Notes on Differential Equations

Tom Duchamp

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Department of Mathematics
University of Washington

duchamp@uw.edu

Preface

Here's an outline of the course:

Part 1 For the first three and a half weeks, we'll study first order differential equations. We'll begin with more general first order differential equations and end by concentrating on first order linear differential equations.

Part 2 For the next three and a half weeks, we'll study second order linear differential equations, beginning with *homogeneous differential equations* of the form

$$ay'' + by' + cy = 0$$

and some applications, including the *harmonic oscillator* and concluding with *nonhomogeneous differential equations* of the form

$$ay'' + by' + cy = f(t)\,.$$

The special case where $f(t) = \cos(\omega t)$ is particularly important.

Part 3 During the final three weeks, we'll study how to solve differential equations using Laplace transforms.

By the end of the course, you should know how to do the following:

- Model simple systems involving first order differential equations.
- Visualize solutions using direction fields.
- Use Euler's method to find approximate solutions to first order differential equations.
- Solve first order linear differential equations and initial value problems via integrating factors.
- Solve constant coefficient second order linear initial value problems using the method of undetermined coefficients.
- Calculate with complex numbers and the complex exponential function, compute derivatives and integrals of the complex exponential function.
- Express the function $y(t) = (a\cos(\omega t) + b\sin(\omega t))e^{ct}$ in the forms

$$y(t) = Ae^{ct}\cos(\omega t + \varphi) \text{ and } y(t) = \mathrm{Re}\left(Ce^{(c+i\omega)t}\right)$$

- Model simple mechanical and electrical systems with linear second order differential equations.
- Compute steady-state solution, its amplitude gain and its phase shift with sinusoidal forcing function.
- Compute resonant frequency.
- Compute Laplace transforms and inverse Laplace transforms of commonly occurring functions.
- Solve constant coefficient linear initial value problems using the Laplace transform together with tables of Laplace transforms.
- Use Laplace transforms to solve initial value problems when the forcing function is piecewise continuous or involves the Dirac delta function.

- Express the solution of constant coefficient second order differential equations in terms of the convolution integral.

These notes differ from the book by Boyce and DiPrima in a number of ways.

- The notes give more emphasis on applications and less on theory than Boyce and DiPrima.
- Complex numbers are used more extensively than in Boyce and DiPrima. There are two reasons for this increased emphasis:
  (1) Using complex-valued functions often simplifies computations.
  (2) They are routinely used in applications involving periodic behavior such as electrical circuits, control theory, signal processing (including image processing), crystallography, etc.

Please let me know of any mistakes, including typographical errors, and of any places where the text is confusing. Suggestions should be sent to me via email at `duchamp@uw.edu`.

Acknowledgments. Much of the material in these notes is based on material from my colleagues. Several problems on first order differential equation were written by Neal Koblitz. The material on complex numbers is based on notes written by Bob Phelps. The chapter on Laplace transforms is based on notes written by John Palmieri. I also made heavy use of John Sylvester's lecture notes for Math 307.

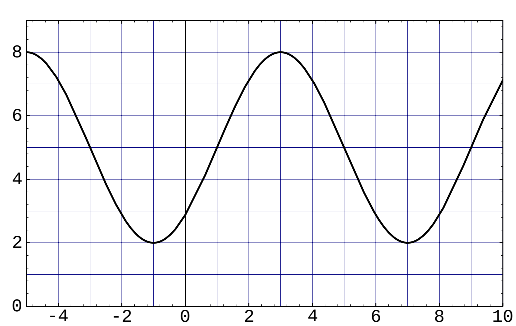
Review Exercises. The prerequisites for Math 307 include high school algebra and trigonometry as well as Calculus at the level of Math 125. Here are a few problems that you should do to check that you are ready to take Math 307. If you have difficulty with some of these, it would be a good idea to review the relevant material. But if you have difficult solving many of them, then you are probably not yet ready to take Math 307.

Algebra Review Problems.

(1) Write down all values of r that are solutions of the quadratic equation $mr^2 + k = 0$, where m and k are both positive real numbers.
(2) Write down all values of r that are solutions to the quadratic equation $mr^2 + cr + k = 0$, where m, c and k are real numbers and $m \neq 0$.
(3) Simplify the expression $\dfrac{\frac{1}{AB}}{\frac{1}{A} + \frac{1}{B}}$, where A and B are non-zero real numbers.
(4) Simplify the expression $\dfrac{x^{1/3}x^{-1/5}}{x^{-a}}$, where x is a positive real number and a is any real number.
(5) Solve the equation $\dfrac{\frac{1}{K} + a}{\frac{1}{K} + b} = e^{-rt}$ for K in terms of the other quantities. Simplify as much as possible.
(6) Solve the equation $\ln(x) - \ln(y) = -2\ln(y) + ax + b$ for y, where x, a and b are positive real numbers. Simplify as much as possible.
(7) Express some basic properties of the natural logarithm and its inverse, the exponential function, by completing the following:
(a) $\ln xy =?$ (b) $e^x e^y =?$ (c) $e^{\ln x} =?$ (d) $\ln e^x =?$ (e) $\ln 1 =?$
(8) Find counterexamples to each of the following "identities"; that is, find specific numbers $x = a, y = b$ such that equality **FAILS** for a and b. (For instance, to show that $(x+y)^2 \neq x^2 + y^2$, it would suffice to take $x = 1, y = 1$, since $(1+1)^2 = 4 \neq 2 = 1^2 + 1^2$.)
(a) $\ln(x + y) = \ln x + \ln y$, where $x > 0, y > 0$.
(b) $\dfrac{1}{x} + y = \dfrac{1}{x} + \dfrac{1}{y}$ $(x, y \neq 0)$.
(c) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ $(x \geq 0, y \geq 0)$.
(d) $e^{x+y} = e^x + e^y$.

Trigonometry Review Problems.

(9) Use the formula for the cosine of the sum of two angles to express $f(t) = \sqrt{3}\sin(7t) - \cos(7t)$ in the form $f(t) = a\cos(bt + c)$, where a, b, and c are real numbers.
(10) The graph below is the graph of a function of the form $y = A\cos(\omega t + \phi) + b$. Find the specific values of A, ω, ϕ and b.



(11) Sketch the curves in the plane given by each of the following two parametric equations:

(a) $\begin{cases} x = 4\cos(3\,t) \\ y = 3\sin(3\,t) \end{cases}, 0 \le t \le \pi/2;$

(b) $\begin{cases} x = e^{-0.5t}\cos(2\pi\,t + \pi) \\ y = e^{-0.5t}\sin(2\pi\,t + \pi) \end{cases}, 0 \le t \le 4.$

Calculus Review Problems.

You'll need to understand the chain rule for differentiation and several basic methods of integration, like substitution, integration by parts, trigonometric substitutions, integration by parts, and rational functions with quadratic denominator. Remember the necessity of adding a constant of integration for indefinite integrals.

(12) Evaluate each of the following:

(a) $\int_0^1 dx/(4 - x^2)$; (b) $\int_1^\infty \frac{dx}{(x^2 + bx)}$, $b > 0$; (c) $\int \frac{dx}{(x-a)(x-b)}$, $a \neq b$; (d) $\int \frac{dx}{1+x^2}$;

(e) $\int \frac{1}{1-x^2}\,dx$; (f) $\int x\cos x\,dx$; (g) $\int_0^\infty \cos(t)e^{-at}\,dt$, $a > 0$; (h) $\int \frac{x-a}{x-b}\,dx$;

(i) $\int_0^\infty te^{-at}\,dt$, $a > 0$; (j) $\int \frac{dx}{\sqrt{x^2 - r^2}}$, $r > 0$; (k) $\int \frac{dx}{\sqrt{r^2 - x^2}}$, $r > 0$; (l) $\int \frac{dx}{x^2 + 6x + 25}$.

Contents

CHAPTER 1

Introduction

Perhaps the most famous differential equation, dating back to 1686, is Newton's second law of motion: "Force equals mass times acceleration." In the special case of an object of mass m moving along a straight line, it can be written in the form

$$my'' = F\left(t, y, y'\right) , \tag{1.1}$$

where $y = y(t)$ is the position of the object at time t and where $F(t, y, y')$ is the force exerted on the object at time t, which may also depend on the position y and velocity y' of the object. Equation (1.1) gives a relation between the function $y(t)$ and its first and second derivatives, but it does <u>not</u> give an explicit formula for $y(t)$. "Solving" this differential equation means finding the formula for $y(t)$. Because (1.1) expresses the second derivative y'' in terms of t, y, and y', it is called a *second order* differential equation.

A good part of this course (more than half!) will be devoted to the study of the solutions of (1.1) in the special case where the force is of the special form

$$F(t, y, y') = f(t) - ky - \gamma y' ,$$

and Newton's second law assumes the special form

$$my'' + \gamma y' + ky = f(t) . \tag{1.2}$$

As we shall see, this differential equation applies to all sorts of mechanical problems such as free-fall with drag taken into account, the simple pendulum, and struts on cars and airplanes. We will also see that the same differential equation models certain electrical circuits (called RLC-circuits), where it is usually written in the form

$$LV'' + RV' + \frac{1}{C}V = V_S(t) , \tag{1.3}$$

where $V = V(t)$ is a voltage, L, R, and C are parameters associated with electronic components, and $V_S(t)$ is an applied voltage (say from a battery or from radio waves).

Newton's Law of Cooling, which Isaac Newton published in 1701, is another important differential equation:

$$T' = -k\left(T - T_A(t)\right) .$$

Here, $T = T(t)$ denotes the temperature of an object at time t, $T_A(t)$ denotes the "ambient temperature" (the temperature of the environment of the object), and $k > 0$ is a constant that measures how well the object is insulated from its environment. A nicer way to write the law of cooling is

$$T' + kT = kT_A(t) . \tag{1.4}$$

If you took Math 125 here at the UW, you've already studied this differential equation, and you may have noticed that (apart from changes in symbols) the same differential equation appears in other modeling problems, such as those involving radioactive decay, exponential population growth, repayment of loans, and some "mixing problems."

