Euler's Method (Pronounced Oiler's)

Method for writing a computer program for solving an IVP.

$\dot{y} = f(t,y)$ Example $\dot{y} = t(4-y)$

$y(0) = 0$ $y(0)=0$

Problem Approximate y on the interval $[0,2]$ using a step size of $h=\frac{1}{2}$.

Motivation.

Linear Approximation - For any function $g(t)$, if h is small, $g(t+h) \approx g(t) + \dot{g}(t)h$

For our function, $y(t)$, the DE says that

$$\dot{y}(t) = f(t,y) = t(4-y(t))$$

so $y(t+h) \approx y(t) + t(4-y(t))h$

we know that $y(0)=0$ so $\dot{y}(0) = 0(4-0) = 0$

$$y(\tfrac{1}{2}) \approx 0 + 0\cdot(4-0)\tfrac{1}{2} = 0$$

Because $y(\frac{1}{2})=0$, $\dot{y}(\frac{1}{2}) = \frac{1}{2}(4-0) = 2$

so $y(1) = y(\frac{1}{2}+\frac{1}{2}) \approx 0 + \frac{1}{2}(4-0)\frac{1}{2} = 0+1 = 1$

Etc.

Problem Approximate y on the interval $[0,2]$ using a step size of $h=\frac{1}{2}$.

$$\dot{y} = t(4-y)$$
$$y(0)=0$$

Euler's Method Formula

$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ Y_{New} &= Y_{old} + f(t_{old}, Y_{old})\, h \end{aligned}}$$

Short Answer *

$$y\left(\tfrac{1}{2}\right) \approx 0 + \tfrac{1}{2}\left[0(4-0)\right] = 0$$

$$y(1) \approx 0 + \tfrac{1}{2}\left[\tfrac{1}{2}(4-0)\right] = 1$$

$$y\left(\tfrac{3}{2}\right) \approx 1 + \tfrac{1}{2}\left[1(4-1)\right] = \tfrac{5}{2}$$

$$y(2) \approx \tfrac{5}{2} + \tfrac{1}{2}\left[\tfrac{3}{2}\left(4-\tfrac{5}{2}\right)\right] = \tfrac{29}{8}$$

Done

Looking at a Direction Field can help you visualize what's going on. At each step we follow the arrow for a distance h, then follow the next arrow, etc.

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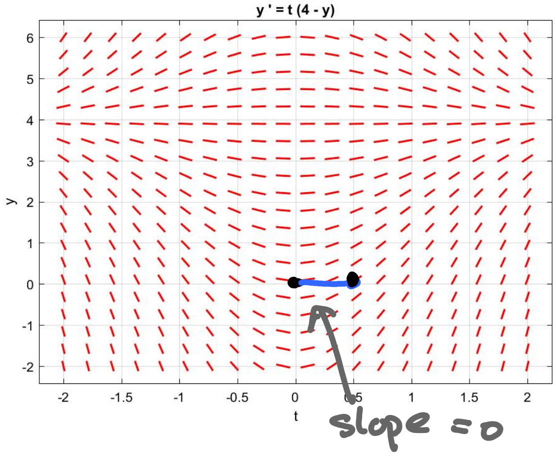
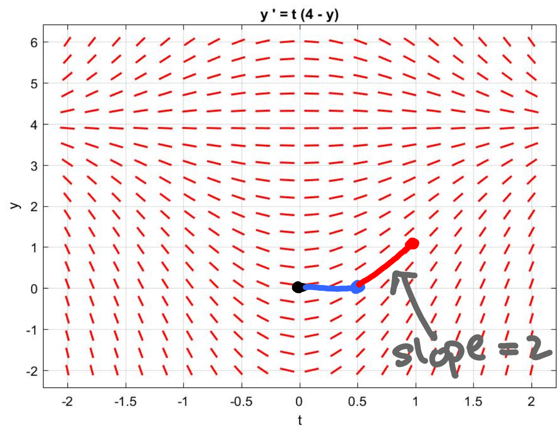
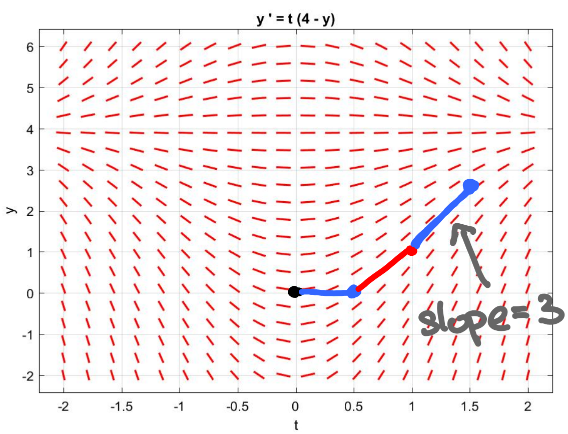
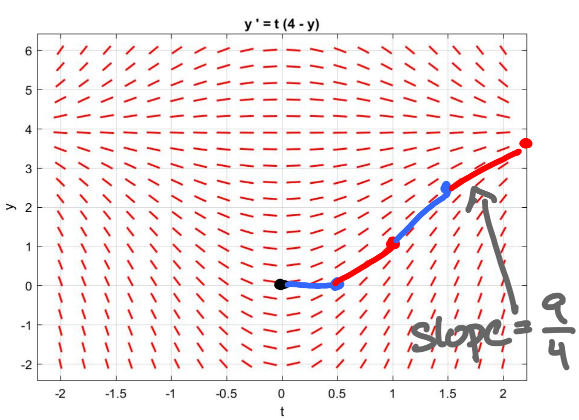
* I expect you to write at least this much on an exam.

