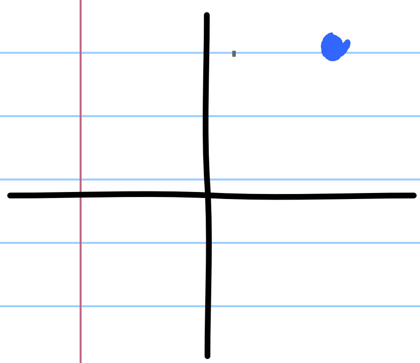
L16ComplexNumbersEulerIdentity

Note Title 8/3/2020

Complex Numbers and Euler's Formula

(pronounced Oiler's)



$z = x + iy$

These are just the x, y coordinates

Multiplication There is no natural way to multiply $x\,y$ coordinates

but there is a way to multiply complex numbers.

$$(a+ib)\cdot(c+id) = ac + iad + ibc + i^2 bd$$

$$\boxed{i^2 = -1} \qquad = (ac - bd) + i(ad + bc)$$

We write $z = a + ib$ and say

a is the real part of z and
b is the imaginary part of z

We also write $\bar{z} = a - ib$ and call

$\bar{z}$ the complex conjugate of z

spoken "zee-bar"

Let $z = a + ib$

$|z| = |a+ib|$ is the length (or modulus) of z.

We calculate $|z|$ using the formula

$$|z|^2 = z \cdot \overline{z} = (a+ib)(a-ib)$$
$$= a^2 + b^2$$

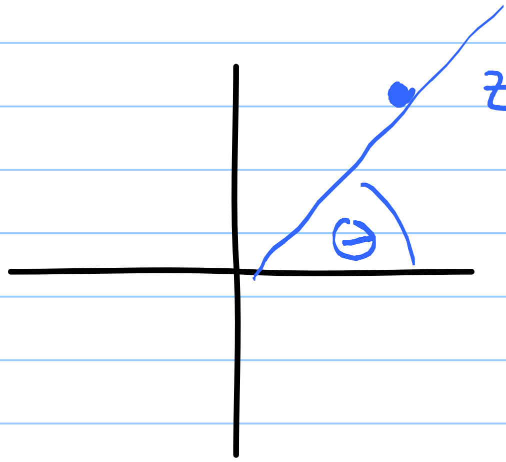
This is the same as the length of the point (a,b) in the plane.

Reciprocals and Division

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} + i \frac{-b}{a^2+b^2}$$

Polar Representation



$z = x + iy$

Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

so $z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= r(e^{i\theta})$

The last step is a consequence of Euler's formula.

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Polar Representation of a Complex Number
(also called phasor representation)
(the same as polar coordinates)

Problem: Find the polar representation of $1+\sqrt{3}\,i$?

This means, find r and θ so that

$$1+\sqrt{3}\,i = r e^{i\theta} \xrightarrow{\text{Euler's formula}} = r\cos\theta + i\, r\sin\theta$$

Real parts must be equal and imaginary parts must be equal

so we must solve $\quad 1 = r\cos\theta$

and $\quad \sqrt{3} = r\sin\theta$

Solve for r

$$1^2+(\sqrt{3})^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$$

$$4 = r^2$$

Solve for θ:

$$\frac{\sqrt{3}}{1} = \tan\theta$$

$$\theta = \operatorname{atan}(\sqrt{3}) = \frac{\pi}{3}$$

so $\boxed{1+\sqrt{3}\,i = 2e^{i\frac{\pi}{3}}}$

<u>Question</u> What are the real and imaginary parts of

$$e^{i\left(2t+\frac{\pi}{3}\right)}$$

Euler's Formula

$$e^{i\left(2t+\frac{\pi}{3}\right)} = \underbrace{\cos\left(2t+\frac{2\pi}{3}\right)}_{\text{Real part}} + i\,\underbrace{\sin\left(2t+\frac{2\pi}{3}\right)}_{\text{Imaginary part}}$$

Justifying Euler's Identity

Note Title 1/22/2020

$$e^{it} = \cos t + i \sin t$$

Main Property of exponential $e^{(t+s)} = e^t e^s$

Characterization of e^t

Theorem 1

If $f(t+s) = f(t)f(s)$

then $f'(t) = f'(0)f(s)$ and $f(0) = 1$.

Proof Differentiate

$f(t+s) = f(t)f(s)$

with respect to s:

$f'(t+s) = f(t)f'(s)$

and set $s=0$.

$f'(t) = f(t)f'(0)$

Next, just set $s=0$ in

$f(t+s) = f(t)f(s)$

$f(t) = f(t)f(0)$

so $f(0) = 1$ ∎

Theorem 2

If $f'(t) = b\, f(t)$

then $f(t) = k\, e^{bt}$

Proof

$f'(t) = b\, f(t)$

so

$\dfrac{df}{f} = b\, dt$

$\ln|f| = bt + C$

$f = k e^{bt}$ ∎

What about $f(t) = \cos t + i \sin t$?

② $f'(t) = -\sin t + i \cos t$

$= i(\cos t + i \sin t)$

$= i\, f(t)$

so $f(t) = k e^{it}$ and $f(0) = 1$ so $f(t) = e^{it}$

Sum of Angle Formulas are hard to remember

Product of Exponential Formula is easy $e^{(t+s)} = e^t e^s$

$\cos(t+s) + i\sin(t+s) = e^{i(t+s)}$

$= e^{it} \cdot e^{is}$ Euler's Formula

$= (\cos t + i \sin t)(\cos s + i \sin s)$ multiply

$= (\cos t \cos s - \sin t \sin s) + i(\cos t \sin s + \sin t \cos s)$

so

$\cos(t+s) = \cos t \cos s - \sin t \sin s$

and

$\sin(t+s) = \cos t \sin s + \sin t \cos s$