Equations (1.2),(1.3), and (1.4) are all examples of *linear differential equations*, the most common class of differential equations. Linear differential equations are differential equations of the form

$$ay' + by = f(t) \text{ or } ay'' + by + cy = f(t)\,,$$

where a, b, and c are constants. From a purely mathematical point of view, we are going to spend most of the time in this course studying these two differential equations from various points of view.

1.1. Some Examples of Modeling: Falling Bodies, the Harmonic Oscillator, Epidemics, and Electrical Circuits

We mentioned two situations where a physical system can be modeled by a differential equation (Newton's Law of Cooling and Newton's second law of motion). Let's explore Newton's second law in more detail with two examples: an object thrown up in the air and an object attached to a spring.

Example 1.1. Suppose we want to construct a mathematical model for the motion of an object when it is thrown up in the air. The first step it to decide exactly what we want to study. Let's assume that we only want to know how high the object gets when thrown.

We have to make some assumptions. Assume that the object has a mass m of 15 kilograms (so $m = 15$ kg). Let's let y denote the height of the object above ground level, measured in meters. Let's let t denote the time, measured in seconds after the object is thrown. So y depends on t, which we indicate by writing $y = y(t)$. We could compute everything about the motion of the object if we knew the formula for $y(t)$. So our goal is to determine $y(t)$.

If we believe Newton, then we believe that $y(t)$ "satisfies" the differential equation

$$15y''(t) = \text{Force at time } t\,.$$

To say that $y(t)$ "satisfies" the differential equation means that if we knew the formula for $y(t)$ and computed its second derivative, then we would get the formula for the force on the object at time t.

But we don't know $y(t)$, so we have to work backwards: we need a formula for the forces on the object and then we need to somehow determine the formula for $y(t)$ from that. In reality, all sorts of forces might act on the object: the wind might be blowing, rain might be falling on the object, if the object is made of metal and there is a large magnet nearby it would pull on the object. So we have to make simplifying assumptions!

Let's assume that gravity is the only force acting on the object and that it is constant. The gravitational field of the Earth then exerts a force $F_{grav} = -mg$ on the object, where $g \approx 9.8\text{m/sec}^2$ denotes the acceleration due to gravity. (In the British system, $g \approx 32\text{ft/sec}^2$). The negative sign is necessary because the gravitational force points down.

Ignoring all other forces exerted on the object, Newton's second law of motion ("$F = ma$") shows that $y = y(t)$ satisfies the differential equation

$$15\frac{d^2y}{dt^2} = -15 \times 9.8 \text{ or } \frac{d^2y}{dt^2} = -9.8\,.$$

We now have a simple <u>mathematical model</u> for the motion that we can use to determine the height reached by the object. We can do this in two steps: Let $v(t) = y'(t)$ (the velocity in meters per second). Then

$$v'(t) = y''(t) = -9.8 \quad \text{(meters per second)}$$

Integration then yields the formula

$$v(t) = \int -9.8\,dt = -9.8\,t + C_1\,,$$

where C_1 is some constant. Since $\dfrac{dy}{dt} = v(t)$, another integration leads to a formula for $y(t)$:

$$y(t) = \int v(t)\,dt = -4.9\,t^2 + C_1\,t + C_2\,,$$

where C_2 is another constant. At the maximum height,

$$y'(t) = -9.8t + C_1 = 0 \implies t = C_1/9.8$$

So the maximum height is

$$-4.9(C_1/9.8)^2 + C_1(C_1/9.8) + C_2 = 0.153C_1^2 + C_2\,.$$

This is not a particularly instructive answer, since we don't know C_1 and C_3! Clearly, just knowing Newton's second law isn't enough. We need more information. Specifically, we need to know the height above ground of the object when it was thrown, and we need to know the velocity of the object when it was thrown.

We could make up numbers for those, but it's better to use symbols.

So let's set

$$y(0) = y_0 \text{ and } y'(0) = v_0$$

for the initial height and velocity. And it's better to use the symbol g rather than it's value $(9.8\,\mathrm{m/sec}^2$. Since $v(t) = -gt + C_1$, we find that $v(0) = v_0 = C_1$. So $v(t) = -gt + v_0$. Then

$$y(t) = -\frac{1}{g}t^2 + v_0 t + C_2$$

So $y(0) = C_2 = y_0$ and we have $y(t) = \dfrac{1}{2}gt^2 + v_0 t + y_0$. The maximum height is

$$-\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + y_0 = \frac{1}{2g}v_0^2 + y_0\,.$$

We now have an expression for the maximum height in terms of the physically meaningful quantities y_0 and v_0 (initial height and initial velocity) rather than in terms of uninteresting constants. In addition, by replacing 9.8 by g, we have a general formula that applies to an object thrown on Mars (where $g = 3.71\,\mathrm{m/sec}^2$ or on the Moon (where $g = 1.625\,\mathrm{m/sec}^2$.

Remark 1.1. (A note on units) In the *British system,* the unit of mass is the *slug* and the unit of force is the *pound*. A mass of one slug has a weight of about 32 pounds. In general, if a body weighs w pounds (so the downward force of gravity is w pounds) then its mass m is w/g slugs (this follows from the formula $mg = w =$ force of gravity). In the *mks (meter-kilogram-second) system,* the unit of mass is the *kilogram (kg)* and the unit of force is the *Newton (N)*. In the *cgs (centimeter-gram-second) system* , the unit of mass is the *gram* and the unit of force is the *dyne*. Here are some conversion factors: 1 slug = 14.6 kilograms, 1 foot = 0.305 meter, 1 pound = 4.45 Newtons

Exercise 1.1. A boater and a motor boat together weigh 640 lbs. Suppose that the thrust of the motor is equal to a constant force of 20 lb. in the direction of motion, and that the resistance of the water to the motion is equal numerically to twice the speed in feet per second and that the boat is initially at rest. Denote the speed of the boat at time t by $v = v(t)$.

(a) What is the mass of the boater and boat in slugs?
(b) Write down the differential equation satisfied by v.

Hint: *Write Newton's second law in the form* $m\dfrac{dv}{dt}$ = *Force.*

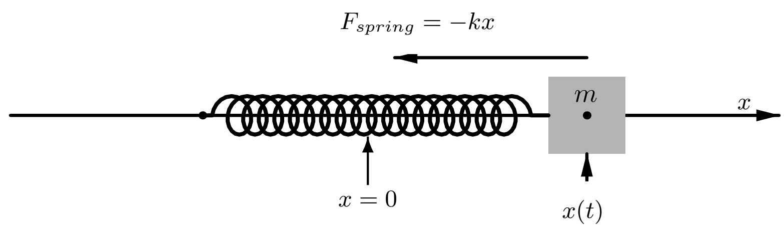


FIGURE 1.1. Hooke's Law states that the force a spring exerts on an object is proportional to the amount that the spring is stretched or compressed. The x axis is positioned so that when $x = 0$ the spring is in its equilibrium position and exerts no force on the object. For $x > 0$ the force points to the left (negative) and for $x < 0$ the force points to the right.

EXAMPLE 1.2. (THE HARMONIC OSCILLATOR) Consider an object of mass m attached to a spring and free to move to the right and left without friction, as illustrated in Figure 1.1. As we did in the previous example, we model the motion of the object using Newton's second law of motion. The only change is the formula for the force on the object. Hooke's Law states that the force F_{spring} that the spring exerts on the object is proportional to the amount that the spring is stretched (or compressed) relative to its equilibrium position[1] If $x = x(t)$ denotes the position of the object relative to its rest position, Hooke's law can be expressed as

$$F_{spring} = -kx\,.$$

The constant $k > 0$ is called the *spring constant* and measures the strength of the spring. The units of k in the mks system are Newtons per meter and pounds per foot in the British system. In this situation, Newton's second law of motion takes the form

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0\,. \tag{1.5}$$

In the mks system, the units of $\dfrac{d^2x}{dt^2}$ are meters/sec^2. It follows that the units of $\dfrac{k}{m}$ are 1/sec^2.

As you would suspect, the object will oscillate back and forth along the x-axis, which is why this mechanical system is called the *harmonic oscillator*. It's easy to check directly that the function

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) \text{ with } \omega_0 = \sqrt{\frac{k}{m}}$$

[1]It it important to keep in mind that Hooke's Law models an ideal spring and only approximately applies to actual springs, where the force is only approximately proportional to the amount of stretching.

is a solution of (1.5) for any constants C_1 and C_2. To show this we have to substitute the above formula for $x(t)$ into the left hand side of the differential equation and verify that it sums to 0. Let's do it:

$$\begin{aligned}\frac{d^2x(t)}{dt^2}+\frac{k}{m}x(t) &= (C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))''+\frac{k}{m}(C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))\\ &= -\omega_0^2 C_1\cos(\omega_0 t)-\omega_0^2 C_2\sin(\omega_0 t)+\frac{k}{m}(C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))\ .\\ &= -\omega_0^2(C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))+\frac{k}{m}(C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))\ .\\ &= \left(\frac{k}{m}-\omega_0^2\right)(C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t))\\ &= 0\,,\end{aligned}$$

where in the last step we used the identity $\omega_0=\sqrt{k/m}$.