How it looks in Dfield









Details

$y(0) = 0$

$t_{New} = 0 + \frac{1}{2}$

$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ y_{New} &= y_{old} + f(t_{old}, y_{old})\, h \end{aligned}}$$

$y_{new} = 0 + (0(4-0)) = 0$

$y(\frac{1}{2}) \approx 0 + \frac{1}{2}[0(4-0)] = 0$

$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ y_{New} &= y_{old} + f(t_{old}, y_{old})\, h \end{aligned}}$$

$t_{new} = \frac{1}{2} + \frac{1}{2} = 1$

$y_{new} = 0 + \frac{1}{2}(\frac{1}{2}(4-0)) = 1$

$y(1) \approx 0 + \frac{1}{2}[\frac{1}{2}(4-0)] = 1$

$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ y_{New} &= y_{old} + f(t_{old}, y_{old})\, h \end{aligned}}$$

$t_{new} = 1 + \frac{1}{2} = \frac{3}{2}$

$y_{new} = 1 + \frac{1}{2}(\frac{3}{2}(4-1)) = \frac{5}{2}$

$y(\frac{3}{2}) \approx 1 + \frac{1}{2}[1(4-1)] = \frac{5}{2}$

$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ y_{New} &= y_{old} + f(t_{old}, y_{old})\, h \end{aligned}}$$

$t_{new} = \frac{3}{2} + \frac{1}{2} = 2$

$y_{new} = \frac{5}{2} + \frac{1}{2}(\frac{3}{2}(4-\frac{5}{2})) = \frac{29}{8}$

$y(2) \approx \frac{5}{2} + \frac{1}{2}[\frac{3}{2}(4-\frac{5}{2})] = \frac{29}{8}$

