L03 IVP and Dfield

Note Title 7/22/2020

Initial Value Problem

Example

(DE) $\dot{y} = f(t, y)$ $\qquad$ $\dot{y} = t(4-y)$

(IC) $y(t_0) = y_0$ $\qquad$ $y(0) = 0$

DE = Differential Equation

IC = Initial Condition

Theorem - There is exactly one solution to the IVP.

The solution is a function $y(t)$

To be discussed later

① There are conditions:

$f(t, y)$ = differentiable function

② Solution may not last forever.

" There is a unique solution defined in some interval about t_0. "

3 Topics

This lecture

Direction Fields - Graphical representation of a Differential Equation that helps us to understand the behavior of solutions.

Upcoming

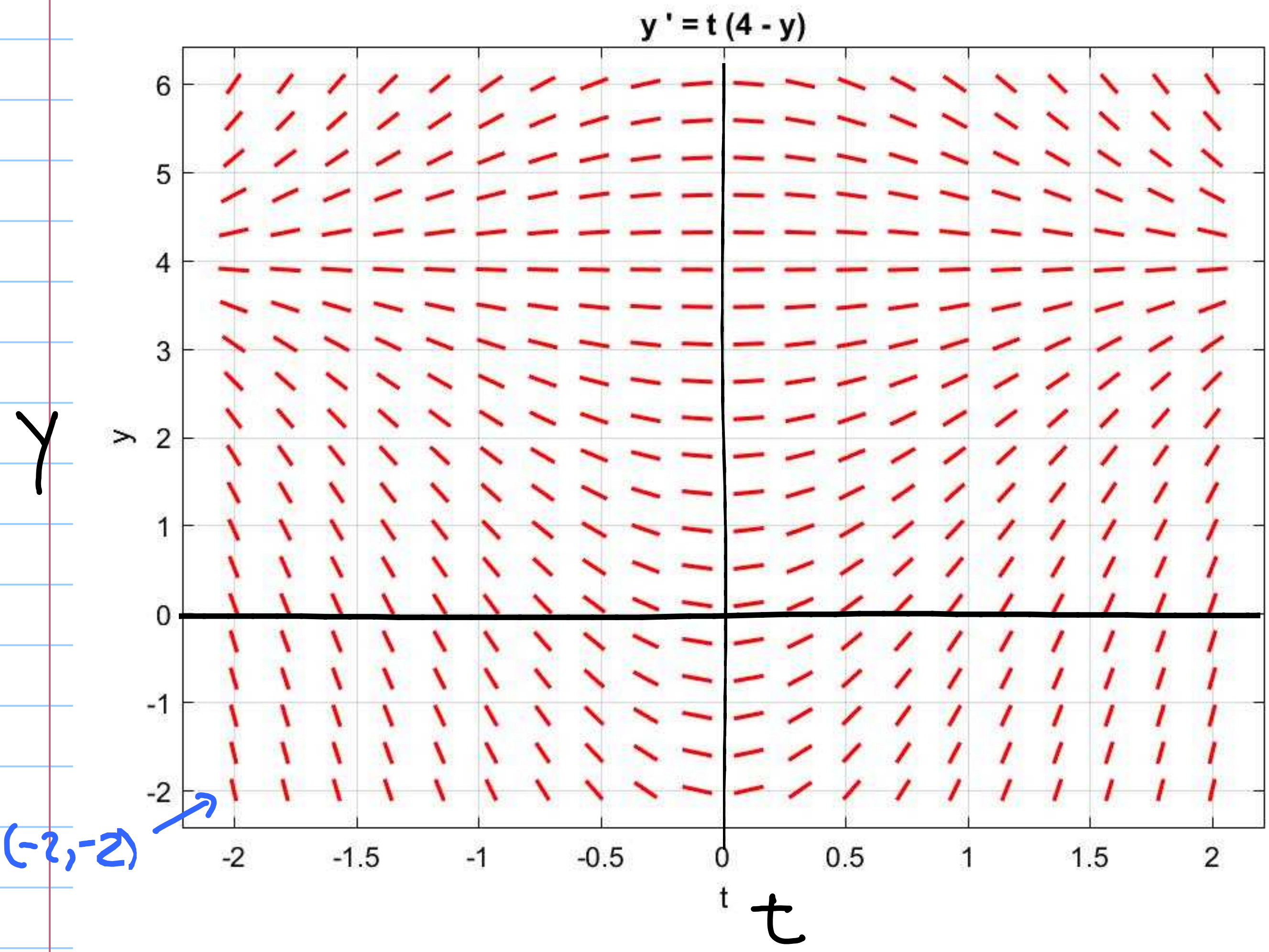
Euler's Method

Method for writing a computer program for solving an IVP.