Just as in the previous example, to determine C_1 and C_2 more information is needed. Suppose both the position and the velocity of the object at a given time, say $t=0$, are known. This *initial data* is sufficient to uniquely determine the function $x(t)$. In other words, the data

$$mx''+kx=0\,,\quad x(0)=x_0\,,\quad x'(t_0)=v_0\,,$$

comprised of a differential equation together with the position $x(0)=x_0$ and the velocity $x'(0)=v_0$ of the object at time $t=0$, are sufficient to determine the position of the object for all t. This is easy to see, for the initial conditions are

$$x(0)=C_1\cos(0)+C_2\sin(0)=C_1=x_0 \text{ and } x'(0)=-C_1\omega_0\sin(0)+C_2\omega_0\cos(0)=C_2\omega_0=v_0\,,$$

which gives

$$x(t)=x_0\cos(\omega_0 t)+\frac{v_0}{\omega_0}\sin(\omega_0 t)\,.$$

In this form, it is difficult to picture the motion. But we can use the *phase-shift formula* (see Appendix A), to write the solution in the better form

$$x(t)=A\cos(\omega_0 t+\phi)\,,$$

where $A=\sqrt{C_1^2+C_2^2}=\sqrt{x_0^2+\left(\frac{v_0}{\omega_0}\right)^2}$, and $\tan(\phi)=-C_2/C_1=-\frac{v_0/\omega_0}{x_0}$. With a little trigonometry, you can see the reasoning behind this formula. From high school trigonometry,

$$A\cos(\omega_0 t+\phi)=A\cos(\phi)\cos(\omega_0 t)-A\sin(\phi)\sin(\omega_o t)$$

So $C_1=A\cos(\phi)$ and $C_2=-A\sin(\phi)$. Think of (C_1,C_2) as a point in the plane. Then $A=\sqrt{C_1^2+C_2^2}$ is the distance from the origin to the point and if $-\phi$ is the angle between the positive horizontal axis and the line from the origin to the point (see Figure A.1) then $C_1=A\cos(-\phi)$ and $C_2=A\sin(-\phi)$. So we can compute as follows:

$$\begin{aligned}x(t) &= C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)\\ &= (A\cos(\phi))\cos(\omega_0 t)-(A\sin(\phi))\sin(\omega_0 t)\\ &= A(\cos(\phi)\cos(\omega_0 t)-\sin(\phi)\sin(\omega_0 t))\\ &= A\cos(\omega_0 t+\phi)\,.\end{aligned}$$

The units of ω_0 are 1/sec, so the product $\omega_0 t$ is dimensionless. Newton's second law thus predicts that the object will oscillate with period $T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{m}{k}}$ seconds and an amplitude $A=\sqrt{x_0^2+\left(\frac{v_0}{\omega_0}\right)^2}=\sqrt{x_0^2+\left(\frac{mv_0^2}{k}\right)}$.

Example 1.3. (Epidemics) Differential equations are also used to model epidemics. The most simple model is a "two compartment" model where the population is divided into two populations:

$$x = \text{the proportion susceptible to infection ("well")}$$
$$y = \text{the proportion infected ("sick")}$$

We'll make the following additional assumptions:

(a) The disease spreads through contact between sick individuals and well individuals, and the rate of the spread dy/dt is proportional to the number of such contacts per unit of time.
(b) Members of both groups move about freely among each other, so the number of contacts per unit time is proportional to the product of x and y.
(c) 'Sick' individuals don't recover and so do not become susceptible.

It follows from (a) and (b) that

$$\frac{dy}{dt} = \alpha x \cdot y\,,$$

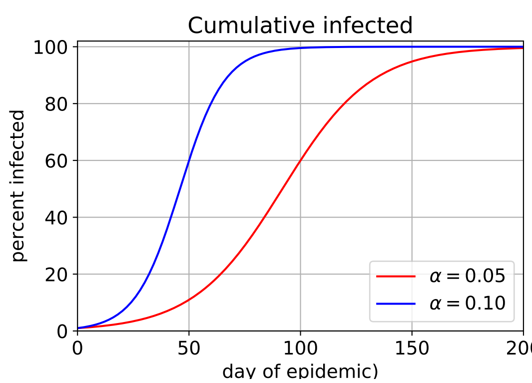
where α is a positive constant related to the frequency of contact and the probability of infection upon contact. Because everyone is assumed to be either well or sick, it follows that $x + y = 1$ (100% of the population). Consequently, $x = 1 - y$ and so

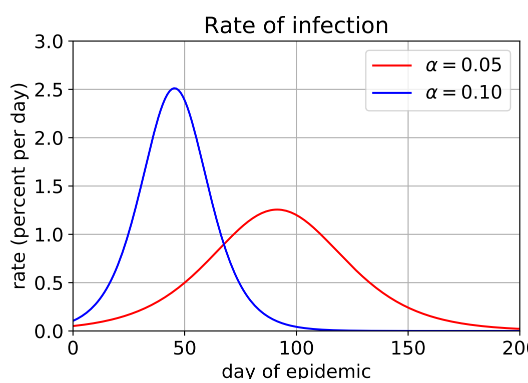
$$\frac{dy}{dt} = \alpha\,(1-y)y\,.$$

If $y(0) = y_0$ is the fraction of the total population that is initially infected, the spread of the epidemic can be predicted by solving the *initial value problem*

$$\frac{dy}{dt} = \alpha\,(1-y)y\,, \quad y(0) = y_0\,.$$

The graphs below show the cumulative percent of the total population that is infected ($= 100\,y(t)$) and the rate of infection in percent of the total population per day ($= 100\,y'(t)$) that the model predicts for two values of α if initially 1% of the population is infected (that is, $y_0 = 0.01$).





The larger value of $\alpha = 0.10$ leads to a maximum rate of inflection of around 2.5 percent per day, while the smaller value of $\alpha = 0.05$ lease to a maximum of around 1.25 percent per day, that is a "flatter curve." For a serious disease, the rate of infection is a measure of the daily demands on the hospital system so the public is encouraged to behave in a way the yields a small value of α. Later in these notes, we will discuss this model in more detail and show how we arrived at the graphs above.

The "two compartment" model is clearly limited, and (at best) only applies at the very beginning of an epidemic and under limited conditions. More realistic "multi-compartment" models that divide the population into *suseptibles*, *exposed*, *infected*, and *recovered* (SEIR models) were used early on in the COVID-19 pandemic. Even more realistic models that further divide the population into age groups and location exist. Such models are beyond the scope of these notes.

Example 1.4. (Electrical Circuits) *Electrical circuits* are also modeled using differential equations. An electrical circuit is a collection of electronic components connected by wires through which an electrical current flows. The main components of electrical circuits are *resistors*, *capacitors*, and *inductors*, together with external *voltage sources*, such as batteries, electric generators, and antennas (which detect electromagnetic radiation, e.g. radio signals). Simple electrical circuits are a nice source of examples of systems that are nicely modeled by differential equation. We will only consider very simple circuits made from resistors, capacitors, and inductors. The website https://www.electronics-tutorials.ws/accircuits/passive-components.html has a nice description of these:

- Resistors regulate, impede or set the flow of current through a particular path or impose a voltage reduction in an electric circuit as a result of this current flow. Resistance is denoted by R and is measured in *Ohms* (denoted by Ω).
- The capacitor is a component that has the ability or "capacity" to store energy in the form of an electric charge like a small battery. Capacitance is denoted by C and is measured in *Farads* (denoted by F) or micro[2] Farads (denoted by μF).
- An inductor is a coil of wire that induces a magnetic field within itself or within a central core as a direct result of current passing through the coil. Inductance[3] is denoted by L and is measured in *Henries.* (denoted by H) or in *micro Henries* (denoted by μH)

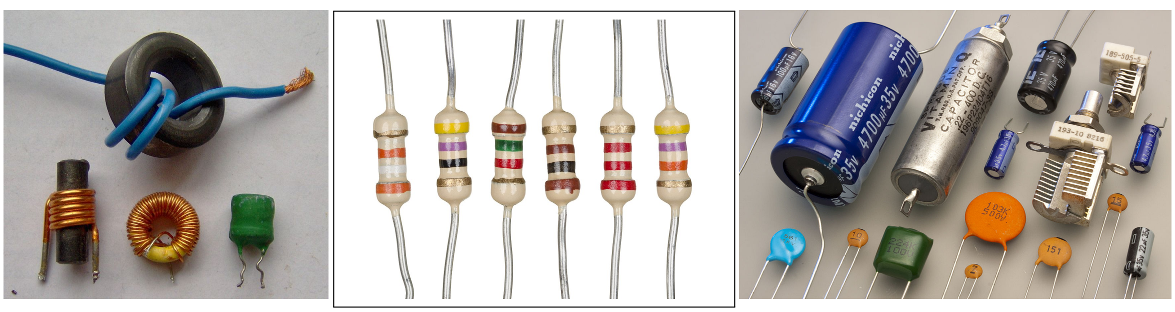


Figure 1.2. An assortment of inductors (left), resistors (center), and capacitors (right). (Photograph of inductors by F. Dominec. Photograph of capacitors by Eric Schrader.)

Figure 1.3 illustrates how a resistor (R), inductor (L), and capacitor (C) might be assembled to form an electrical circuit called an RLC-circuit. By convention, if electrons are flowing counterclockwise in the wires, then the current $I(t)$ (measured in *amperes*) is considered to be flowing in the opposite direction, i.e. clockwise in the figure. Negative charge will accumulate on one "side" of the capacitor and a positive charge $q(t)$ (measured in *coulombs*) will collect on the other side at the rate

$$q'(t) = I(t)\,, \tag{1.6}$$

as illustrated in the schematic diagram shown in Figure 1.3.

The current in the circuit is related to the voltage source $V(t)$, which acts as a pressure causing current to flow along the wires of the circuit. For example, if the voltage source is a 9-volt battery, then

[2]Micro, denoted by μ, means 10^{-6}. For instance, 100μF denotes a capacitance of $100 \times 10^{-6} = 10^{-4}$ Farads.

[3]The symbol 'L' is used in the name of the physicist Heinrich Lenz, who studied inductance.

