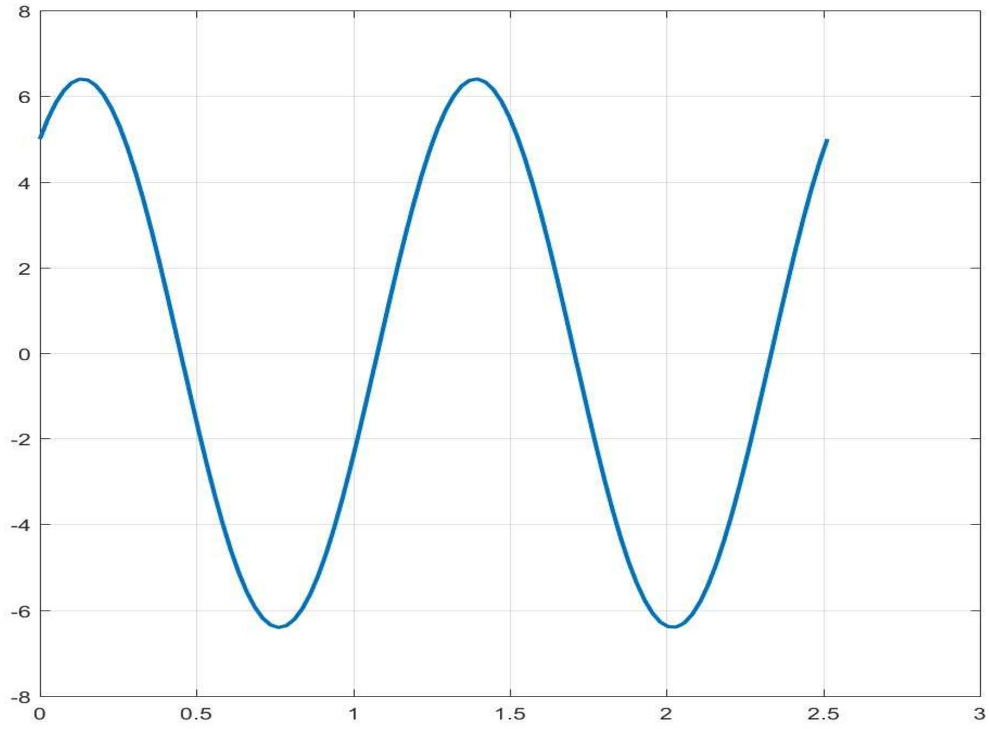
$$5 = x(0) = D_1 \cos(0) + D_2 \sin(0) = D_1$$

$$20 = \dot{x}(0) = -5D_1 \sin(0) + 5D_2 \cos(0) = 5D_2$$

so $D_1 = 5$ and $D_2 = 4$

$$\boxed{x(t) = 5\cos 5t + 4\sin 5t}$$

$x(t) = 5 \cos 5t + 4 \sin 5t$



How do you draw the graph from the Formula?

How do deduce the Formula from the graph?

Frequency Amplitude Phase representation

$$5 \cos 5t + 4 \sin 5t = \sqrt{5^2+4^2} \cos\left(5t - \operatorname{atan}\left(\tfrac{4}{5}\right)\right)$$

$$= A \cos(\omega t - \varphi)$$

$$\boxed{\begin{array}{l} \text{Frequency} = \omega \\ \text{Amplitude} = A \\ \text{Phase} = \varphi \end{array}}$$

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$$5\cos 5t + 4\sin 5t = \sqrt{5^2+4^2}\cos\left(5t + \operatorname{atan}\left(\tfrac{4}{5}\right)\right)$$

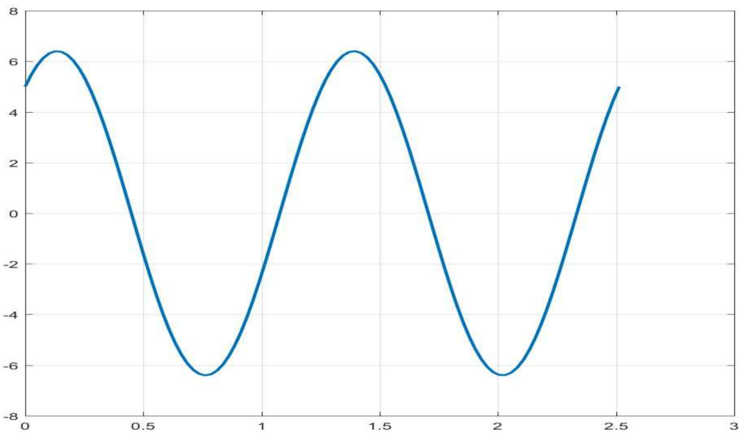
Deriving Frequency, Amplitude, phase from a formula*

Sum of Angles formula for cosine

$$A\cos(\omega t - \alpha) = A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha$$
$$= A\cos\alpha\cos\omega t + A\sin\alpha\sin\omega t$$

