L17OptionalZetaUnderDamped

Note Title 8/9/2020

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0 \quad \text{Mass-Spring}$$

Often rewritten as

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \qquad \text{(DE)}$$

Solution in polar form

$$x(t) = A\, e^{-t\zeta\omega_0}\cos(\omega_d t - \varphi)$$

$$\omega_d = \sqrt{1-\zeta^2}\;\omega_0$$

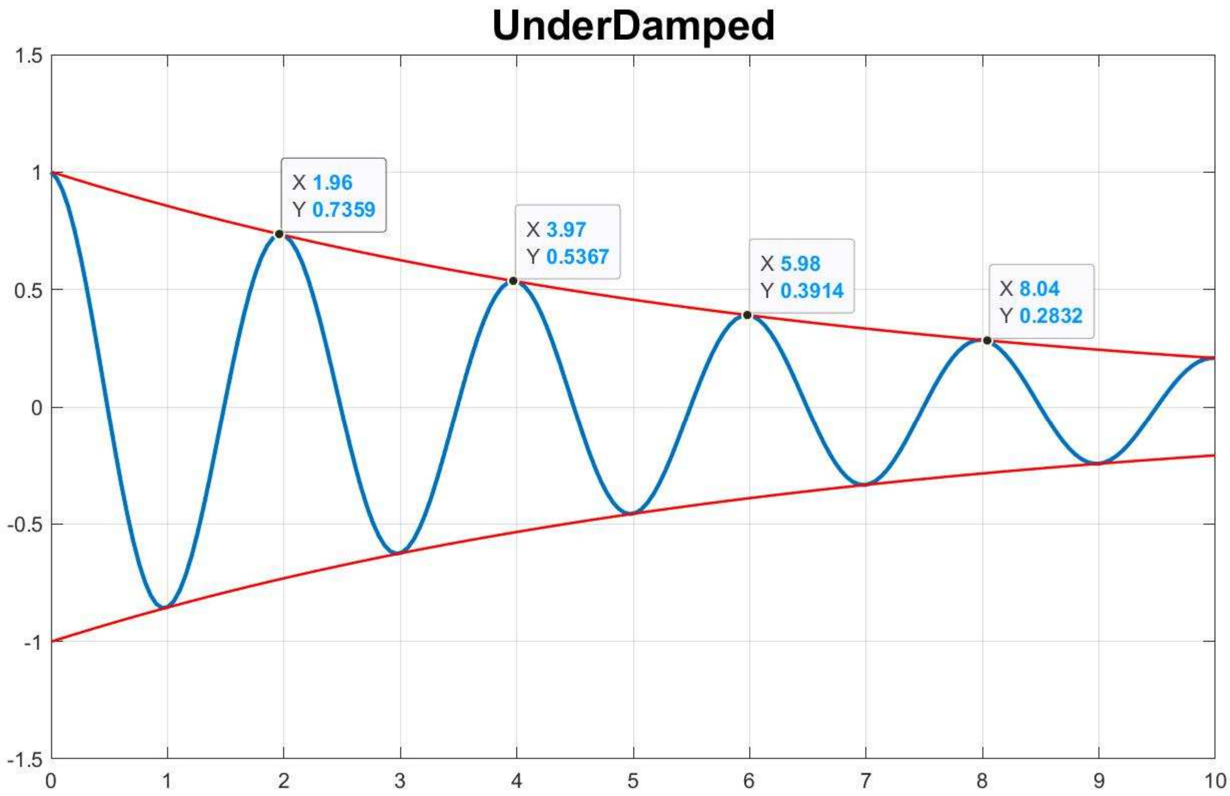
These quantities have standard names.

ω_0 = natural frequency

ω_d = quasi-frequency

ζ = damping ratio

(DE) is written this way because its easy to find ω_d, ζ, ω_0 from the graph.