$V(t) = 9$ Volts. The voltage source from an electrical receptacle in the U.S. is a 60 cycle per second 120 volt source:[4]

$$V(t) = 120\sqrt{2}\cos\left(\frac{2\pi}{60}t\right) \text{ volts.}$$

Moving clockwise around the circuit, the voltage ("pressure") drops across each component by an amount that depends on the component. Denoting the voltage drops across the resistor, capacitor, and inductor by $V_R(t)$, $V_C(t)$ and $V_L(t)$, respectively:

$$V_R(t) = RI(t) \quad V_C(t) = \frac{1}{C}q(t) \quad V_L(t) = L\,\frac{dI(t)}{dt}\,. \tag{1.7}$$

If you are unfamiliar with electrical circuits, please do not be concerned. Eugene Khutoryansky has made several elementary videos explaining the basics of simple electric circuits for people with little or no background in physics. Here are links to four of his videos on YouTube: Battery Energy and Power, Voltage and Current Laws, Ohms Law, Capacitors. These videos are all you need to understand. Please take some time to view them (each one is only a few minutes long).

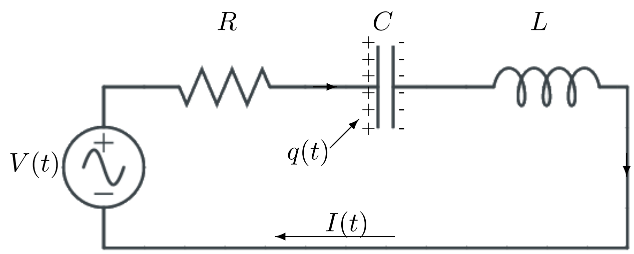


FIGURE 1.3. A schematic diagram of an RLC-circuit with an external (time dependent) voltage source $V(t)$.

In this course, we use electrical circuits only to illustrate applications of differential equations. All of our applications use *Kirchhoff's law*, which states that the voltage drops around a closed circuit sum to zero:

$$\text{Kirchhoff's law:} \quad V_L(t) + V_R(t) + V_C(t) + (-V(t)) = 0\,, \tag{1.8}$$

where, because $V(t)$ is a voltage increase, it becomes $-V(t)$ when viewed as a drop.

Combining Equations (1.6), (1.7), and (1.8) leads to the following second order differential equation for $q(t)$:

$$L\,q'' + R\,q' + \frac{1}{C}\,q = V(t)\,, \tag{1.9}$$

modeling an RLC-circuit. Because $V_C = q/C$, the differential equation (1.9) can be rewritten as a differential equation for voltage across the capacitor:

$$(LC)\,V_C'' + (RC)\,V_C' + V_C = V(t)\,,$$

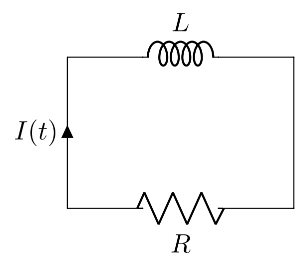
If we remove the resistor and voltage source from the circuit, then the differential equation for V_C reduces to

$$V_C'' + \frac{1}{LC}\,V_C = 0\,,$$

which (apart from the change of symbols) is the same as Equation (1.5) for the harmonic oscillator. This implies that, just as the position of the mass in the mass-spring system oscillates, so does the charge on the capacitor in an LC-circuit.

[4]The maximum voltage is $120\sqrt{2} \approx 170$ volts. The figure 120 is the "root mean square" of the voltage.

EXERCISE 1.2. Suppose we remove both the capacitor and the voltage source from the circuit in Figure 1.3. Then the circuit becomes



Assume that the inductor has an inductance of $L = 10\mu$-Henries ($= 10 \times 10^{-6}$ Henries, and the resistor has a resistor of $R = 100$ Ohms. Writing down the differential equation satisfied by the current I. **Hint:** *For this circuit, Kirchhoff's law says that the sum of the voltage drops across the inductor and the resistor is zero. It might also be worthwhile to look closely at the formulas in* (1.7).

1.2. What's a differential equation and what does it mean to "solve" it?

We now have examples from physics, epidemiology, and engineering where differential equations are used. It's time now to isolate what these examples have in common from a mathematical perspective. This will allow us to develop techniques that apply to many disciplines.

Suppose that t and y are two quantities where y depends on t. It is often useful to think of t as time and y as a quantity that varies in time according to some rule. A *first order ordinary differential equation* or simply a *first order differential equation*[5] is an equation of the form

$$y' = F(t, y)$$

where $F(t, y)$ denotes a function of t and y. A *solution* of the differential equation is a function $y = y(t)$ that satisfies the identity

$$y'(t) = F(t, y(t)) .$$

on some interval.

EXAMPLE 1.5. The function $y(t) = e^{-t^2/2}$ is a solution of the differential equation $y' = -ty$ because

$$y'(t) = e^{-t^2/2}\left(\frac{-2t}{2}\right) = -te^{-t^2/2} = -ty(t) .$$

Another solution solution is $y_1(t) = 5e^{-t^2/2}$.

Similarly, a *second order differential equation* is an equation of the form

$$y'' = F(t, y, y') ,$$

and a *solution* is a function $y = y(t)$ that satisfies the equation

$$y''(t) = F(t, y(t), y'(t))$$

[5] *Partial differential equations* involve functions of more than one variable and partial derivatives. The word "ordinary" refers to differential equations involving functions of only one variable. Since we do not consider partial differential equations in these notes, we will usually drop the word "ordinary."

EXAMPLE 1.6. The function $y(t) = \sin(2t)$ is a solution of the differential equation
$$y'' + 4y = 0$$
because $\sin''(2t) + 4\sin(2t) = -4\sin(2t) + 4\sin(2t) = 0$. The function $\cos(2t)$ is also a solution. In fact, any function of the form
$$y(t) = C_1\cos(2t) + C_2\sin(2t)\,,\ \text{for } C_1 \text{ and } C_2 \text{ constants,}$$
is a solution, as you can see by plugging this formula into the the expression $y''(t) + 4y(t)$, computing the second derivative, and simplifying. If you don't make any algebra errors, you should end up with 0.

EXAMPLE 1.7. It's a fact that the solution of many differential equations can be written in terms of the exponential function (i.e. e^{rt}), sines and cosines ($\cos(\omega t)$ and $\sin(\omega t)$), or a combination of these. This will only become apparent later in the course. But for now, we can use this observation to guess solutions to some differential equations.

For instance, the exponential function $y = e^{rt}$ is a good candidate for the solution of the differential equation
$$y'' + 3y' + 2y = 0\,.$$
Substituting e^{rt} into the differential equation gives
$$(e^{rt})'' + 3(e^{rt})' + 2(e^{rt}) = r^2e^{rt} + 3re^{rt} + 2e^{rt} = (r^2 + 3r + 2)e^{rt} = (r-2)(r-1)e^{rt} = 0\,.$$
This implies that $r = 2$ or $r = 1$. Therefore, $y(t) = e^{2t}$ and $y(t) = e^t$ are both solutions. Armed with these two solutions, one then finds that
$$y(t) = C_1e^t + C_2e^{2t}$$
is also a solution for any two constants C_1 and C_2, as you should check for yourself!

EXERCISE 1.3.
(a) For two values of r, find a solution of the differential equation $y'' + 4y' - 21y = 0$ of the form e^{rt}.
(b) Find a solution of the differential equation $y'' + 4y = 24e^{2t}$ of the form $y = Ae^{rt}$. Hint: *You have to figure out both r and A.*

1.3. What is an Initial Value Problem?

We seen that differential equations such as those above have many solutions. To narrow the possibilities, we've seen that additional information is necessary in the form of "initial data." We codify this in the following definition: A *(first order) initial value problem* or *IVP* is given by a differential equation together with the value of the solution at point of the form:
$$y' = F(t, y) \text{ and } y(t_0) = y_0\,. \tag{1.10}$$
A *solution* of this initial value problem is a function $y(t)$ that satisfied both the differential equation and the initial condition. That is,
$$y'(t) = F(t, y(t)) \text{ for all } t \text{ and } y(t_0) = y_0$$
The *initial condition* is just the equation $y(t_0) = y_0$, giving the value of the solution at a particular value of t.

EXAMPLE 1.8. Consider the initial value problem
$$y' + y = 0\,, \quad y(1) = 4$$
One checks that the function $y(t) = Ce^{-t}$ for C a constant is a solution of the differential equation. The initial condition $y(1) = Ce^1 = 4$ implies that $C = 4e^{-1}$. Consequently,
$$y(t) = 4e^{-1}e^t = 4e^{t-1}$$

is the solution to the initial value problem.

A *(second order) initial value problem* consists of the following data:

$$y'' = F(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0. \tag{1.11}$$

A *solution* of the initial value problem (1.11) is a function $y = y(t)$ satisfying both the differential equation and the *initial conditions*. That is,

$$y''(t) = F(t, y(t), y'(t)), \quad y(t_0) = y_0, \text{ and } y'(t_0) = y'_0.$$

EXAMPLE 1.9. Find the solution of the initial value problem

$$y'' + 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

SOLUTION From Example 1.7, we know that the function

$$y(t) = C_1 e^t + C_2 e^{2t},$$

where C_1 and C_2 are constants, is a solution of the differential equation. For this function, the initial conditions are

$$y(0) = C_1 + C_2 = 2 \text{ and } y'(0) = C_1 + 2C_2 = 3.$$

These two algebraic equations can be solved for C_1 and C_2 using high school algebra to give $C_1 = 1$ and $C_2 = 1$. The solution to the initial value problem is, therefore,

$$y(t) = e^t + e^{2t}.$$

EXAMPLE 1.10. Solve the initial value problem

$$y'' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 6.$$

SOLUTION By Example 1.6, the function $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$ is the solution of the differential equation. The initial conditions then give

$$y(0) = C_1 = 2 \text{ and } y'(0) = 2C_2 = 6,$$

which implies $C_2 = 3$. Therefore, the function $y(t) = 2\cos(2t) + 3\sin(2t)$ is the solution of the initial value problem.