Some people find it helpful to record steps in a table.*

$$\dot{y} = t(4-y) \qquad y(0)=0$$

t	0	$1/2$	1	$3/2$	2
y	0	0	1	$5/2$	$29/8$
$f(t,y)$	0	2	3	$9/4$	

$$t_{new} = t_{old} + h$$

$$y_{new} = y_{old} + f(t_{old}, y_{old})\, h$$

$$y(0) = 0$$

$$y\left(\tfrac{1}{2}\right) = 0 + \tfrac{1}{2}\left[0(4-0)\right] = 0$$

$$y(1) = 0 + \tfrac{1}{2}\left[\tfrac{1}{2}(4-0)\right] = 1$$

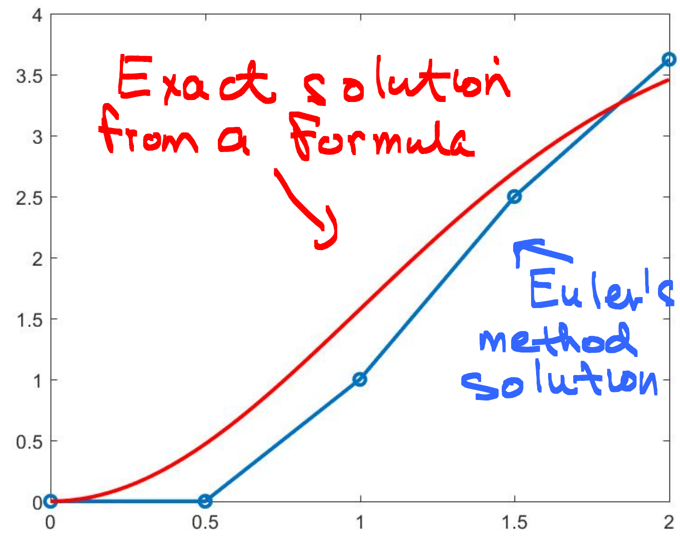
$$y\left(\tfrac{3}{2}\right) = 1 + \tfrac{1}{2}\left[1(4-1)\right] = \tfrac{5}{2}$$

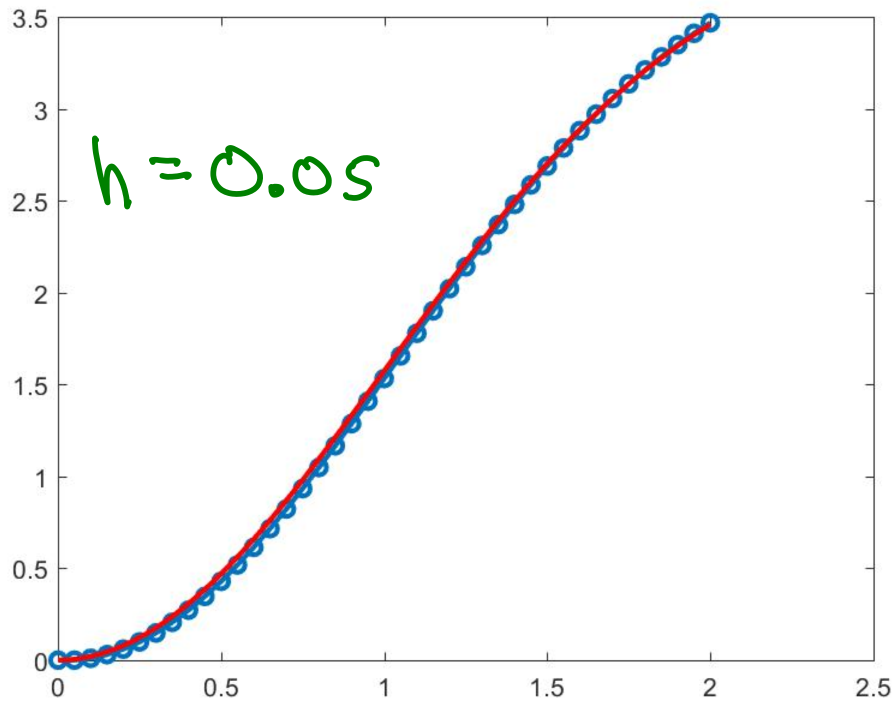
$$y(2) = \tfrac{5}{2} + \tfrac{1}{2}\left[\tfrac{3}{2}\left(4-\tfrac{5}{2}\right)\right] = \tfrac{29}{8}$$

* You don't have to make a table. You only need to do the calculation below the table.

t	0	$1/2$	1	$3/2$	2
y	0	0	1	$5/2$	$\frac{29}{8}$
$f(t,y)$	0	2	3	$9/4$	

$h = 0.5$





We usually choose h much smaller than in the example above and write a program to calculate y and $f(t,y)$

```
%% Euler's method *

% define the DE
f = @(t,y) t*(4-y);

%%
t(1) = 0; %initial time            <- the first time value
y(1) = 0; % initial condition      <- the first y-value

% define h
h = 0.05
% how many steps
numsteps = round(2/h);   <- as many steps as necessary to reach t=2

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% This is Euler's method
for m = 1:numsteps
    t(m+1) = t(m)+h;
    y(m+1) = y(m) + h*f(t(m),y(m));
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plot the results
figure(2)
plot(t,y,'marker','o')

t = linspace(0,2);
ytrue = 4.*(1-exp(-t.^2/2));       <- exact formula. We'll calculate this in the next lecture.
line(t,ytrue,"color",'r')
```

$t_{new} = t_{old} + h$

$y_{new} = y_{old} + h f(t_{old}, y(t_{old}))$

* I won't ask you to write a program