$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^2 x = 0$$

Solution in polar form

$$x(t) = A e^{-\zeta\omega_o t}\cos(\omega_d t - \varphi)$$

Derivation of Solution

Seek $\quad x(t) = e^{rt}$

$$r^2 + 2\zeta\omega_o r + \omega_o^2 = 0$$

$$(r + \zeta\omega_o)^2 = -\omega_o^2(1-\zeta^2)$$

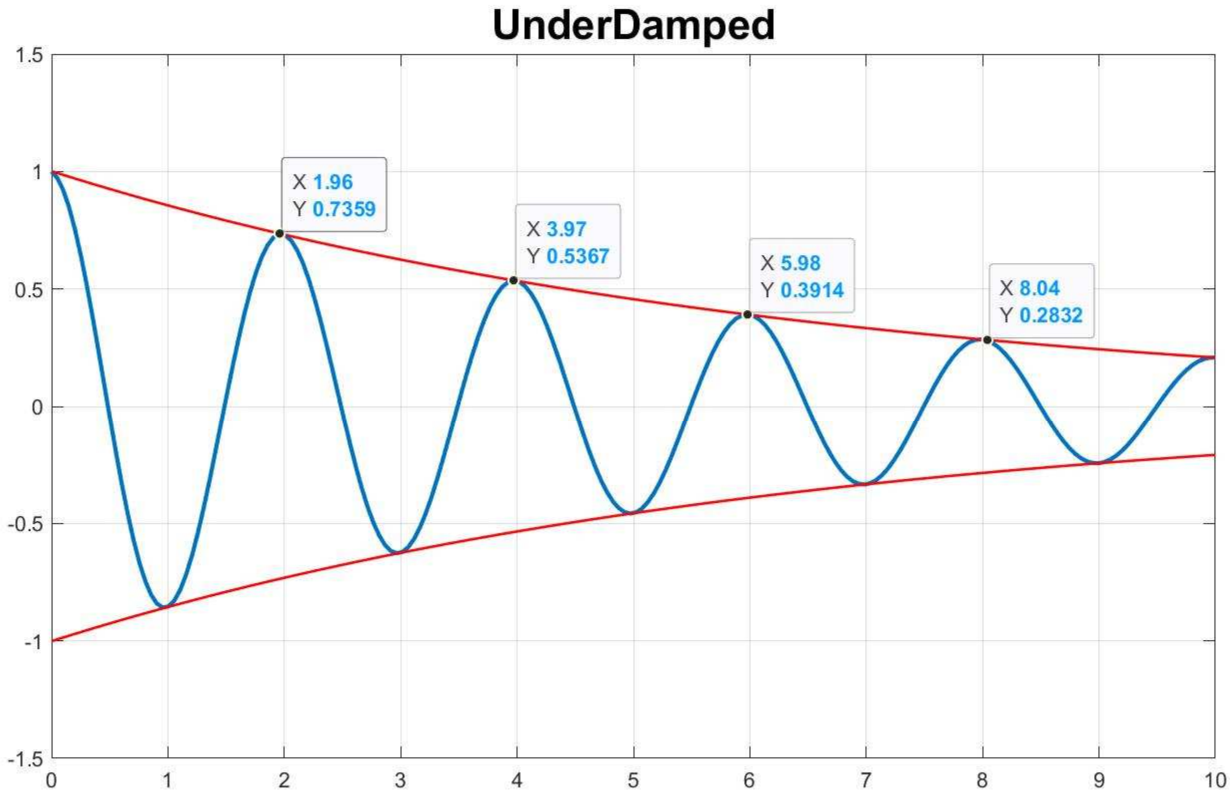
$$r = -\zeta\omega_o \pm i\omega_o\sqrt{1-\zeta^2}$$

Definition $\quad \omega_d = \sqrt{1-\zeta^2}\,\omega_o$

$$x(t) = e^{-\zeta\omega_o t}(c_1\cos\omega_d t + c_1\sin\omega_d t)$$

$$= e^{-\zeta_o\omega t} A\cos(\omega_d t - \varphi)$$

Problem: Find ω_d and ζ $\left(\omega_0 = \frac{\omega_d}{\sqrt{1-\zeta^2}}\right)^3$



