An Underdamped Harmonic Oscillator

Problem A 2kg mass is suspended from a spring with spring constant (stiffness) ~~24~~ 40 kg/sec² and a damping coefficient of ~~16~~ 8 kg/sec. The mass is set in motion from its equilibrium position with an initial (downward) velocity of 1 m/sec. Formulate and solve the initial value problem.*

Solution

$$2\ddot{y} = -8\dot{y} - 40y \qquad y(0) = 0 \qquad \dot{y}(0) = 1$$

Seek $y = e^{rt}$ then $\dot{y} = re^{rt}$ and $\ddot{y} = r^2e^{rt}$

$$\ddot{y} + 4\dot{y} + 20y = 0$$

$$r^2e^{rt} + 4re^{rt} + 20e^{rt} = 0$$

$$r^2 + 4r + 20 = 0$$

$$r^2 + 4r + 4 = -16$$

$$(r+2)^2 = -16$$

$$\boxed{r = -2 \pm 4i}$$

Solutions to "the indicial equation" are complex numbers.

$r = -2 \pm 4i$

* stiffer spring, less damping

General Solution

$$y(t) = c_1 e^{(-2+4i)t} + c_2 e^{(-2-4i)t}$$

[We need to interpret (i.e. say what $e^{(-2+4i)t}$ means) but let's just go ahead for the time being.]

Initial Conditions

$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = c_1(-2+4i) + c_2(-2-4i)$$

so $c_1 = -c_2$ and

$$1 = c_1(-2+4i) - c_1(-2-4i)$$

$$= -2c_1 + 4ic_1 + 2c_1 + 4ic_1$$

$$= \cancel{-2c_1} + 4ic_1 \cancel{+2c_1} + 4ic_1$$

$$1 = 8ic_1$$

$c_1 = \frac{1}{8i}$ and $c_2 = \frac{-1}{8i}$

$$y(t) = \frac{1}{8i} e^{(-2+4i)t} - \frac{1}{8i} e^{(-2-4i)t}$$

This is a real physical problem. It should always have a real solution. The displacement shouldn't be a complex number.

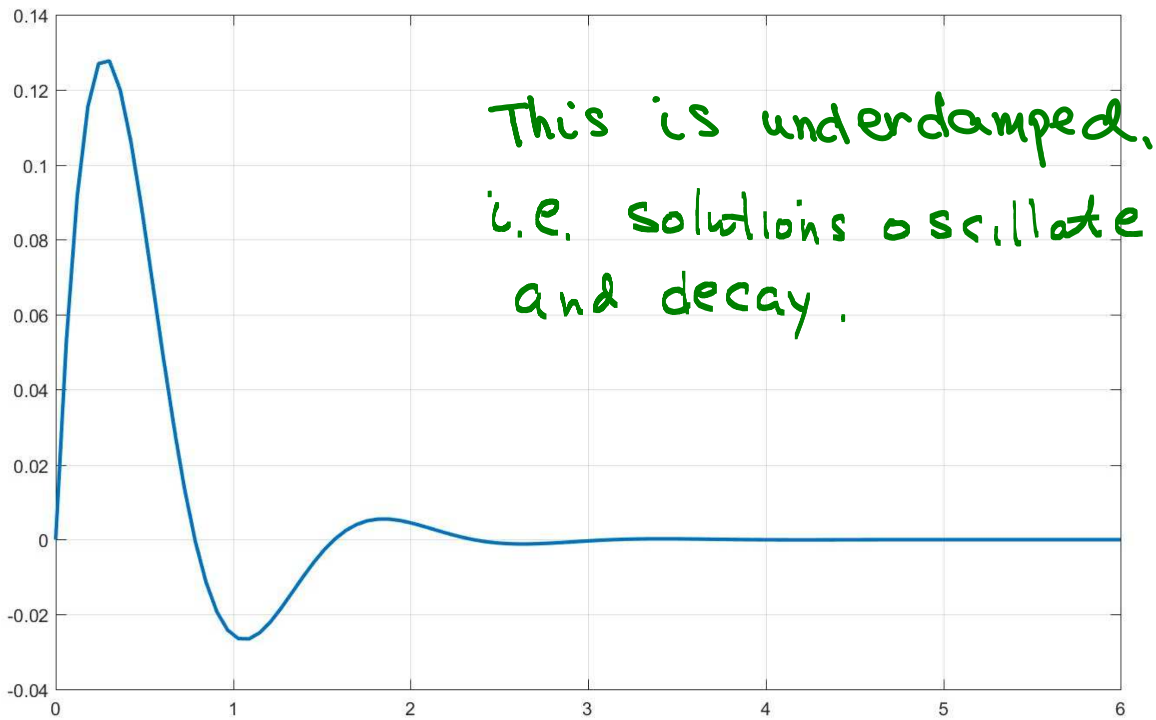
$$y(t) = \frac{1}{8i} e^{(-2+4i)t} - \frac{1}{8i} e^{(-2-4i)t}$$

$$= \frac{e^{-2t}}{4} \left(\frac{e^{4it} - e^{-4it}}{2i} \right)$$

I'll show you that

$$\left(\frac{e^{4it} - e^{-4it}}{2 \cdot i} \right) = \sin 4t$$

$$y(t) = \frac{e^{-2t}}{4} \sin 4t$$



Additional question: write down the equation that you must solve to find the maximum displacement (you won't be able to solve with pencil and paper)