L14WhatDoesLinearMean

Note Title 8/4/2020

What does Linear Mean? Linear Equations

$3x + 4y = w$ is a linear equation for the three variables (x, y, w). This means:

You can add solutions

If $(1,0,3)$ is a solution, i.e. $3\,1 + 4\,0 = 3$

[$(1,0,3)$ means $x=1, y=0, w=3$]

and $(0,1,4)$ is a solution, i.e. $3\,0 + 4\,1 = 4$

then $(1,1,7)$ is a solution, i.e. $3(1+0)+4(0+1) = (3+4)$

You can multiply by constants

If $(1,0,3)$ is a solution, i.e. $3\,1 + 4\,0 = 3$

then $(2,0,6)$ is a solution, i.e. $3(2\cdot 1) + 4(2\cdot 0) = (2\cdot 3)$

Nonlinear means this doesn't work

$x^2 + y^2 = w^2$ is NOT a linear equation

$(1,0,1)$ solves $1^2 + 0^2 = 1^2$

$(0,1,1)$ solves $0^2 + 1^2 = 1^2$

But

$(1,1,2)$ doesn't $(1+0)^2 + (0+1)^2 \neq (1+1)^2$

Linear DE's $\ddot{y} + a\dot{y} + by = 0$

[a and b could be constants or functions of t]

Two Principles

① If $y_1(t)$ solves and $y_2(t)$ solves

then $c_1 y_1(t) + c_2 y_2(t)$ solves

② If $y_1(t)$ and $y_2(t)$ are

independent ($y_1(t) \neq C\, y_2(t)$) solutions

then every solution $y(t)$ can be written

as $y(t) = c_1\, y_1(t) + c_2\, y_2(t)$

I will explain ① on the following pages. ② is harder; I won't explain now.

I will expect you to use these two principles, not to prove them.

We said that $\ddot{y} + 8\dot{y} + 12y = 0$ was a linear DE.

We checked that e^{-6t} and e^{-4t} were solutions,

and concluded that, because the DE was linear,

$y(t) = c_1 e^{-6t} + c_2 e^{-4t}$ was also a solution.

To check this, we need to see that

$$(c_1 e^{-6t} + c_2 e^{-4t})^{\cdot\cdot} + 8(c_1 e^{-6t} + c_2 e^{-4t})^{\cdot} + 12(c_1 e^{-6t} + c_2 e^{-4t}) \overset{?}{=} 0$$

The main ingredient is the fact that the derivative, $\frac{d}{dt}$, is linear. You take the derivative of the sum of two functions by taking the derivative of each one and then adding the two derivatives. Also, the derivative of a constant times a function is the constant times the derivative of the function. This means that

$$(c_1 e^{-6t} + c_2 e^{-4t})^{\cdot} = c_1 (e^{-6t})^{\cdot} + c_2 (e^{-4t})^{\cdot}$$

and that

$$(c_1 e^{-6t} + c_2 e^{-4t})^{\cdot\cdot} = c_1 (e^{-6t})^{\cdot\cdot} + c_2 (e^{-4t})^{\cdot\cdot}$$

We can put these facts together:

$$12\left(c_1 e^{-6t}+c_2 e^{-4t}\right) = c_1\left(12 e^{-6t}\right)+c_2\left(12 e^{-4t}\right)$$

$$8\left(c_1 e^{-6t}+c_2 e^{-4t}\right)' = c_1\cdot 8\left(e^{-6t}\right)'+c_2\cdot 8\left(e^{-4t}\right)'$$

$$+\ \left(c_1 e^{-6t}+c_2 e^{-4t}\right)'' = c_1\left(e^{-6t}\right)''+c_2\left(e^{-4t}\right)''$$

$$\begin{aligned}&\left(c_1 e^{-6t}+c_2 e^{-4t}\right)'' + \\ &8\left(c_1 e^{-6t}+c_2 e^{-4t}\right)' + \\ &+12\left(c_1 e^{-6t}+c_2 e^{-4t}\right)\end{aligned} = \begin{aligned}&c_1\left[\left(e^{-6t}\right)''+8\left(e^{-6t}\right)'+12\left(e^{-6t}\right)\right] \\ &+ \\ &c_2\left[\left(e^{-4t}\right)''+8\left(e^{-4t}\right)'+12\left(e^{-4t}\right)\right]\end{aligned}$$

$$\boxed{\begin{array}{ll}\left(e^{-6t}\right)''+8\left(e^{-6t}\right)'+12\left(e^{-6t}\right)=0 & \text{because } e^{-6t} \text{ solves DE} \\ \left(e^{-4t}\right)''+8\left(e^{-4t}\right)'+12\left(e^{-4t}\right)=0 & \text{because } e^{-4t} \text{ solves DE}\end{array}}$$

$$= c_1\cdot 0 + c_2\cdot 0 = 0$$

So we have checked that $y(t) = c_1 e^{-6t} + c_2 e^{-4t}$ solves the DE.

A slightly more general calculation

Note Title 8/4/2020

for a different DE $\ddot{y}+4y=f(t)$

Suppose that

$$\ddot{y}_1 + 4y_1 = f_1$$
$$\ddot{y}_2 + 4y_2 = f_2$$

and a and b are constants, then

$$w = a y_1 + b y_2$$

Solves $\ddot{w} + 4w = a f_1 + b f_2$

Proof

$$a\ddot{y}_1 + a\cdot 4\, y_1 = a f_1$$
$$+\quad b\ddot{y}_2 + b\cdot 4\, y_2 = b f_2$$

$$a\ddot{y}_1 + b\ddot{y}_2 + 4\cdot(a y_1 + b y_2) = a f_1 + b f_2$$

derivative is linear

$$(a y_1 + b y_2)^{\cdot\cdot} + 4\cdot(a y_1 + b y_2) = a f_1 + b f_2$$

$$\ddot{w} + 4w = a f_1 + b f_2$$

So far, we have only used this in the case that $f_1 = 0$ and $f_2 = 0$