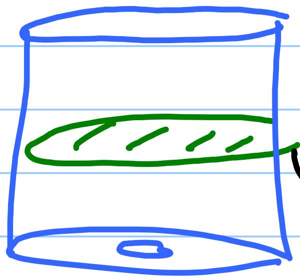
Toricelli's Law (Hwk Problem)

cross section



cylindrical Tank

Water leaks out hole

Determine $h(t)$ = height of water at time t

Motivation – Build a water clock by marking $h(t)$ on the sides of the cylinder.

Two Equations

h = height of water
v = speed of water exiting

Water Out = decrease in Volume

$$A_{\text{hole}} \cdot v = A_{\text{cylinder}} \cdot \frac{dh}{dt} \qquad (T1)$$

cross section

Increase in kinetic energy = { Decrease in potential energy

$$\frac{1}{2} \Delta m \, v^2 = \Delta m \, g \, h \qquad (T2)$$

$\Delta m = A_{\text{hole}} \rho \, v \, dt$ = mass that exits in time dt

Find DE for h(t)

$$\frac{1}{2}\cancel{\Delta m}\, v^2 = \cancel{\Delta m}\, g h \qquad (T2)$$

$$v = \pm\sqrt{2gh}$$

water flows down through the hole, so v is negative

$$A_{cyl} \cdot \frac{dh}{dt} = -A_{hole}\, v \qquad (T1)$$

$$A_{cyl}\, \frac{dh}{dt} = -A_{hole}\sqrt{2gh}$$

$$\frac{dh}{dt} = -\underbrace{\frac{\pi r_{hole}^2}{\pi r_{cyl}^2}}_{\text{constant}} \cdot \sqrt{2000}\; h^{1/2}$$

Notice, that $\frac{dh}{dt}$ is negative, as it should be, confirming that we chose the sign of v correctly.

Question Find DE for $v(t)$

Answer

minus, because up is positive

$$v^2 = 2gh \Rightarrow v = -\sqrt{2g}\sqrt{h}$$

$$\cancel{2}v\frac{dv}{dt} = \cancel{2}g\frac{dh}{dt}$$

$$= g\frac{r_{hole}^2}{r_{cyl}^2}\cdot\sqrt{2g}\quad h^{1/2}$$

$$\cancel{v}\frac{dv}{dt} = g\frac{r_{hole}^2}{r_{cyl}^2}\cancel{\sqrt{2g}}\left(\frac{-\cancel{v}}{\cancel{\sqrt{2g}}}\right)$$

$$\frac{dv}{dt} = -g\frac{r_{Hole}^2}{r_{cylinder}^2}$$

A pond has an initial volume of 10,000 m^3 water and no salt. Two streams flow in and one stream flows out.

Stream A
- $\frac{500\ m^3}{day}$ influx
- water contains $\frac{5\ kg}{1000\ m^3}$ salt

Stream B
- $\frac{750\ m^3}{day}$ influx
- No salt

Stream C $\frac{1300\ m^3}{day}$ outflux

Find the differential Equation for S = total amount of salt in the pond

$$\frac{dS}{dt} = \text{rate of Salt influx} - \text{rate of Salt outflux}$$

$$= \frac{500\,m^3}{day} \cdot \frac{5\,kg}{1000\,m^3} - \frac{1300\,m^3}{day} \cdot \text{concentration of Salt}$$

$$\text{concentration of Salt} = \frac{S}{\text{Volume of lake}} = \frac{S}{10{,}000 - 50t}$$

$$\text{Volume} = \underbrace{10{,}000}_{\text{initial Volume}} + \underbrace{500t}_{\text{stream A in}} \underbrace{+750t}_{\text{stream B in}} \underbrace{-1300t}_{\text{stream C out}}$$

$$\frac{dS}{dt} = \frac{5}{2}\,\frac{kg}{day} - \frac{1300}{10{,}000-50t}\,S$$

$$\frac{dS}{dt} - \frac{1300}{50t-10000}\,S = \frac{5}{2}$$

$$\boxed{\frac{dS}{dt} - \frac{26}{(t-200)}\,S = \frac{5}{2} \qquad S(0) = 0}$$

$$\frac{dS}{dt} - \frac{26}{(t-200)}S = \frac{5}{2} \qquad S(0) = 0$$

No salt at time = 0

Find Integrating Factor

$$\frac{dm}{dt} = \frac{-26}{(t-200)}m$$

$$\frac{dm}{m} = \frac{-26}{t-200}dt$$

$$\ln|m| = -26\ln|t-200| + c$$

$$= \ln|(t-200)^{-26}| + c$$

$$m = (t-200)^{-26}$$

Multiply Both sides by m

$$(t-200)^{-26}\frac{dS}{dt} - 26(t-200)^{-27} = \frac{5}{2}(t-200)^{-26}$$

$$\frac{d}{dt}\left[(t-200)^{-26}S\right] = \frac{5}{2}(t-200)^{-26}$$

Integrate

$$(t-200)^{-26}S = \frac{5}{2}\cdot\frac{(t-200)^{-25}}{-25} + c$$

$$S = -\frac{1}{10}(t-200) + c(t-200)^{26}$$

$$S = 20 - \frac{t}{10} + c(t-200)^{26}$$

Find c $\qquad 0 = 20 + c(200)^{26}$

$$\boxed{S = 20 - \frac{t}{10} - \frac{20}{(200)^{26}}(t-200)^{26}}$$

$$\frac{dS}{dt} - \frac{26}{(t-200)} S = \frac{5}{2}$$

Introduce a new time variable

$$x = t - 200$$

$$\frac{dx}{dt} = 1 \qquad \text{so}$$

$$\frac{dS}{dx} = \frac{dS}{dt} = \frac{26}{x} S + \frac{5}{2}$$

Then we can solve for $S(x)$

$$\frac{dS}{dx} - \frac{26}{x} S = \frac{5}{2} \quad ; \quad S(-200) = 0$$

Now we use the same methods as before, but we can just write x, instead of $(t-200)$