As we've already seen, when written in the form $y(t) = 2\cos(2t) + 3\sin(2t)$, it isn't clear what the graph of the function $y(t)$ looks like. A better way to visualize it is to express it in the form $y(t) = A\cos(2t+\phi)$. To do this use the "phase-shift formula" in Appendix A) to write $y(t)$ as follows:

$$y(t) = 2\cos(2t) + 3\sin(2t) = \sqrt{2^2 + 3^2}\cos\left(2t - \arctan(3/2)\right) \approx 3.6\cos\left(2(t - 0.49)\right).$$

In this form, the graph of $y(t)$ is easily sketched (see Figure 1.4). It's a cosine function of amplitude 3.6, shifted to the left by 0.49 units, and of period $T = \dfrac{2\pi}{2} \approx 3.14$.

EXERCISE 1.4. Find the solution of the initial value problem

$$y' + 3y = 0, \quad y(1) = 2.$$

Hint: *Look for a solution of the form* $y(t) = Ce^{rt}$.

EXERCISE 1.5. Find the solution of the initial value problem

$$y'' + 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

Hint: *Look for a solution of the form* $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

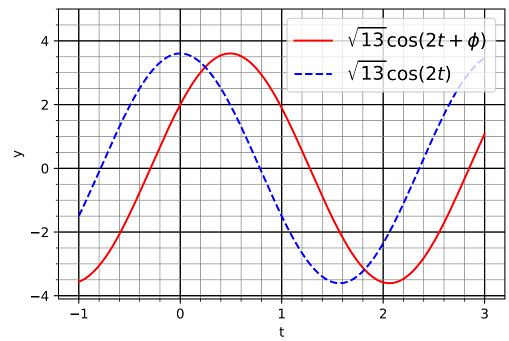


FIGURE 1.4. The graph of $y = \sqrt{13}\cos(2t - \arctan(3/2))$.

Part 1

First Order Differential Equations

CHAPTER 2

The Geometry of First Order Differential Equations

In this chapter, we begin a study of the geometry of first order differential equations. It introduces the direction field of a differential equation, giving a way to visualize solutions of first order differential equations and initial value problems. It ends with Euler's method for finding approximate solutions of the initial value problem

$$y' = F(t, y)\,, \quad y(t_0) = y_0$$

in cases where we can't find an explicit solution.

In the following chapters, we focus on two particular classes of first order differential equations (separable and linear differential equations) where explicit methods for finding solutions are known. We conclude our study of first order differential equations with some applications of the theory.

2.1. The Direction Field of a Differential Equation

In this section, we present a geometric description of the differential equation

$$\frac{dy}{dt} = F(t, y)$$

that is useful for understanding the behavior of its solutions.

FUNDAMENTAL OBSERVATION: Suppose we already know that $y = y(t)$ is a solution to this differential equation. We can evaluate the derivative $y'(a)$ <u>without</u> differentiating:

$$y'(a) = F(a, y(a))\,.$$

Therefore, if $y(a) = b$, then the slope m of the tangent line to the curve $y = y(t)$ at (a, b) is $m = y'(a) = F(a, b)$; and the equation of the tangent line to $y = y(t)$ at $t - a$ is

$$y = m(t - a) + b \text{ where } m = F(a, b).$$

We can encode this by drawing a short line segment of slope $m = F(a, b)$ through the point (a, b). This line segment is called a *direction element* or *line element* of the differential equation.

EXAMPLE 2.1. For instance, suppose that $y = y(t)$ is a solution of the differential equation $y' = \dfrac{2}{t+1} - y$ and we know that $y(1) = 2$. Then $y'(1) = \dfrac{2}{1+1} - 2 = -1$. The direction element for this differential equation is, therefore, a line segment through the point $(1, 2)$ of slope -1. The graph of solution is the dark curve in Figure 2.1.

The *direction field* of the differential equation is the picture obtained by drawing a direction element through each point in the (t, y)-plane. The effect of drawing lots of direction elements is a picture that resembles a collection of iron filings in a magnetic field (the filings line up parallel to the magnetic field).

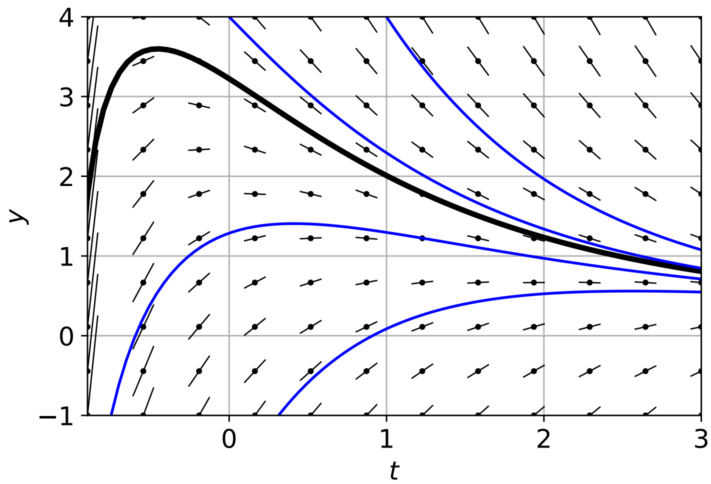


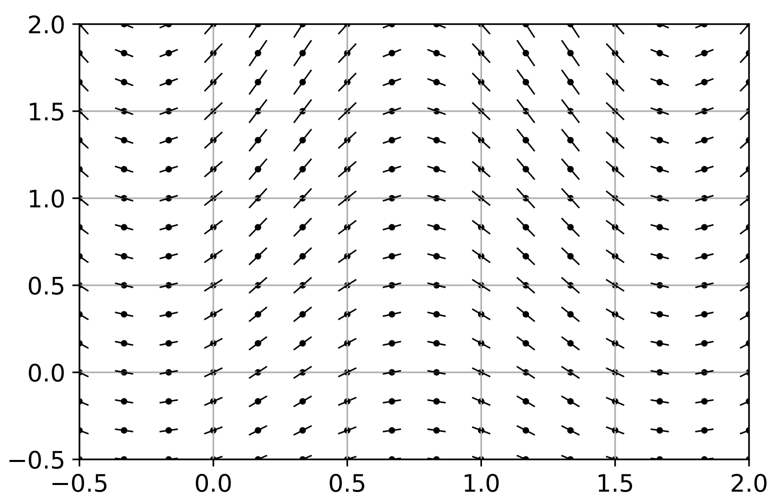
FIGURE 2.1. The direction field of the differential equation $y' = \dfrac{2}{t+1} - y$, together with several integral curves.

By construction, if $y = y(t)$ is a solution of the differential equation, then at every point $(a, y(a)$ the slope of the tangent line to the curve agrees with the slope of the line element $F(a,b)$. This forces the graphs of solutions to conform with the direction field of the differential equation (see Figure 2.1),

The graphs of solutions of a differential equation are called *integral curves* of the differential equation. The fact that "nice" differential equations have unique solutions has a geometric interpretation: *the integral curves of a first order differential equation never cross*; and *there is a unique integral curve through each point* (t_0, y_0) *in the* (t,y)*-plane.*

EXERCISE 2.1. The direction field of the differential equation $v' = F(t,v)$ is shown in the figure below.
(a) On the figure, carefully sketch the solution to the initial value problem

$$v' = F(t,v)\,, \quad v(0) = 0.5\,.$$



(b) Let $y = y_1(t)$ be the solution of the initial value problem in part (a). What is the approximate value of $y_1(1.5)$?
(c) Now sketch the solution of the initial value problem $v' = F(t, v)\,, \quad v(0) = 0$. Let $y = y_2(t)$ be the solution of this initial value problem.
(d) Assume that the solutions $y_1(t)$ and $y_2(t)$ are defined for all values of t between 0 and 5. Is it possible for the graphs of $y_1(t)$ and $y_2(t)$ to cross at $t = 5$? Explain your answer.

2.2. Euler's Method

Although there are a number of techniques for solving special classes of differential equations, there is no general algorithm for solving all differential equations. Consequently, mathematicians have developed a number of numerical methods for finding approximate solutions of many differential equations that appear in applications. Finding better numerical methods remains an active area of research. Of these, *Euler's method* is the simplest and the easiest to describe.

The method is based on the tangent line approximation. Suppose that $y = y(t)$ is a differentiable function of t and we know both the value of $y(t)$ and the slope of the tangent line to $y = y(t)$ at $t = t_0$. The *tangent line approximation* of $y(t)$ at $t = t_0$ is the linear function

$$y = y_0 + y'(t_0)\,(t - t_0)\,,$$

which approximates $t = y(t)$ for values of t near t_0 (see Figure 2.2).

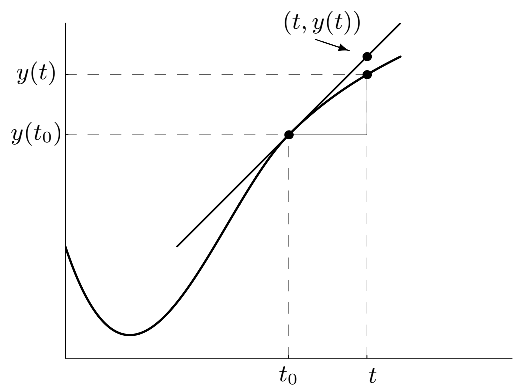


FIGURE 2.2. The tangent line approximation

EXAMPLE 2.2. To understand how to use the tangent line approximation to approximate the solution of a differential equation, consider the following initial value problem:

$$y' = y\,, \quad y(0) = 1\,.$$

Ignore for the moment that the solution is $y(t) = e^t$. Choose a small step size, say $h = 0.1$. We will use the tangent line approximation to find approximate values for $y(h) = y(0.1)$, $y(2h) = y(0.2)$, $y(3h) = y(0.3)$, etc.

