# L2 Modelling Free Fall

## Topics

How to model free fall as an Initial Value Problem.

IVP = DE + Initial Value

How to check that a function satisfies a DE and solve for parameters

How to use the chain rule to write the DE in different units

**Problem** Suppose an small heavy object is thrown upwards with an initial velocity of 44.7 meters/sec.

(a) write a differential equation for the velocity as a function of time [before it hits the ground]
[neglecting air resistance]
[up is the positive direction]

(b) Write the Initial Value Problem.

**Problem** Suppose an small heavy object is thrown upwards with an initial velocity of 44.7 meters/sec.

(a) write a differential equation for the velocity as a function of time [before it hits the ground]
[neglecting air resistance]
[up is the positive direction]

(b) write the Initial Value Problem.

**Solution** Newton - "Force = mass · acceleration"

acceleration = derivative of velocity

$$m\frac{dv}{dt} = \text{gravitational force}$$

$$\cancel{m}\dot{v} = -\cancel{m}g \;; \quad g = 9.8 \text{ m/s}$$

Minus sign ($-mg$) because we follow the convention that up is positive and gravity pulls down.

(a) DE is $\frac{dv}{dt} = -9.8$

(b) IVP is DE plus Initial Condition

$$\frac{dv}{dt} = -9.8 \qquad v(0) = 44.7 \text{ m/s}$$

Problem part (c) - solve the IVP

Solution

$$\int \frac{dv}{dt}\,dt = \int (-9.8)\,dt$$

$$v(t) = -9.8t + C$$

$$44.7 = v(0) = \cancel{-9.8 \cdot 0} + C$$

$$v(t) = 44.7 - 9.8t$$

**Problem** A 100 kg sky diver drops from a great height. air resistance is proportional to his velocity. The constant of proportionality is 20 kg/sec.

(a) Formulate the Initial Value Problem

(b) Find constants A, B, C so that

$$v(t) = A + Be^{-ct}$$ solves the IVP

**Solution:**

(a) mass · acceleration = gravitational force + force of air resistance

$$100\,\dot{v} = -100g \pm 20\cdot v$$

? Warning - easy to get this wrong

$$\dot{v} = -9.8 - \frac{20}{100}v$$

"drops" means $v(0) = 0$

"$-v$" means, the "opposite of $v$"

If $v$ is positive (object rising), air resistance pushes down

If $v$ is negative (object falling), air resistance pushes up

**IVP**
$$\frac{dv}{dt} = -9.8 - \frac{v}{5}$$
$$v(0) = 0$$

(b) $v(t) = A + Be^{-ct}$ must solve

$$\frac{dv}{dt} = -9.8 - \frac{v}{5} \quad \text{and} \quad v(0) = 0$$

$$\| $$

$$-Bce^{-ct} = -9.8 - \frac{(A + Be^{-ct})}{5}$$

$$0 + (-Bce^{-ct}) = \left(-9.8 - \frac{A}{5}\right) - \left(\frac{B}{5}e^{-ct}\right)$$

so $0 = -9.8 - \frac{A}{5}$ and $-Bce^{-ct} = -\frac{B}{5}e^{-ct}$

Conclude $A = -49$ and $c = 1/5$

$$v(t) = -49 + Be^{-t/5}$$

Initial Condition $-49 + B = v(0) = 0$

so $B = 49$

$$v(t) = -49 + 49e^{-t/5}$$

Remark - We will discuss methods for finding the formula $v(t) = A + Be^{ct}$ soon. For now, we concentrate on formulating the (DE) and checking that a function solves it.

Comment - Suppose $A, B, C, D$ are constants and

$$A + Be^{-t} = C + De^{-t}$$

for every $t > 0$, then $A = C$ and $B = D$

Why? It's not easy to "prove" that A must be equal B and C must equal D. I won't ask you to do this.

**Problem** with units of meters and seconds

$$\frac{dv}{dt} = -9.8 - \frac{v}{5} \qquad v(0) = 0$$

Suppose I measure distance in miles and time in hours. What is the new IVP.

**Solution**

Let $V$ = velocity in miles/hour

and let $s$ = time in hours

$$\frac{dV}{ds} = ?$$

Facts: 1610 meters/mile
3600 seconds/hour

$$V \frac{\text{miles}}{\text{hour}} = v \frac{\cancel{\text{meters}}}{\cancel{\text{second}}} \cdot \frac{1 \text{ miles}}{1610 \cancel{\text{meter}}} \cdot \frac{3600 \cancel{\text{seconds}}}{1 \text{ hour}}$$

$$s = t \text{ seconds} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}}$$

$$\boxed{V = \frac{3600}{1610} v \qquad s = \frac{t}{3600}}$$

First Step: Relate $V$ and $s$ to $v$ and $t$

Second Step : Relate $\frac{dV}{ds}$ to $\frac{dv}{dt}$

$$\frac{dV}{ds} = \frac{d\left(\frac{3600}{1610}v\right)}{ds}$$

$$= \frac{3600}{1610}\frac{dv}{ds}$$ chain rule

$$= \frac{3600}{1610}\frac{dt}{ds}\cdot\frac{dv}{dt}$$ use DE for $\frac{dv}{dt}$

$$= \frac{3600}{1610}\cdot 3600\left(-9.8 - \frac{v}{5}\right)$$ Replace $v$ with $V$

$$\boxed{\frac{dV}{ds} = \frac{(3600)^2}{1610}\left(-9.8 - \frac{\left(\frac{1610}{3600}V\right)}{5}\right)}$$

$$\frac{dV}{ds} = -\frac{(3600)^2 \cdot 9.8}{1610} - \frac{3600}{5}V$$

$$V(0) = 0$$