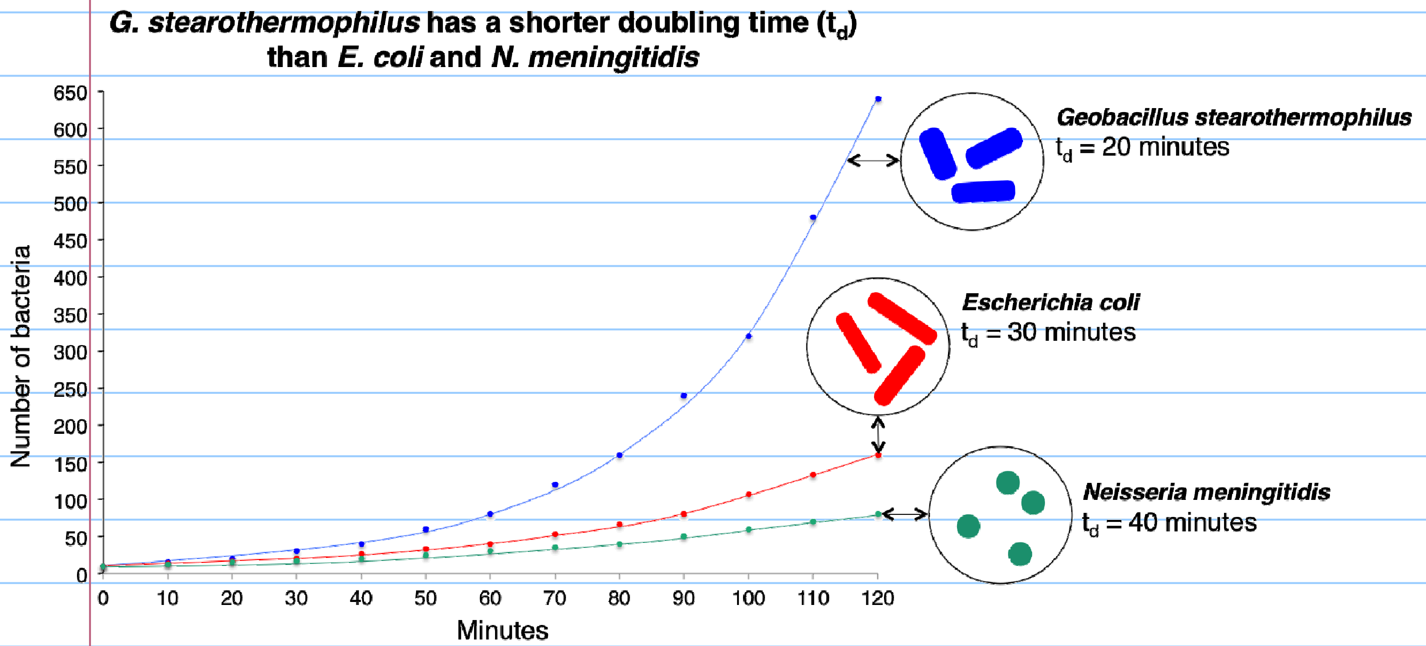
L08 Population Models

Note Title 7/29/2020

Unlimited Resources (Exponential Growth)



$P(t)$ = Population at time t

r = growth rate = birth rate − death rate

$$\left.\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}\right\}$$ Initial Value Problem

<u>Question</u> Suppose the doubling time is 30 minutes. Find r.

r is sometimes called the "proportionality constant

$$\left.\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}\right\}$$ Initial Value Problem

