Differential Equations

1. Modelling (i.e. Word Problems $F = ma$)
2. Calculus ← Easiest Part
3. Algebra
4. Interpretation

First Order Differential Equation

is a formula for the derivative of a function $\frac{dy}{dt}$ in terms of y and t.

Examples

$$\frac{dy}{dt} = y + t$$

$$\frac{dy}{dt} = \sin t$$

$$\frac{dy}{dt} = y$$

or

$$\frac{dy}{dx} = y + x$$

$$\frac{dy}{dx} = \sin x$$

$$\frac{dy}{dx} = y$$

First order Differential Equation

is a formula for the derivative of a function $\frac{dy}{dt}$ in terms of y and t.

The solution to a first order DE is a function $y(t)$.

Example

(DE) $$\frac{dy}{dt} = y$$

A solution: $y(t) = e^t$

We will concentrate first on how to find the DE from the model (e.g. a word problem).

We will discuss methods for finding solutions later.

A newly constructed fish pond contains 2000 liters of water. Unfortunately the pond has been contaminated with 5 kg of a toxic chemical during the construction process. The pond's filtering system removes water from the pond at a rate of 200 liters/minute, removes 40% of the chemical, and returns the same volume of water to the pond. Write a differential equation for the time (measured in minutes) evolution of:

The total mass (in kilograms) of the chemical in the pond:

$$\frac{dm}{dt} = ?$$

I want $\frac{kg}{min}$ removed

I know 200 $\frac{liters}{min}$ pass thru filter

How many kg in each of those liters?
(this depends on m)

$\frac{m}{2000}$ kg/liter

How many of those kg's are removed each minute?

$\underbrace{0.4}_{\text{unitless}} \cdot \underbrace{\frac{m}{2000}}_{\text{kg/}\cancel{\text{liter}}} \cdot \underbrace{200}_{\cancel{\text{liters}}\text{/minute}}$

$$\boxed{\frac{dm}{dt} = -\frac{0.4 \cdot 200}{2000}\, m}$$

Write the DE for

The concentration (in kg/liter) of the chemical in the pond:

We will use the DE for total mass and the Chain Rule to relate $\frac{dc}{dt}$ to $\frac{dm}{dt}$.

$$\frac{dm}{dt} = -0.04\, m$$

write c in terms of m

$$c = \frac{m}{2000}$$

Relate $\frac{dc}{dt}$ to $\frac{dm}{dt}$

$$\frac{dc}{dt} = \frac{1}{2000}\frac{dm}{dt} = \frac{-1}{2000}\, 0.04\, m$$

Now eliminate m from the equation.

$$\frac{dc}{dt} = -0.04\left(\frac{m}{2000}\right)$$

$$\boxed{\frac{dc}{dt} = -0.04\, c}$$

Remark: c satisfies the same DE as m. This is a coincidence. It is not typical.

Let s = time in hours, write a DE for the concentration as a function of time in hours

Solution: We will use the Chain Rule to relate $\frac{dc}{ds}$ to $\frac{dc}{dt}$.

Relate s and t

$t = 60s$ minutes = hours · 60

Warning: It's really easy to get this wrong!!

1 hour = 60 minutes so $\cancel{s = 60t}$ No!!

when $s=1$, its 1 hour later, so its 60 minutes later, $t = 60$.

Recall $\quad \frac{dc}{dt} = -0.04c$

Chain rule: $\quad \frac{dc}{ds} = \frac{dc}{dt} \cdot \frac{dt}{ds} = (-0.04c) \cdot 60$

$$\frac{dc}{ds} = -2.4c$$