Because $y'(t) = y(t)$, we know that $y'(0) = y(0) = 1$. The tangent line approximation then gives the approximation

$$y(0.1) \approx y(0) + y'(0)(0.1 - 0) = 1 + (1)(0.1) = 1.1\,.$$

The tangent line approximation can be used again to approximate $y(0.2)$: From the differential equation, $y'(0.1) = y(0.1) \approx 1.1$. Therefore,

$$y(0.2) \approx y(0.1) + y'(0.1)(0.1) = 1.1 + (1.1)(0.1) = 1.21\,.$$

We can repeat this as many times as we like. The following table summarizes the result for the first ten iterations of this process:

$n =$	0	1	2	3	4	5	6	7	8	9	10
$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y_n =$	1.000	1.100	1.210	1.331	1.464	1.611	1.772	1.949	2.144	2.358	2.594
$y(t) = e^t =$	1.000	1.105	1.221	1.349	1.492	1.649	1.922	2.014	2.226	2.460	2.718

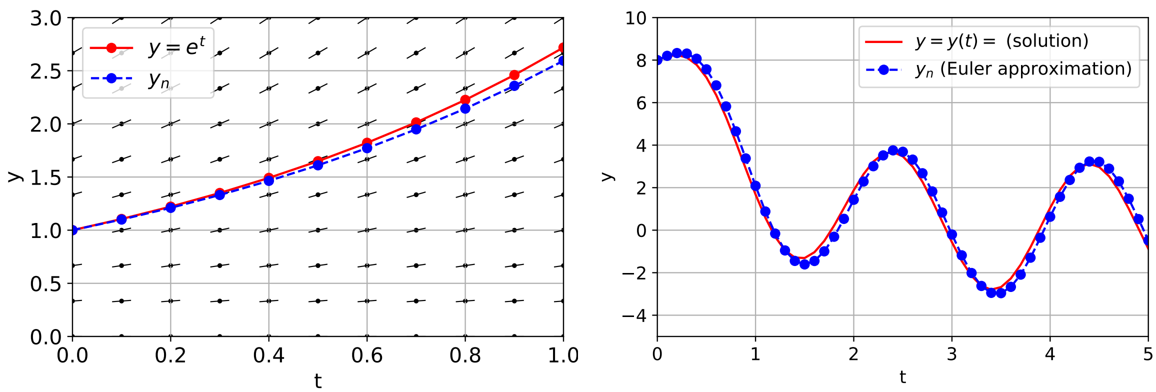


FIGURE 2.3. The figure on the left shows Euler's method applied to the initial value problem $y' = y$, $y(0) = 1.0$, with step size $h = 0.1$. The figure on the right shows the result of applying Euler's method to the initial value problem $y' + y = 10\cos(\pi t)$, $y(0) = 8.0$, with step size $h = 0.1$.

The general method to find an approximate solution of the initial value problem $y' = F(t, y), \quad y(t_0) = y_0$, proceed as follows:

(0) Choose a step size $h > 0$.
(1) Set $n = 0$.
(2) Set $y'_n = F(t_n, y_n)$.
(3) Set $t_{n+1} = t_n + h$.
(4) Set $y_{n+1} = y_n + y'_n \cdot h$
(5) Increase n by one and go to step (2).

EXERCISE 2.2. Suppose that $y = y(t)$ is the solution of the initial value problem

$$y' = t + \cos(y)\,, y(0) = 1\,.$$

Use Euler's method with a step size of $h = 0.2$ to estimate $y(1)$.

CHAPTER 3

Solving First Order Differential Equations

Euler's method only gives approximate solutions to differential equations. In this chapter, we discuss two classes of differential equations in which it is possible to obtain exact solutions: *separable differential equations* and *linear differential equations*.

A *(first order) separable differential equation* is one of the form[6]

$$h(y)y' = g(t)\,, \tag{3.1}$$

where $g(t)$ and $h(y)$ are continuous functions, is called a *separable* differential equation..

A *(first order) linear differential equation* is one of the form

$$y' + p(t)y = f(t)\,,\text{where } p(t) \text{ and } f(t) \text{ are continuous} \tag{3.2}$$

In the special case when $f(t) = 0$, the differential equation is called a *homogeneous linear differential equation* ; otherwise, it is said to be a *nonhomogeneous* differential equation. The function $f(t)$ on the right-hand side of a differential equationis called the *forcing function*

EXAMPLE 3.1. Here's an alarming example of an initial value problem that doesn't have a unique solution: consider the initial value problem

$$\frac{dy}{dt} = F(y)\,, \quad y(0) = 0\,,$$

where $F(y) = \sqrt{2|y|}$. For any real number $a > 0$, consider the function $y_a(t)$ defined as follows:

$$y_a(t) = \begin{cases} (t-a)^2/2 & \text{for } t \geq a \\ 0 & \text{for } t \leq a\,. \end{cases}$$

By construction, $y_a(t)$ satisfies the initial condition $y_a(0) = 0$. It also satisfies the differential equation

$$y_a'(t) = F(y_a(t)) \text{ for all } t\,.$$

This is clear because

$$y_a'(t) = 0 = F(0) = F(y_a(t)) \text{ for } t \leq a\,,$$

and

$$\frac{d(t-a)^2/2}{dt} = (t-a) = \sqrt{2(t-a)^2/2} = F(y_a(t)) \text{ for } t \geq a\,.$$

REMARK 3.1. The previous example naturally leads to the following question:

What conditions on $F(t, y)$ are sufficient to guarantee that the initial value problem

$$y' = F(t, y)\,, \quad y(t_0) = y_0$$

has a unique solution?

---

[6]If you took Math 125 here at the UW, then you've already worked with separable differential equations, and much of the material in the next section will be a review for you.

Although this is a fascinating question, it will not be addressed here. Suffice it to say that if the function $F(t, y)$ is sufficiently nice[7] (which is the case in virtually all differential equations encountered in practice), then the initial value problem has a unique solution. That means two things: (1) there is a solution, and (2) there is only one solution to the initial value problem.

This fact has an important geometric interpretation, which it is very important to keep in mind: (Provided $F(t, y)$ is sufficiently nice)

the integral curves of a first order differential equation never cross.

To see this, we can argue as follows: If two integral curves crossed, say at the point (t_0, y_0), then they there would be at least two solutions of the initial value problem

$$y' = F(t, y)\,, \quad y(t_0) = y_0\,,$$

which would contradict the claim that there is only one solution.

3.1. Separable Differential Equations

Consider the *separable* differential equation

$$h(y)y' = g(t)\,,$$

where $g(t)$ and $h(y)$ are continuous functions. Our goal is to find a technique for solving it.

Start by assuming that the function $y = y(t)$ is a solution of the differential equation. Then, by definition, it satisfies the equation

$$h(y(t))y'(t) = g(t)\,,$$

for all values of t in some interval, say $a < t < b$. It follows that the integral of the left-hand side differs from the integral of the right-hand side by a constant:

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt + C\,.$$

Let[8]

$$H(y) = \int^y h(z)\,dz \text{ and } G(t) = \int^t g(s)\,ds$$

be anti-derivatives of $h(y)$ and $g(t)$, respectively. Thus, the function $y = y(t)$ is implicitly defined by the equation

$$H(y) = G(t) + C\,. \tag{3.3}$$

This process can be reversed. Suppose that $y = y(t)$ satisfies (3.3). The computation

$$\frac{dH(y(t))}{dt} = H'(y(t))\,y'(t) = h(y(t))\,y'(t) = G'(t) = g(t)\,.$$

then shows that it is $y(t)$ a solution of the differential equation (3.1), called the *general solution* of the differential equation.

REMARK 3.2. The function $y = y(t)$ is said to be *implicitly defined* by (3.3) because for a given value of t it is in general not clear how to determine $y(t)$. Unfortunately, Equation (3.3) cannot always be *explicitly solved* for y in terms of t. It is, however, best to obtain an explicit solution when possible.

---

[7]The function $F(t, y)$ in Example 3.1 is not differentiable with respect to y for $y = 0$; it is not "nice."

[8]We use the notation $\int^x f(s)\,ds$ to denote a specific anti-derivative of the function $f(x)$. For instance, $\int^x \cos(s)\,ds = \sin(x)$ rather than $\sin(s) + C$.

EXAMPLE 3.2. *Solve the differential equation* $\dfrac{dy}{dt} = (1+y^2)e^{t/2}$.

SOLUTION Rewrite the differential equation in the form $\dfrac{1}{1+y^2}\dfrac{dy}{dt} = e^{t/2}$.
Integrate

$$\int^y \frac{dz}{1+z^2} = \int^t e^{s/2}\,ds + C$$

to arrive at the *implicit solution* $\tan^{-1}(y) = 2e^{t/2} + C$, where C is an arbitrary constant. In this case, it is possible to solve for y to obtain the *explicit solution*

$$y(t) = \tan\left(2e^{t/2} + C\right) .$$

3.1.1. Solving Initial Value Problems. The solution of the initial value problem

$$h(y)y' = g(t) \quad y(t_0) = y_0 \,,$$

can be found by first finding the general solution:

$$H(y) = G(t) + C \,,$$

where $H'(y) = h(y)$ and $G'(t) = g(t)$; and then setting $y = y_0$ and $t = t_0$ to solve for C:

$$H(y_0) = G(t_0) + C \text{ or } C = H(y_0) - G(t_0)$$

to obtain the solution

$$H(y) - H(y_0) = G(t) - G(t_0) \,. \tag{3.4}$$

EXAMPLE 3.3. *Solve the initial value problem* $y' = (1+y^2)e^t$, $y(0) = 1$.

SOLUTION Write the differential equation in the form $\dfrac{y'}{1+y^2} = e^t$.
Now integrate to get

$$\tan^{-1}(y) - \tan^{-1}(1) = e^t - e^0 \,.$$

Since $\tan^{-1}(1) = \pi/4$ and $e^0 = 1$, it follows that

$$\tan^{-1}(y) - \pi/4 = e^t - 1 \,.$$

Solving for y yields the final result:

$$y(t) = \tan\left(e^t - 1 + \pi/4\right) .$$

EXERCISE 3.1. Solve each of the following first order differential equations and initial value problems. (Note that in some of these, the independent variable is x rather than t.