Separable Differential Equations

method for deriving a formula for the solution to one type of IVP.

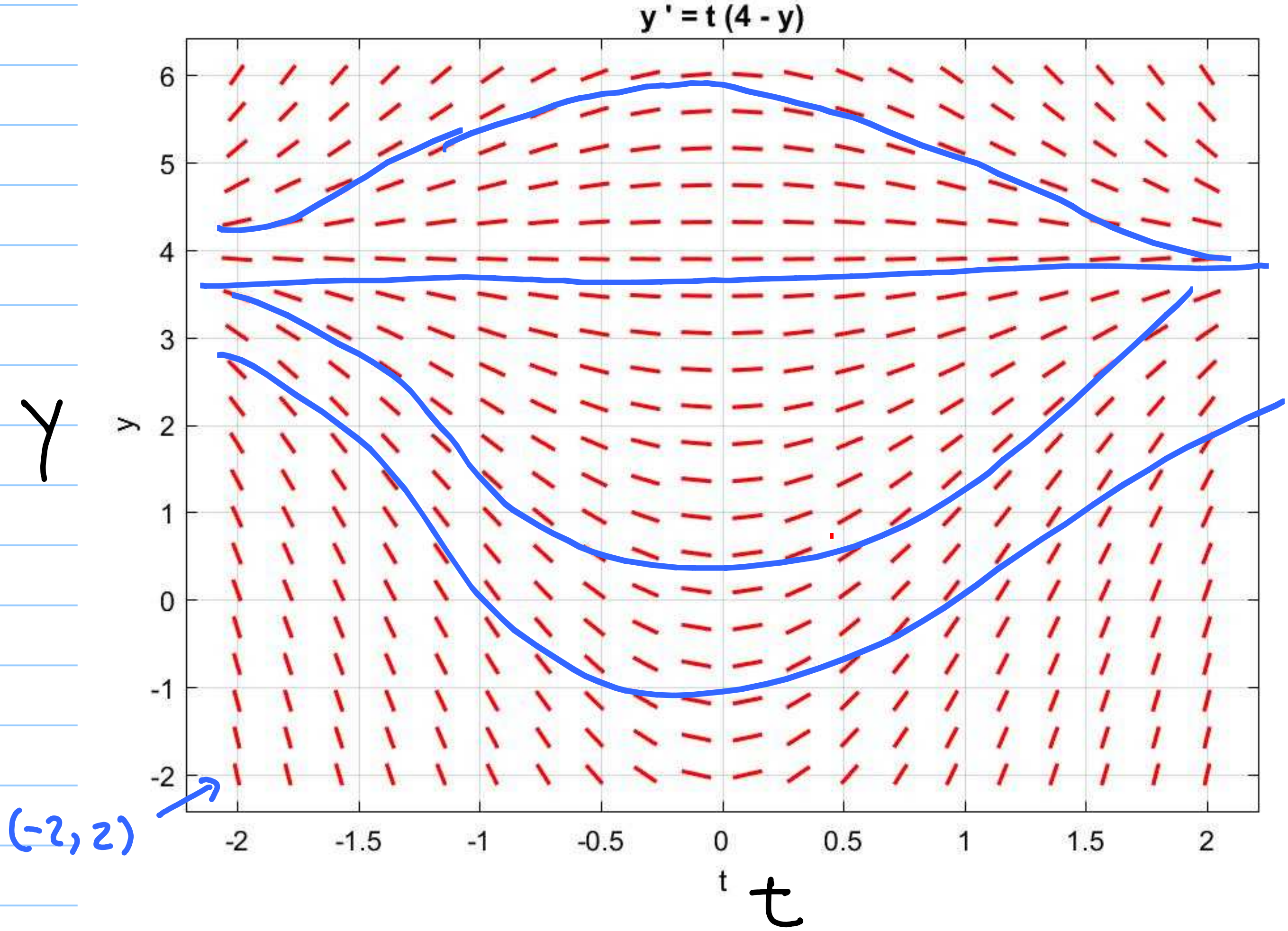
Direction Field $\dot{y} = t(4-y)$



The red line at the point (t,y) has slope $t(4-y)$

- All the lines at $(t,4)$ have slope ? **Ans** 0
- All the lines at $(0,y)$ have slope ? 0
- The line at $(-2,-2)$ has slope $-2(4+2) = -12$