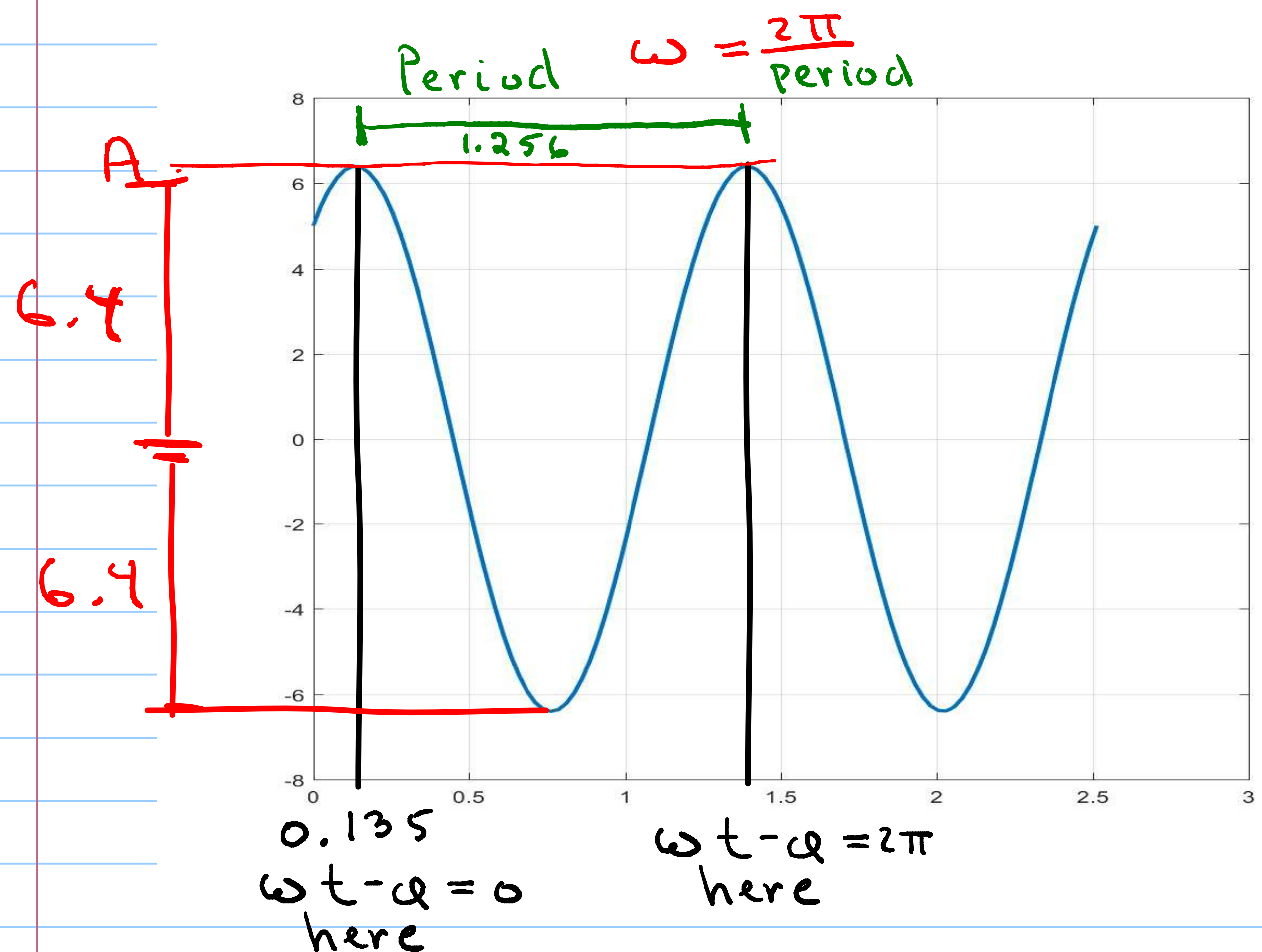
$$5\cos 5t + 4\sin 5t = A\cos\alpha\cos\omega t + A\sin\alpha\sin\omega t$$

$$\left.\begin{aligned} 5 &= A\cos\alpha \\ 4 &= A\sin\alpha \end{aligned}\right\} \Rightarrow A = (5^2+4^2)^{1/2} \qquad \alpha = \operatorname{atan2}(4,5) = \pm\operatorname{atan}\left(\tfrac{4}{5}\right)$$



* You may notice that the steps here are very similar to writing a complex number in polar form.

Frequency, Amplitude, Phase from the graph



$x(t) = A\cos(\omega t - \varphi)$

$A = 6.4 \qquad \omega = \dfrac{2\pi}{1.256} = 5$

$x(t)$ achieves its first maximum at $t = 0.135$

and $\cos(\omega t - \varphi)$ achieves its first

maximum when $\omega t - \varphi = 0$, i.e. $x(0.135) = A\cos(0)$

So $0.135 \cdot \omega - \varphi = 0$ and $\varphi = 5 \cdot 0.135 = 0.675$

$$x(t) = 6.4\cos(5t - 0.675)$$

* Adding multiples of 2π to φ also gives correct answers.