<u>Question</u> Suppose the doubling time is 30 minutes. Find r

<u>Answer</u>

$$\frac{dP}{P} = r\,dt$$

$$\ln|P| = rt + C$$

$$P = k e^{rt}$$

$$P_0 = P(0) = k$$

$$P(t) = P_0 e^{rt}$$

<u>Calculate r</u>

$P(30)$ equals twice $P(0)$

$$P(30) = P_0 e^{30r}$$

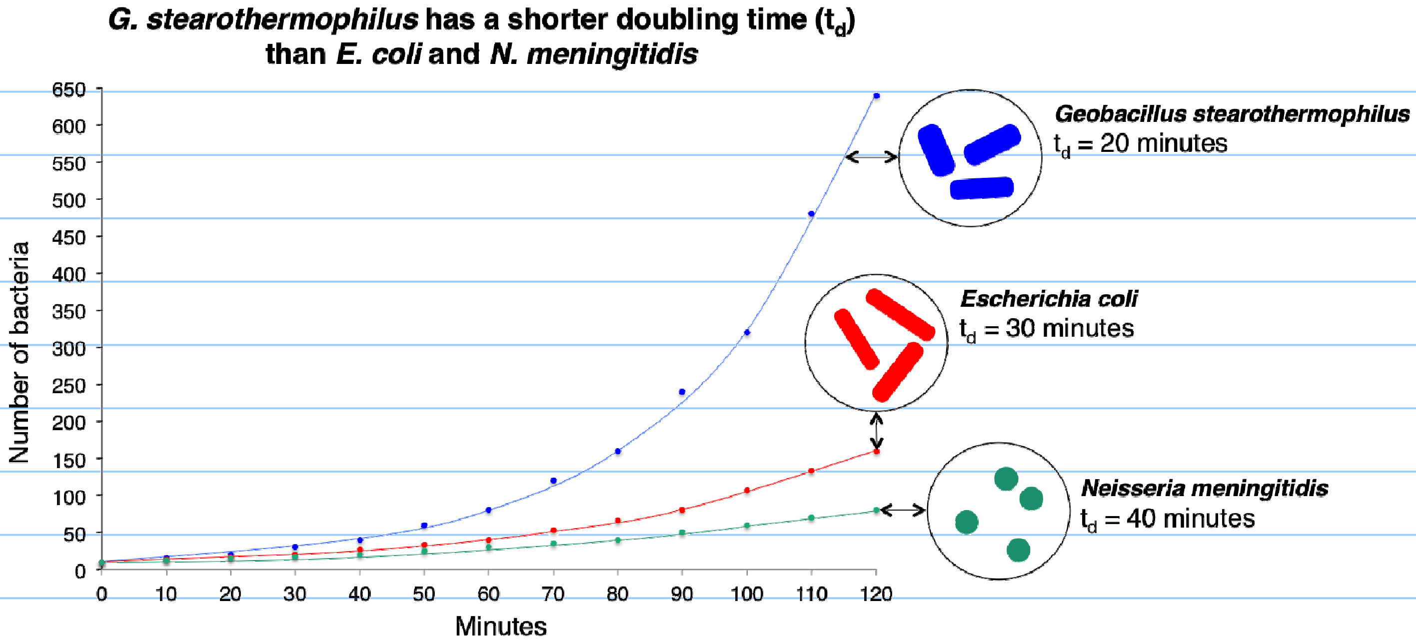
$$\underset{\downarrow}{2P(0)}$$

$$2\cancel{P_0} = P(30) = \cancel{P_0} e^{r \cdot 30}$$

$$2 = e^{30r}$$

$$\frac{\ln 2}{30} = r$$

<u>Note:</u> The value of P_0 didn't matter for this problem.



Remark - Its important to work with "letters" (r, P_0) rather than just numbers.

In real applications, we almost never measure parameters like r directly.

Warning: You will see many word problems in textbooks or online where its not clear if they are telling you r or some data from which you should determine r.