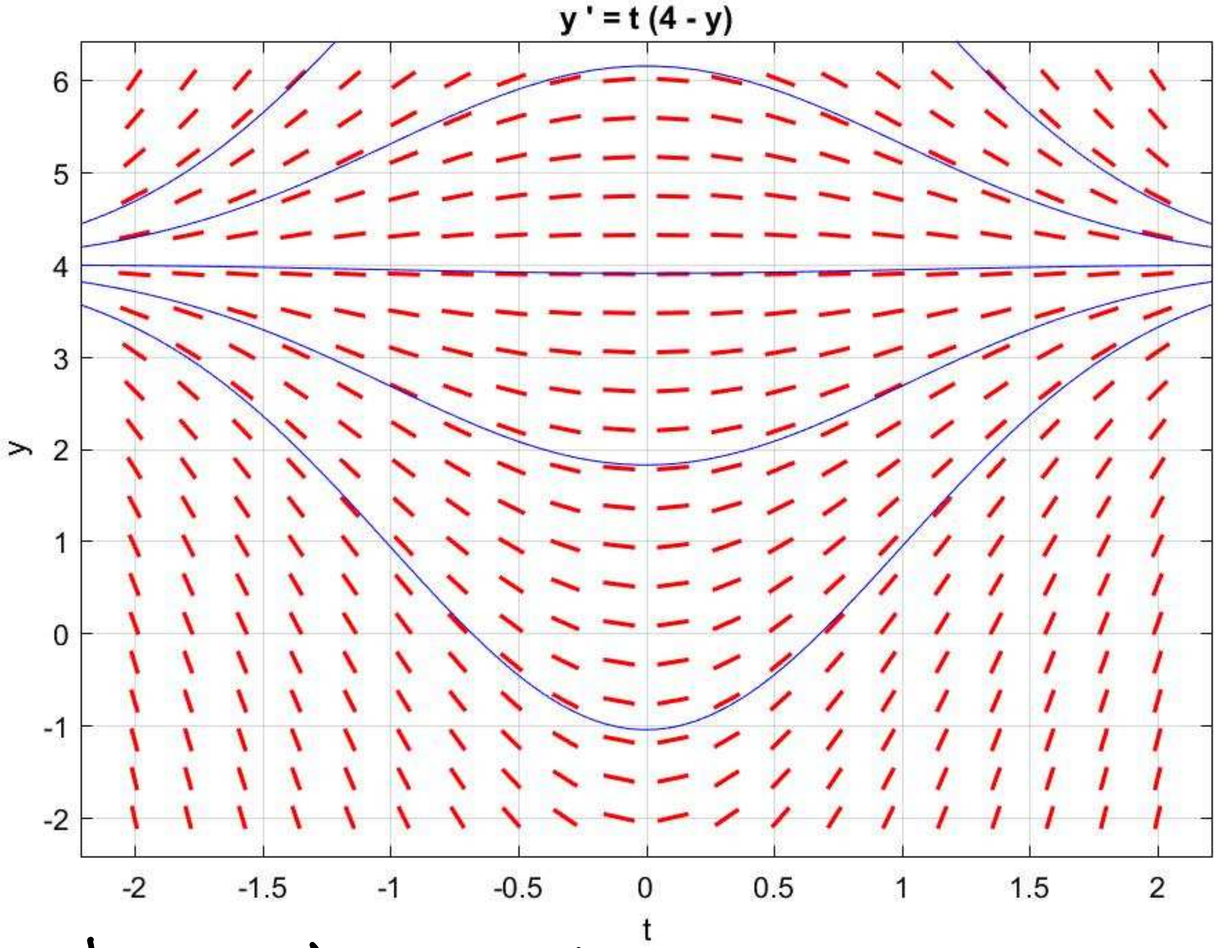
Direction Field $\dot{y} = t(4-y)$



The red line at the point (t, y) has slope $t(4-y)$

Curves that are tangent to the lines are solutions to the DE.

Draw Some Solutions



What important properties of solutions to the DE can we see from this picture?