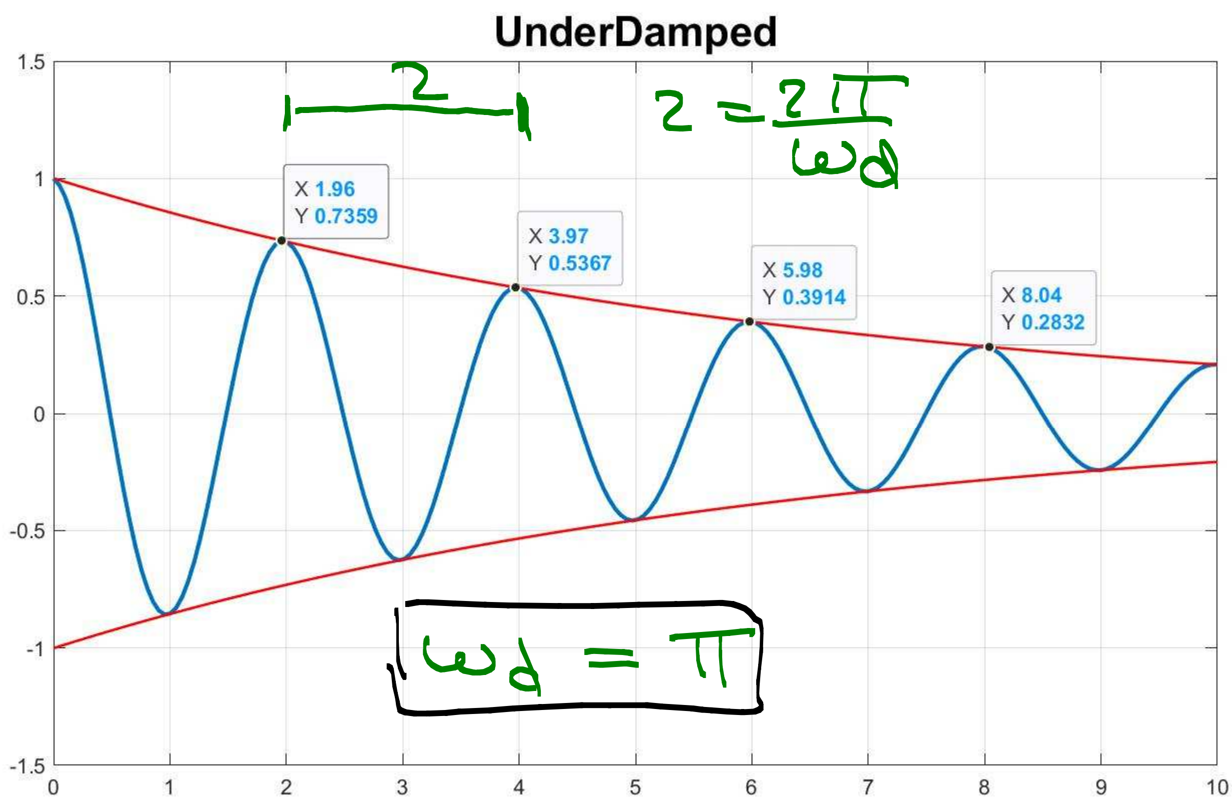
Relevant Facts

① The time between peaks (local maxima) is $\frac{2\pi}{\omega_d}$

② The cosine term $\cos(\omega_d t - \alpha)$ has the same value at all peaks

③ $\omega_d = \sqrt{1-\zeta^2}\,\omega_0 \approx \omega_0$ [because, typically, ζ is small.]

① The time between peaks is $\frac{2\pi}{\omega_d}$



$$e^{-\zeta\omega_0(1.96)}\cos(\omega_d\cdot 1.96-\alpha) = 0.7359$$

$$e^{-\zeta\omega_0(3.97)}\cos(\omega_d\cdot 3.97-\alpha) = 0.5367$$

② The cosine term $\cos(\omega_d t-\alpha)$ has the same value at all peaks

$$\frac{e^{-\zeta\omega_0(1.96)}\cos(\cancel{\omega_d}\cdot 1.96-\alpha)}{e^{-\zeta\omega_0(3.97)}\cos(\cancel{\omega_d}\cdot 3.97-\alpha)} = \frac{0.7359}{0.5367}$$

$$\frac{e^{-\zeta\omega_0(1.96)}}{e^{-\zeta\omega_0(3.97)}} = \frac{0.7359}{0.5367}$$

$$e^{\zeta\omega_0 \cdot 2.01} = 1.3712$$

$$\zeta\omega_0 = \frac{\ln(1.3712)}{2.01} = 0.1578$$

Usual Approximation $\omega_0 \approx \omega_d$

$$\omega_0 = \omega_d = \pi$$

$$\zeta = \frac{0.1578}{\pi} = 0.0502$$

Exact Calculation

$$(\zeta\omega_0)^2 + \omega_d^2 = (\zeta\omega_0)^2 + (1-\zeta^2)\omega_0^2$$

$$= \omega_0^2$$

$$(0.1578)^2 + \pi^2 = \omega_0^2$$

$$3.1455 = \omega_0$$

$$0.0502 = \frac{0.1578}{3.1455} = \zeta$$

Proof of ① and ②

Max occurs at t_0 where $\frac{dy}{dt} = 0$

$$\frac{d}{dt}\left(e^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)\right) = 0$$

$$-\zeta\omega_0 \cancel{e^{-\zeta\omega_0 t}} \cos(\omega_d t - \phi) = \cancel{e^{-\zeta\omega_0 t}}\, \omega_d \sin(\omega_d t - \phi)$$

