Limited Resources (Logistic Growth Model)

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) \qquad \text{(DE)}$$

r = growth rate

k = carrying capacity

$k = \begin{cases} \text{maximum population that} \\ \text{available resources (e.g. food supply)} \\ \text{can sustain} \end{cases}$

Question $\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$

Suppose $r = 1$ and $k = 10$

and $P(0) = 1$. Solve the IVP.

Aside Solutions to the logistic equation are called sigmoid functions. They play an important role in Biology and, more recently, in machine learning and neural networks.

(DE) $$\frac{dP}{dt} = P\left(1-\frac{P}{10}\right)$$ (IC) $P(0)=1$

Separate variables

$$\frac{dP}{P\left(1-\frac{P}{10}\right)} = dt$$

Make it look a little neater

$$\frac{dP}{P(10-P)} = \frac{dt}{10}$$

partial fractions

(details not shown)

$$\frac{1}{10}\frac{dP}{P} - \frac{1}{10}\frac{dP}{P-10} = \frac{dt}{10}$$

Integrate both sides,

Note the minus sign

$$\frac{1}{10}\ln|P| - \frac{1}{10}\ln|P-10| = \frac{1}{10}t + c$$

$$\ln\left|\frac{P}{P-10}\right| = t + c$$

$$\left|\frac{P}{P-10}\right| = c_2 e^t$$

$$\left|\frac{P}{P-10}\right| = C_2 e^t$$

Now, worry about absolute values

$$\frac{P}{P-10} = \pm C_2 e^t$$

Fortunately, its easy, set $C_3 = \pm C_2$

The $\pm$ just changes the sign of the constant

$$\frac{P}{P-10} = C_3 e^t$$

Initial Condition:

$$-\frac{1}{9} = \frac{1}{1-10} = C_3$$

$$\frac{P}{P-10} = -\frac{1}{9} e^t$$

Now solve for P

$$P = \frac{1}{9}(10-P)e^t$$

$$P = \frac{10e^t}{9} - P\frac{e^t}{9}$$

$$9P + e^t P = 10e^t$$

$$P(9 + e^t) = 10e^t$$

$$\boxed{P(t) = \frac{10e^t}{9 + e^t}}$$

Partial Fractions

Fact: As long as $a \neq b$, $\frac{cx+d}{(x-a)(x-b)}$ can always be rewritten as $\frac{A}{x-a} + \frac{B}{x-b}$

How do we find A and B ?

$$\frac{1}{P(P-10)} = \frac{A}{P} + \frac{B}{P-10}$$

Eliminate denominators *

$$1 = A(P-10) + BP \qquad **$$

The equality must hold for every P, so we choose values for P that simplify the equation

Set $P=0$

$$1 = -10A \quad \text{so} \quad A = \frac{-1}{10}$$

Set $P=10$

$$1 = 10B \quad \text{so} \quad B = \frac{1}{10}$$

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* If you look carefully at the step where we "eliminated denominators", you will see that, for $P=0$ and $P=10$, we multiplied both sides of the equation by zero. To justify this step, we argue that the equation ** is valid for every P except $P=0$ or 10, but $A(P-10)+BP$ is a continuous function, so we can take limits to see that ** is also valid at $P=0$ and $P=10$.

Reminder

Partial Fractions - cover up method

$$\frac{1}{(s-2)(s-3)(s-4)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} + \frac{C}{s-4}$$

Find A, B, C

To find A Cover up $(s-2)$ & set $s=2$

$$A = \left.\frac{1}{\cancel{(s-2)}(s-3)(s-4)}\right|_{s=2} = \frac{1}{(2-3)(2-4)}$$

To Find B Cover up $(s-3)$ & set $s=3$

$$B = \left.\frac{1}{(s-2)\cancel{(s-3)}(s-4)}\right|_{s=3} = (2-3)(2-4)$$

Justification - multiply both sides by $(s-2)$

$$\frac{1\,(\cancel{s-2})}{(\cancel{s-2})(s-3)(s-4)} = \frac{A\,(\cancel{s-2})}{(\cancel{s-2})} + \frac{B\,(s-2)}{(s-3)} + \frac{C\,(s-2)}{s-4}$$

Set $s=2$

$$\frac{1}{(2-3)(2-4)} = A + 0 + 0$$

$$\frac{1}{(2-3)(2-4)} = A \qquad \frac{1}{(3-2)(3-4)} = B \qquad \frac{1}{(4-2)(4-3)} = C$$