(a) $y' = (1-t)(2-t)$, $y(1) = 1$.
(b) $y' = (1-t)(2-y)$.
(c) $y' = 2y(1-y)$, $y(0) = 2$.
(d) $\dfrac{dy}{dt} = ay(b-y)$, $y(0) = y_0$, where $a > 0$ and $b > 0$ are constants.
(e) $y' = 1-y^2$, $y(0) = 0$.
(f) $y' = \cos(x)(y^2+1)$.
(g) $\dfrac{dy}{dx} = x\sqrt{4-y^2}$.

3.2. Linear First Order Differential Equations

A *linear first order differential equation* is a differential equation that can be written in the form

$$y' + p(t)y = f(t)\,,$$

where $p(t)$ and $f(t)$ are continuous or piecewise continuous on some interval. Remember also that when $f(t) = 0$, the equation is said to be a *homogeneous* differential equation, otherwise, it is said to be *nonhomogeneous differential equation*. The function $f(t)$ is called the *forcing function*.

EXAMPLES 3.4. The following differential equations are all linear:

$$\begin{aligned}
&\frac{dy}{dt} + 2y = 0 && y' + 2y = e^t \\
&\frac{dy}{dt} + ty = 0 && y' + ry = k, \quad r, k \quad \text{constant} \\
&(1+t^2)y' + y = 0 && ty' = te^t - y, \quad t > 0 \\
&y' + t^{-1}\,y = 1, \quad t > 0 && y' + y = \sin^{-1}(t), \quad |t| < 1
\end{aligned}$$

Note: the equation $(1+t^2)y' + y = 0$ is linear because it can we rewritten as $y' + \dfrac{1}{1+t^2}\,y = 0$. Similarly, the differential equation $ty' = te^t - y$ is linear because it can be rewritten as $y' + \dfrac{1}{t}y = e^t$.

If a differential equation cannot be rewritten in the form $y' + p(t)y = g(t)$, then we say that it is a *nonlinear differential equation*. Here are some examples of nonlinear differential equations:

$$\begin{aligned}
&\frac{dy}{dt} + 2y^2 = 0 && y' + 2\frac{1}{y} = e^t && y\frac{dy}{dt} + ty = 2 \\
&(y')^2 + y = t && y' + \sqrt{y} = 0 && y' + ty = (1+y^2)
\end{aligned}$$

REMARK 3.3. Some differential equations are both linear <u>and</u> separable. Here are a few examples:

$$\begin{aligned}
y' + 5y = 3 &\text{ can we rewritten as } \frac{y'}{3-5y} = 1 \\
y' + \cos(t)y = 0 &\text{ can be rewritten as } \frac{y'}{y} = -\cos(t) \\
e^t y' + ty = 0 &\text{ can be rewritten as } \frac{y'}{y} = -te^{-t}
\end{aligned}$$

In these cases, the differential equation can be solved either by separation of variable or by the method we are about to explain.

3.2.1. Undetermined coefficients for first order differential equations.

When $p(t) = k$, for k a constant, the differential equation has the form

$$y' + ky = f(t)\,.$$

When, in addition, the forcing function $f(t)$ is one of the following forms:

$$f(t) = a\,, \quad ae^{bt}\,, \quad at^n\,, \quad a\cos(\omega t)\,, \text{ or } a\sin(\omega t)\,, \tag{3.5}$$

where a, b, n, and ω are constants it is possible to quickly solve the differential equation, as we now show.

First notice that if $y(t)$ is a solution than so is $y(t) + Ce^{-kt}$, where C is a constant. We can see this by computing. Suppose that $y(t)$ is a solution. That means that $y'(t) + ky(t) = f(t)$. Now compute as follows:

$$\left(y(t) + Ce^{-kt}\right)' + k\left(y(t) + Ce^{-kt}\right) = y'(t) - kCe^{-k} + ky(t) + kCe^{-kt} = y'(t) + ky(t) = f(t)\,,$$

which shows that $y(t) + Ce^{-kt}$ is also a solution. Since this always happens, all we need to do if guess a particular solution to the differential equation and add Ce^{-kt} to it. To make this clear, I'll denote a *particular solution* by $y_p(t)$. The *general solution* of the differential equation $y' + ky = f(t)$ is then

$$y(t) = y_p(t) + Ce^{-kt}\,.$$

(By "general solution" we mean that by choosing the appropriate value of C we can express any solution this way.)

Example 3.5.
(a) Find the general solution of the differential equation $y' + y = t$.
(b) Find the solutions that satisfy each of the following initial conditions:

$$y(0) = -3,\ y(0) = 2,\ \textit{and}\ y(0) = 0.$$

(c) Sketch the direction field for the differential equation along with the graphs of the three solutions you found in part (b).

Solution (a) Try $y_p(t) = At + B$. Then

$$y_p'(t) + y_p(t) = A + (At + B) = At + (A + B) = t\,.$$

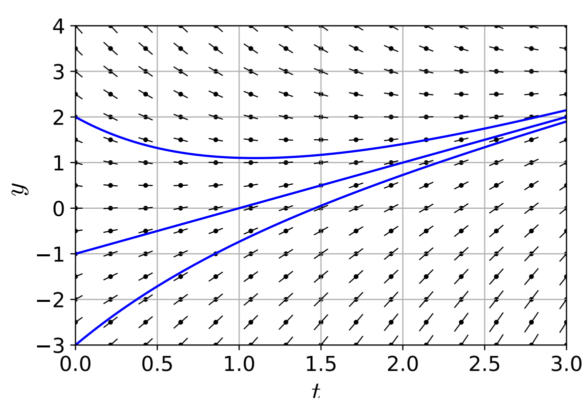
This forces $A = 1$ and $B = -A = -1$, so the general solution is

$$y(t) = t - 1 + Ce^{-t}\,.$$

(b) Since $y(0) = C - 1$, the three initial conditions force C to have the values -2, 0, and 3, respectively. So the solutions to the three initial values problems are

$$y(t) = (t-1) - 2e^{-t}\,, \quad y(t) = t - 1\,,\ \text{and}\ y(t) = t - 1 + 3e^{-t}\,.$$

(c) To sketch the direction field, write the differential equation in the form $y' = t - y$. Notice that $y' = 0$ along the line $y = t$; $y' < 0$ above the line, and $y' > 0$ below the line. This alone is enough to give a rough picture of the direction field. I've used computer software to generate the more detailed picture along with the graphs of the three solutions shown below.



Now suppose that the forcing function is $f(t) = ae^{bt}$, with $b \neq -k$. We want to solve the differential equation

$$y' + ky = ae^{bt}\ \text{ for } b \neq -k.$$

Substituting $y_p(t) = Ae^{-bt}$ into the differential equation and computing as follows:

$$(Ae^{bt})' + k(Ae^{bt}) = (b+k)Ae^{bt} = ae^{bt}$$

shows that $A = \frac{a}{k+b}$ and, therefore $y_p(t) = \dfrac{1}{k+b}e^{at}$. So the general solution is

$$y(t) = Ce^{-kt} + \frac{1}{k+b}e^{bt}\,, \quad \text{for } C \text{ a constant.}$$

This solution is not valid when $b = -k$ and Ae^{-kt} is a solution of the homogeneous differential equation $y' + ky = 0$. In this case, there is a particular solution of the form $y_p(t) = Ate^{-kt}$

EXAMPLE 3.6. Let's work out a concrete example:
(a) Find the general solution of the differential equation $y' + 7y = 8e^{-3t}$.
(b) Solve the initial value problem $y' + 7y = 8e^{-3t}$, $y(0) = 6$.
(c) Find the general solution of the differential equation $y' + 7y = 3e^{-7t}$.

SOLUTION (a) Let $y_p = Ae^{-3t}$, and compute as follows to find A:

$$(Ae^{-3t})' + 7(Ae^{-3t}) = ((-3)+7)Ae^{-3t} = 4Ae^{-3t} = 8e^{-3t}\,.$$

So $4A = 8$ or $A = 2$. Hence, a particular solution is $y_p(t) = 2e^{-3t}$ and

$$y(t) = 2e^{-3t} + Ce^{-7t}$$

is the general solution of the differential equation.

(b) We have to find C. But $y(0) = 2 + C = 6$, so $C = 4$ and the solution of the initial value problem is

$$y(t) = 2e^{-3t} + 4e^{-7t}\,.$$

(c) Let $y_p(t) = Ate^{-7t}$, and compute as follows:

$$\left(Ate^{-7t}\right)' + 7\left(Ate^{-7t}\right) = A(1-7t)e^{-7t} + 7Ate^{-7t} = Ae^{-7t} = 3e^{-7t}\,.$$

So $A = 3$, and the general solution is $y(t) = 3te^{-7t} + Ce^{-7t} = (3t + C)e^{-7t}$.