$$-\frac{\zeta\omega_0}{\omega_d} = \tan(\omega_d t - \phi)$$

So at 2 consecutive peaks t_0 & t_1

$$\tan(\omega_d t_0 - \phi) = \tan(\omega_d t_1 - \phi)$$

$$\omega_d t_0 - \cancel{\phi} = \omega_d t_1 - \cancel{\phi} + N\pi$$

At consecutive maxima

$$\omega_d t_0 - \omega_d t_1 = 2\pi$$

② If $\tan(\omega_d t_0 - \phi) = \tan(\omega_d t_1 - \phi)$

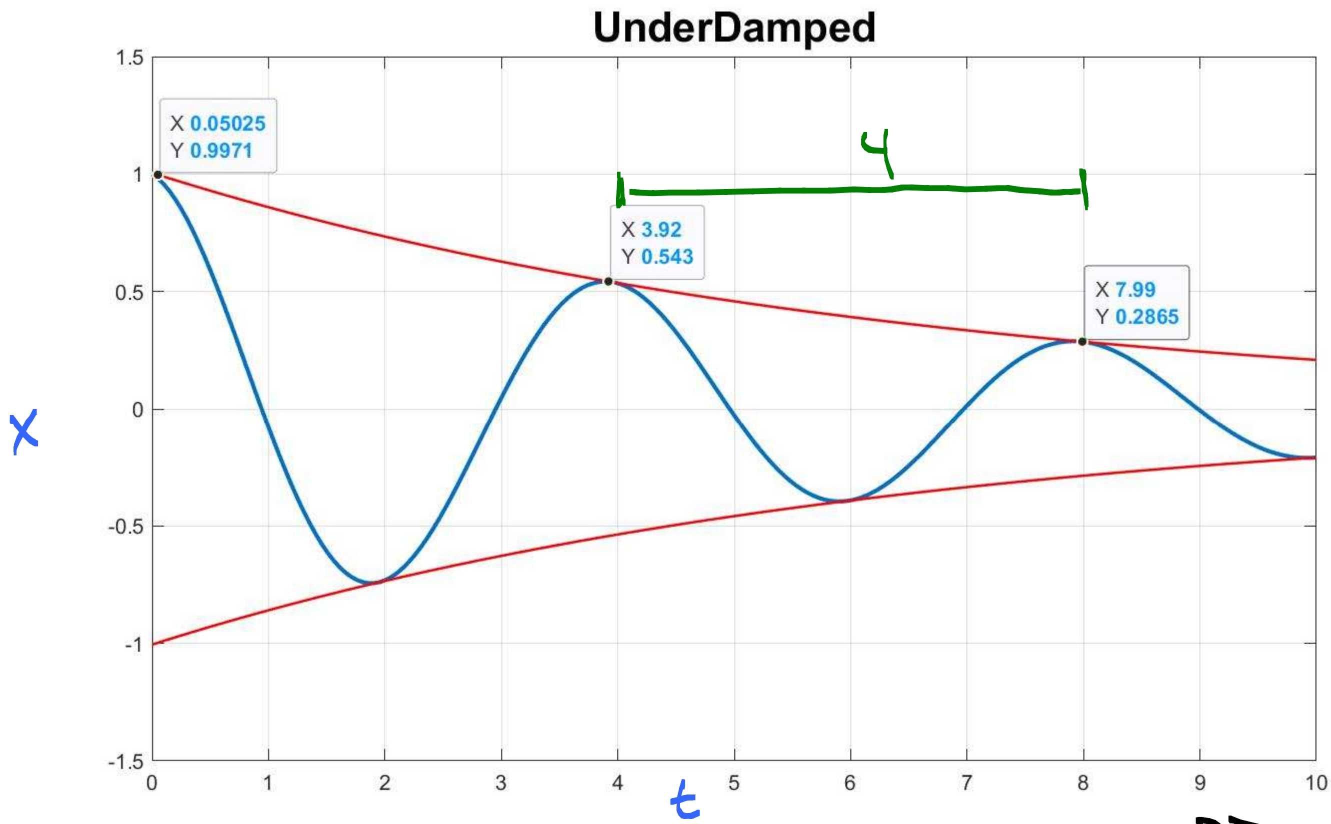
then $\cos(\omega_d t_0 - \phi) = \pm \cos(\omega_d t_1 - \phi)$

at maxima $\cos(\omega_d t_0 - \phi) = \cos(\omega_d t_1 - \phi)$

Problem 2

Find ξ and ω_0

$$x(t) = A\, e^{-\xi \omega_0 t} \cos(\omega_d t - \phi)$$



The time between peaks is $\frac{2\pi}{\omega_d}$

$$\frac{2\pi}{4} = \omega_d \approx \omega_0$$

The cosine term has the same value at all peaks

$$\frac{e^{-\xi\omega_0 \cdot 3.92} \cos(\cancel{\tfrac{\pi}{2}} 3.92 - \phi)}{e^{-\xi\omega_0 \cdot 7.99} \cos(\cancel{\tfrac{\pi}{2}} 7.99 - \phi)} = \frac{0.543}{0.286}$$

$$4\xi\omega_0 = \ln\left(\frac{0.543}{0.286}\right) = 0.64$$

$$\xi = \frac{0.64}{2\pi} = 0.102$$