L06 First Order Linear DE's

Note Title 7/28/2020

Examples

$$y' = -ty + \cos t$$

y and derivatives of y OK

$$y' = 3y + e^{-2t}$$

$$y' = y\cos t + \sin t$$

General Example

$$y' = p(t)\,y + g(t)$$

Non-Examples These DE's are NOT linear

$$y' = ty^2 + \sin t$$

other functions of y or derivative NOT OK

$$(y')^2 = ty + \cos t$$

$$y' = \sin y$$

Method of Integrating factors

$$\dot{y} + 4y = 1$$

$$y(0) = 0$$

Multiply both sides by e^{4t} ← Not telling you why yet.

$$e^{4t}\dot{y} + 4e^{4t}y = e^{4t}$$

Recognize product rule

$$(e^{4t}y)^{\cdot} = e^{4t}$$

Integrate

$$e^{4t}y = \frac{e^{4t}}{4} + c$$

$$y = \frac{1}{4} + ce^{-4t}$$ ← This is the "general solution"

$$0 = y(0) = \frac{1}{4} + c$$

$$c = -1/4$$

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}$$ ← This is the solution to the Initial Value Problem

How did we find the integrating Factor?

Call it $m(t)$

$$\dot{y} + 4y = 1$$

Multiply by a function $m(t)$

$$m\dot{y} + 4my = m(t)\cdot\text{Something}$$

We want to choose m so that

$$\frac{d}{dt}(my) = m\frac{dy}{dt} + 4my$$

$$m\cancel{\frac{dy}{dt}} + \frac{dm}{dt}y = m\cancel{\frac{d}{dt}} + 4ym$$

so we need

$$\frac{dm}{dt} = 4m$$

which is separable

$$\frac{dm}{m} = 4\,dt$$

$$\ln|m| = 4t + C$$

$$m = e^{4t}\cdot k$$

k doesn't matter so we set $k=1$

Question

Find the general solution to:

$$\dot{y} = \frac{-2}{t} y + e^t \qquad (DE)$$

Solution

write as

$$\frac{dy}{dt} + \frac{2}{t} y = e^t$$

Find Integrating factor

We want

$$\cancel{m\frac{dy}{dt}} + m \cdot \frac{2}{t} y = \frac{d}{dt}(m\, y)$$

$$= \cancel{m\frac{dy}{dt}} + \frac{dm}{dt} y$$

$$\frac{d}{dt} m = \frac{2}{t} m$$

$$\frac{dm}{m} = \frac{2}{t} dt$$

$$\ln|m| = 2 \ln|t| = \ln(t^2) + c$$

$\nwarrow$ Choose $C = 0$ for convenience

$$m = t^2$$

Now, multiply DE by t^2

$$t^2 \dot{y} + t^2 \frac{2y}{t} = t^2 e^t$$

$$t^2 \frac{dy}{dt} + 2t y = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

Integrate $\quad t^2 y = \int t^2 e^t$ $\quad$ Integration by part's

$$t^2 y = t^2 e^t - 2te^t + 2e^t + C$$

multiply by $\frac{1}{t^2}$

$$\boxed{y = e^t - \frac{2}{t} e^t + \frac{2}{t^2} e^t + \frac{C}{t^2}}$$

General Solution

Aside $\quad$ I remember

$$\int t^2 e^t = t^2 e^t + a\,te^t \quad b\,e^t$$

$$\downarrow \frac{d}{dt}$$

$$t^2 e^t \stackrel{?}{=} (t^2 e^t + 2t e^t) + (a\,te^t + a e^t) + b\,e^t$$

$$t^2 e^t \stackrel{?}{=} t^2 e^t + \underbrace{(2+a)}_{=0} t e^t + \underbrace{(a+b)}_{=0} e^t$$

-2

$a = -2 \qquad b = -a = 2$

Faster than integration by parts ?
