L13SecondOrderOverdamped

Note Title 8/3/2020

Problem A 2kg mass is suspended from a spring with spring constant (stiffness) 24 kg/sec^2 and a damping coefficient of 16 kg/sec. The mass is set in motion from its equilibrium position with an initial (downward*) velocity of 1 m/sec. Formulate and solve the initial value problem.

Solution

y = displacement from equilibrium**

$$m\ddot{y} = -\gamma\dot{y} - ky$$

$$2\ddot{y} = -16\dot{y} - 24y$$

starts from equilibrium $y(0)=0$

initial velocity $\dot{y}(0)=1$

$$2\ddot{y} + 16\dot{y} + 24y = 0$$

$$y(0)=0 \quad \dot{y}(0)=1$$

(IVP)

* Remember that down is positive

** I used the letter $x(t)$ for displacent in the last lecture

IVP

$$\ddot{y} + 8\dot{y} + 12y = 0 \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 1 \qquad \text{(IC)}$$

Seek solutions of the form $y(t) = e^{rt}$

This is an important step that we will use repeatedly.

All solutions to Second Order Constant Coefficient (the constant coefficients in this example are 8 and 12) differential equations are sums of exponential functions.*

Set $y(t) = e^{rt}$

then $\dot{y}(t) = r e^{rt}$ and $\ddot{y}(t) = r^2 e^{rt}$

Insert into (DE)

$$\ddot{y} + 8\dot{y} + 12y = 0$$

$$r^2 e^{rt} + 8 r e^{rt} + 12 e^{rt} = 0$$

$$r^2 + 8r + 12 = 0 \quad \longleftarrow \text{Textbook calls this the "indicial equation".}$$

$$(r+6)(r+2) = 0$$

$$r = -6 \quad \text{or} \quad r = -2$$

so $y(t) = e^{-6t}$ and $y(t) = e^{-2t}$ solve the DE

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* The exponentials may be complex and there may be an occasional polynomial thrown in. Details later.

Because $\ddot{y} + 8\dot{y} + 12y = 0$ is a Linear DE,

1) constant multiples of solutions are also solutions
2) sums of solutions are solutions

General Solution $\quad y_G(t) = c_1 e^{-6t} + c_2 e^{-2t}$

Every solution to the DE can be written as $y_G(t)$ with appropriately chosen constants.

We have initial conditions:

$$0 = y(0) = c_1 e^{-6\cdot 0} + c_2 e^{-2\cdot 0}$$

$$1 = \dot{y}(0) = c_1(-6) e^{-6\cdot 0} + c_2(-2) e^{-2\cdot 0}$$

$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = -6c_1 - 2c_2$$

$$\begin{aligned} 0 &= 2c_1 + 2c_2 \\ + \quad 1 &= -6c_1 - 2c_2 \\ \hline 1 &= -4c_1 \end{aligned}$$

so $c_1 = -\frac{1}{4}$ and $c_2 = \frac{1}{4}$

so $\quad y(t) = -\frac{1}{4} e^{-6t} + \frac{1}{4} e^{-2t}$

Summary (This is what I would write on an exam)

$$\ddot{y} + 8\dot{y} + 12y = 0 \quad \text{(DE)}$$

$$y(0) = 0 \quad \dot{y}(0) = 1 \quad \text{(IC)}$$

seek $y(t) = e^{rt}$

$$r^2 e^{rt} + 8re^{rt} + 12e^{rt} = 0$$

$$r^2 + 8r + 12 = 0$$

$$(r+6)(r+2) = 0$$

$$r = -6 \quad \text{or} \quad r = -2$$

$$y_G(t) = c_1 e^{-6t} + c_2 e^{-2t}$$

$$0 = y(0) = c_1 + c_2$$

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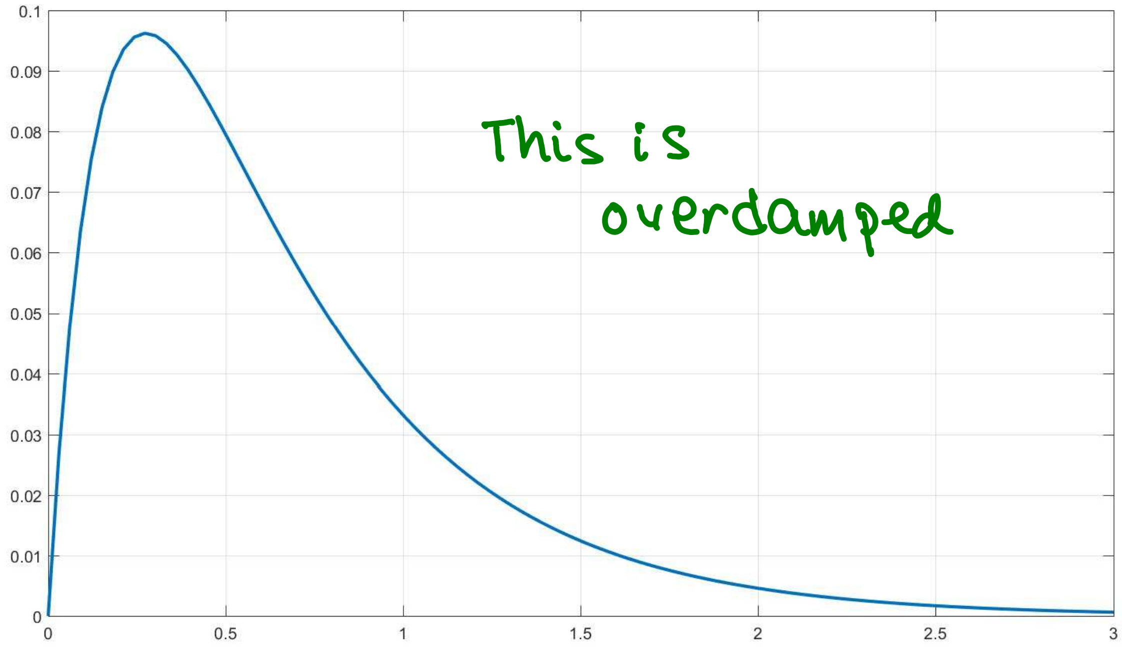
Graph of Displacement



Additional Questions

What is the maximum displacement?

After a long time*, $y(t)$ is extremely close to one of the two terms in the formula. Which one is it?

* For this question, a long time means a few seconds.

Equations of Motion for Damped Harmonic Oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \quad \text{(DE)}$$

Accompanied by Initial Conditions

$$x(0) = x_0 \qquad \dot{x}(0) = v_0 \quad \text{(IC)}$$

initial position

initial velocity

Up is positive in these graphs