Limited Resources (Logistic Growth Model)

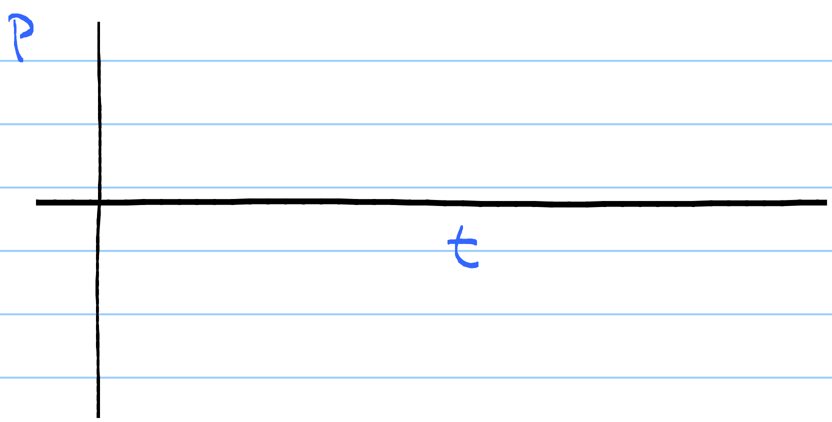
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) \qquad \text{(DE)}$$

r = growth rate

k = carrying capacity

$$k = \begin{cases} \text{maximum population that} \\ \text{available resources (e.g. food supply)} \\ \text{can sustain} \end{cases}$$

Question: Sketch direction field for the DE, label equilibrium solutions and classify as stable or unstable.



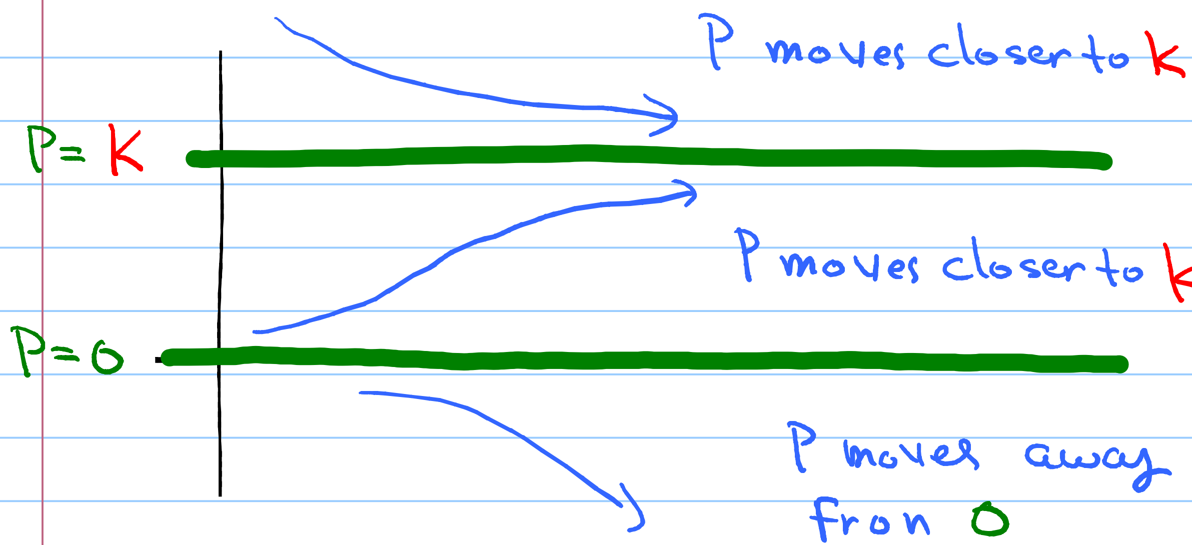
What are the two equilibrium solutions

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$P < 0$ $\quad rP\left(1-\frac{P}{K}\right) < 0$

$0 < P < K$ $\quad rP\left(1-\frac{P}{K}\right) > 0$

$K < P$ $\quad rP\left(1-\frac{P}{K}\right) < 0$



<u>Recall</u> – Theorem says IVP has a unique (exactly one) solution, so curves cannot cross.

$P = K$ is a stable equilibrium

$P = 0$ is an unstable equilibrium or a threshhold