Solutions that start below 4, ? stay below 4

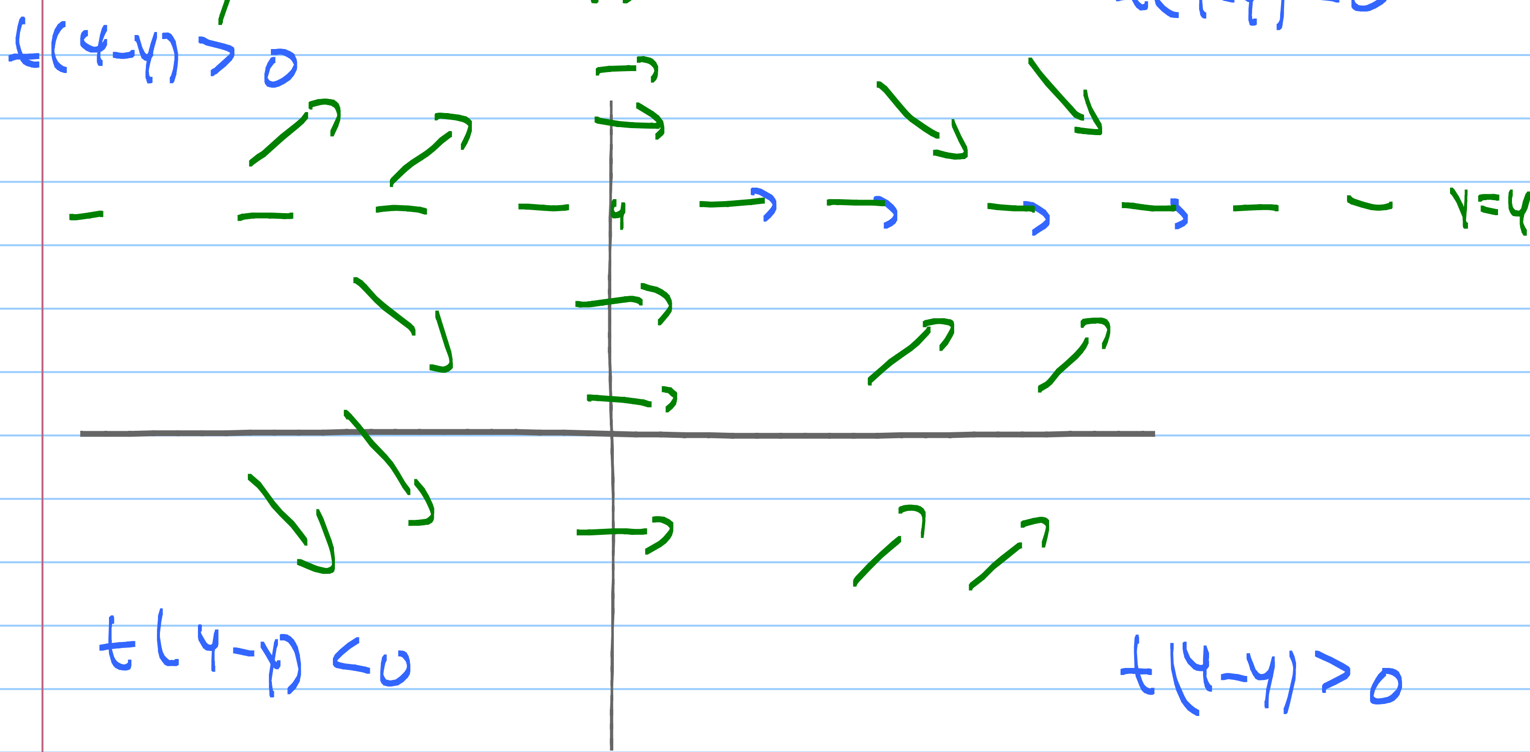
Solutions that start above 4, ? stay above 4

$\lim_{t \to \infty} y(t) = ?\ 4$ $\lim_{t \to -\infty} y(t) = ?\ 4$

The solution $y=4$ is a "stable equilibrium".

Draw Dfield by hand

$\dot{y} = t(4-y)$



Example (a) Draw the Dfield for $\frac{dy}{dt} = (y-2)(y-5)$ by hand

Next, use the picture on the following page to:

(b) Draw some solutions

(c) What can you say about the behavior of solutions as $t \to \infty$

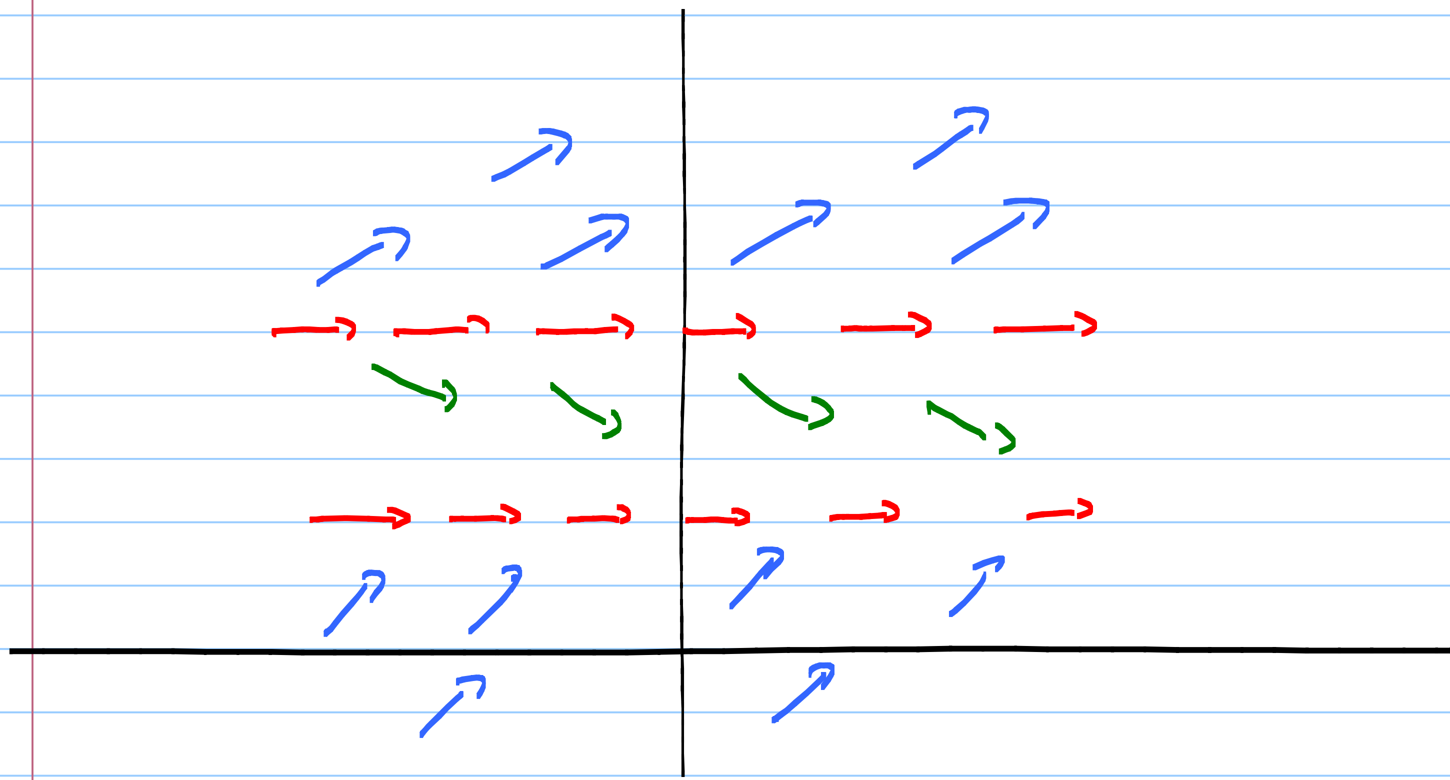
Solution to (a) $\frac{dy}{dt} = (y-2)(y-5)$

Observations:

slope $= 0$ at $y=2$ and $y=5$

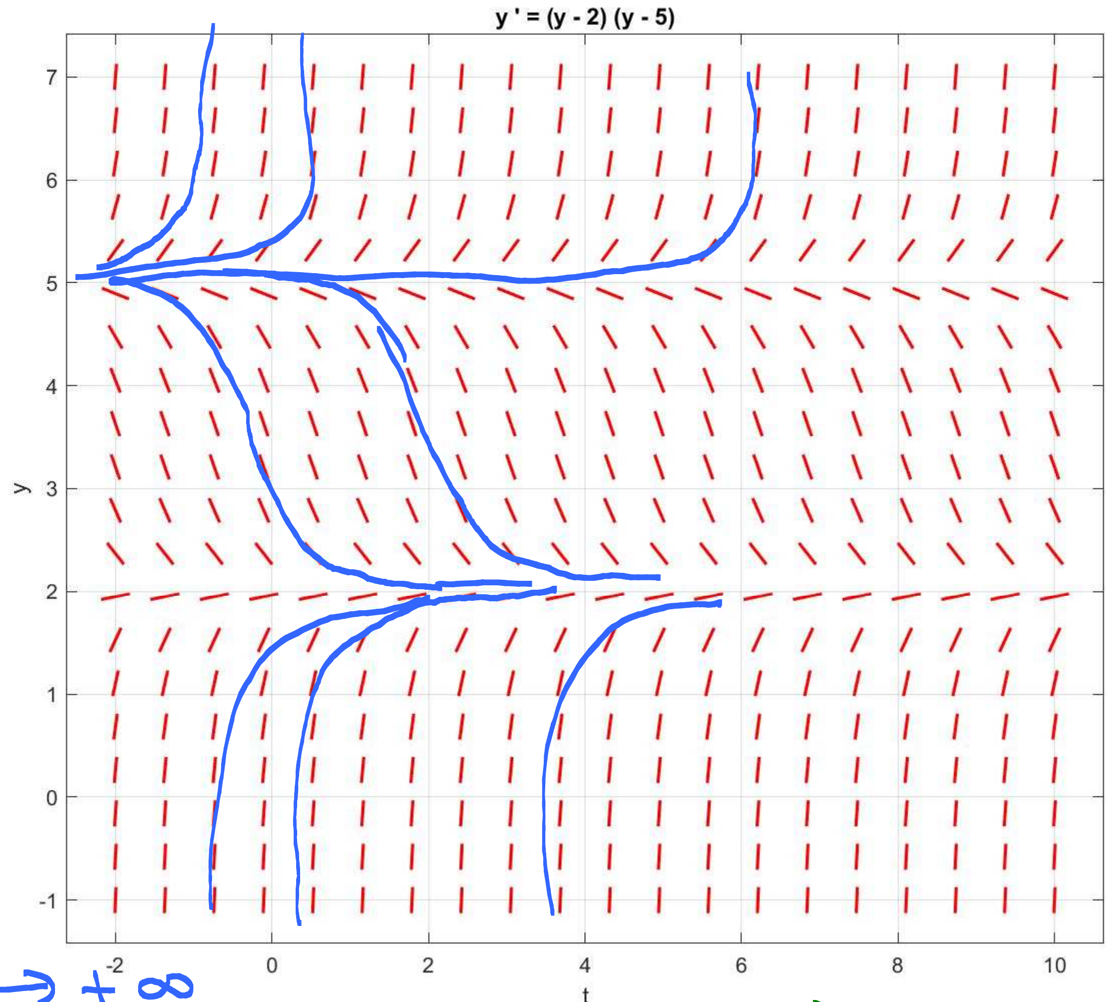
slope > 0 if $y > 5$ or $y < 2$

slope < 0 if $2 < y < 5$



(b) Draw some solutions

(c) What can you say about the behavior of solutions as $t \to \infty$



(c) As $t \to +\infty$

If $y(0) < 5$, then $\lim_{t \to \infty} y(t) = 2$

If $y(0) > 5$ then $\lim_{t \to \infty} y(t) = \infty$ *

* This statement isn't quite correct. It turns out that $y(t)$ approaches ∞, but it increases so fast that it "blows up in finite time", i.e. t never gets to ∞. You can't see this by looking at the DE or the direction field. All I expect you to see is that $y(t)$ increases to ∞.

Here's a brief explanation for completeness. It's a bit subtle, and not the main focus of this course, so you can skip it if you want.

$$\boxed{\frac{dy}{dt} = (y-2)(y-5)}$$

For $y(0) > 5$, $y(t) \longrightarrow \infty$, but it does so in "finite-time".

For example, if $y(0) = 6$, then a formula for $y(t)$ is

$$y(t) = \frac{2 - 20e^{-3t}}{1 - 4e^{-3t}}$$

[I'm not telling how I found the formula. We will see this later.]

Notice that the denominator becomes zero when $t = \frac{\ln(4)}{3}$. This is what we mean by "blow up in finite time".

$$\lim_{t \to \frac{\ln(4)}{3}} y(t) = \infty$$

This solution doesn't last forever; it grows so fast that it "blows up at $t = \frac{\ln(4)}{3}$". The blow up time depends on the initial condition, $y(0)$.

__Conclusion:__ When $y(t)$ is approaching ∞, it may do so in "finite time" (example $y(t) = \frac{1}{2-e^t}$ "blows up at $t = \ln(2)$") or infinite time (example $y(t) = e^t$).

Either is possible, and I don't expect you to be able to tell the difference by looking at the DE.