Another commonly occurring class of equations consists of differential equations of the form

$$y' + ky = a\cos(\omega t) + b\sin(\omega t)\,.$$

EXAMPLE 3.7. Rather than deriving a general formula, it's easier to work by example.
(a) Find the general solution of the differential equation $y' + 2y = 4\cos(3t)$
(b) Solve the initial value problem $y' + 2y = 4\cos(3t)$, $y(0) = 0$. *Graph the solution.*
(c) Find the general solution of the differential equation $y' + 2y = 4\cos(3t) + 2\sin(3t)$

SOLUTION (a) We know that the solution is of the form $y(t) = y_p(t) + Ce^{-2t}$, so we only need to find a particular solution $y_p(t)$, which we are going to assume to be of the form

$$y_p(t) = A\cos(3t) + B\sin(3t)\,,$$

where A and B are real numbers that we have to find. Under this assumption, we can compute as follows:

$$\begin{aligned} y_p'(t) + 2y_p(t) &= (A\cos(3t) + B\sin(3t))' + 2\,(A\cos(3t) + B\sin(3t)) \\ &= -3A\sin(3t) + 3B\cos(3t) + 2A\cos(3t) + 2B\sin(3t) \\ &= (-3A + 2B)\sin(3t) + (2A + 3B)\cos(3t) \\ &= \qquad 0\,\sin(3t) + 4\,\cos(3t) = 4\cos(3t)\,. \end{aligned}$$

This shows that $y_p(t)$ is a solution provided we choose A and B to satisfy the equations

$$-3A + 2B = 0 \text{ and } 2A + 3B = 4\,.$$

This is a problem in high school algebra: From the first equation, $B = 3A/2$. Substituting this into the second equation gives $2A + 3(3A/2) = (13/2)A = 4$, which implies $A = 8/13$ and $B = (3/2)(8/13) = 12/13$. Therefore, $y_p(t) = \dfrac{8}{13}\cos(3t) + \dfrac{12}{13}\sin(3t)$, and

$$y(t) = \frac{8}{13}\cos(3t) + \frac{12}{13}\sin(3t) + Ce^{-2t}$$

is the general solution of the differential equation.

(b) We have to find the value of C. But

$$y(0) = \frac{8}{13}\cos(0) + \frac{12}{13}\sin(0) + Ce^0 = \frac{8}{13} + C = 0 \text{ implies } C = -\frac{8}{13}\,.$$

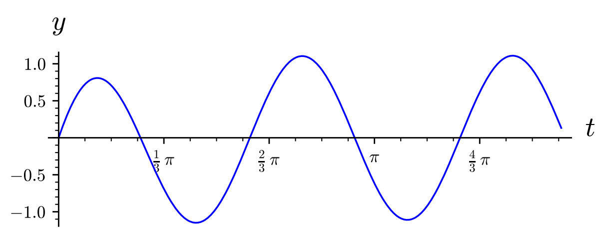
So

$$y(t) = \frac{8}{13}\cos(3t) + \frac{12}{13}\sin(3t) - \frac{8}{13}e^{-2t}$$

is the solution of the initial value problem. Using the phase-shift formula, we can write the solution in the alternate form

$$y(t) = \frac{4}{\sqrt{13}}\cos\left(3t - \tan^{-1}(3/2)\right) - \frac{8}{13}e^{-2t} \approx 1.109\cos(3t - 0.983) + 0.615\,e^{-2t}\,,$$

in which it's easier to visualize the graph of the solution, which is shown below.



(c) The computation is almost the same as the one we did in part (a):

$$\begin{aligned} y_p'(t) + 2y_p(t) &= (A\cos(3t) + B\sin(3t))' + 2\,(A\cos(3t) + B\sin(3t)) \\ &= -3A\sin(3t) + 3B\cos(3t) + 2A\cos(3t) + 2B\sin(3t) \\ &= (-3A + 2B)\sin(3t) + (2A + 3B)\cos(3t) \\ &= \qquad 2\,\sin(3t) + 4\,\cos(3t)\,. \end{aligned}$$

So A and B must now satisfy the equations $-3A + 2B = 2$ and $2A + 3B = 4$, from which we find that $A = \dfrac{2}{13}$ and $B = \dfrac{16}{13}$. The general solution is, therefore,

$$y(t) = \frac{2}{13}\cos(3t) + \frac{16}{13}\sin(3t) + Ce^{-2t}\,.$$

3.2.2. The General Case. When $p(t)$ is not a constant and/or when the forcing function $f(t)$ is more general than in the previous section, we have to adopt a more systematic approach that applies to <u>all</u> initial value problems of the form

$$y' + p(t)y = f(t)\,, \quad y(t_0) = y_0\,, \tag{3.6}$$

where $p(t)$ and $f(t)$ are continuous functions on an interval $a < t < b$ and t_0 is between a and b. (When a and b are not explicitly given, we take the largest interval on which $p(t)$ and $f(t)$ are continuous.)

In this approach, multiplying the differential equation by a so-called *integrating factor* transforms the differential equation into one that can be solved by Riemann integration.

The computations are easier to understand in the *constant coefficient case*, where $p(t) = k$ and where $t_0 = 0$:

$$y' + ky = f(t)\,, \quad y(0) = y_0\,. \tag{3.7}$$

Begin by assuming that the function $y = y(t)$ is a solution of (3.7). Then, by definition, it satisfies the equation

$$y'(t) + ky(t) = f(t)\,.$$

Multiply both sides of this equation by e^{kt} (this is the "integrating factor") to obtain the equation

$$e^{kt}y'(t) + ke^{kt}y(t) = e^{kt}f(t)\,;$$

and notice that, by the product rule for differentiation,

$$\left(e^{kt}y(t)\right)' = e^{kt}y'(t) + ke^{kt}y(t)\,.$$

This shows that the solution satisfies the equation

$$\left(e^{kt}y(t)\right)' = e^{kt}f(t)\,,$$

which can be integrated to obtain the equality

$$\int_0^t \left(e^{ks}y(s)\right)'\,ds = \int_0^t e^{ks}f(s)\,.$$

By the Fundamental Theorem of Calculus, the integral on the left can be explicitly evaluated to yield the equation

$$e^{kt}y(t) - y(0) = \int_0^t e^{ks}f(s)\,ds\,, \tag{3.8}$$

which, in turn, can be solved for $y(t)$ to yield a formula for $y(t)$:

$$y(t) = e^{-kt}\left(\int_0^t e^{ks}f(s)\,ds + y(0)\right)\,. \tag{3.9}$$

Finally, recall that $y(t)$ satisfies the initial condition $y(0) = y_0$ and simplify to obtain the formula

$$y(t) = e^{-kt}\int_0^t e^{ks}f(s)\,ds + y_0e^{-kt}\,. \tag{3.10}$$

Remark 3.4. To can check directly that $y(t)$ is a solution of the initial value problem, first notice that the computation

$$y(0) = e^{-k0}\int_0^0 e^{ks}f(s)\,ds + y_0e^{-k0} = 0 + y_0 = y_0\,,$$

shows that $y(t)$ satisfies the initial condition. Next notice that (by the Fundamental Theorem of Calculus):

$$\begin{aligned} y'(t) &= -ke^{-kt}\left(\int_0^t e^{ks}f(s)\,ds\right) + e^{-kt}\left(e^{kt}f(t)\right) - ky_0e^{-kt} \\ &= -\,k\left(e^{-kt}\left(\int_0^t e^{ks}f(s)\,ds\right) + y_0e^{-kt}\right) + f(t) \\ &= -ky(t) + f(t)\,. \end{aligned}$$

Consequently, $y'(t) + ky(t) = (-ky(t) + f(t)) + ky(t) = f(t)$, showing that $y(t)$ is, indeed, a solution of the differential equation.

This analysis accomplished three goals:

(i) It shows that the initial value problem (3.7) has a solution.
(ii) Because it started with an unknown solution $y(t)$ and arrived at the formula (3.10), it shows that there is only one solution.
(iii) The formula (3.10) gives an explicit algorithm for solving the initial value problem: to find $y(t)$, one need only evaluate one definite integral.

A similar trick applies to the general case,

$$y' + p(t)y = f(t)\,,$$

but with one change: the function e^{kt} must be replaced by a more complicated expression and the computations are messier.

Let[9] $P(t) = \displaystyle\int^t p(s)\,ds$, and replace the "integrating factor" e^{kt} by the function $e^{P(t)}$. Then, as before we can compute as follows

$$e^{P(t)}\left(y'(t) + p(t)y(t)\right) = \left(e^{P(t)}y(t)\right)' = e^{P(t)}f(t)\,,$$

which can be integrated to yield the formula

$$e^{P(t)}y(t) = \int^t e^{P(s)}f(s)\,ds + C\,.$$

This, in turn, can be solve for $y(t)$ to obtain the formula

$$y(t) = y_p(t) + Cy_h(t)\,, \tag{3.11a}$$

where

$$y_h(t) = e^{-P(t)}\,, y_p(t) = e^{-P(t)}\int^t e^{P(s)}f(s)\,ds\,, \text{ and } P(t) = \int^t p(s)\,, ds. \tag{3.11b}$$

The value of the constant C is determined from the initial condition $y(t_0) = y_0$ by solving the equation

$$y(t_0) = y_p(t_0) + Cy_h(t_0) = y_0$$

for C.

Remark 3.5. If $f(t) = 0$, then $y_p(t) = 0$, and so $y = Cy_h(t)$ is the general solution of the homogeneous differential equation $y' + p(t)y = 0$. The function $y_p(t)$ is called a *particular solution* of the nonhomogeneous differential equation, since it does not involve any arbitrary constants. The function $y_h(t)$ is a particular solution of the homogeneous differential equation.

Our computation began with the assumption that a solution $y(t)$ existed and then *solved* for $y(t)$. This showed (1) that there is a solution and (2) that every solution of the differential equation is of the form (3.11a). Since the initial condition $y(t_0) = y_0$ is sufficient to determine the constant C, it follows that there is only one solution to the initial value problem. The only assumption that made in the computations was that the two integrals

$$\int^t p(s)\,ds \text{ and } \int^t e^{P(s)}g(s)\,ds$$

made sense. This is certainly the case if $p(t)$ and $g(t)$ are continuous (or even piecewise continuous) functions. The following theorem summarizes the above discussion.

[9]The expression $\displaystyle\int^t p(s)\,ds$ denotes a specific anti-derivative of $p(t)$. For instance $\displaystyle\int^t sds = t^2/2$ rather than the indefinite integral $\int sds = t^2/2 + C$.

THEOREM 1. *Let $p(t)$ and $g(t)$ be continuous functions defined on the interval $a < t < b$ and suppose that $a < t_0 < b$. Then there is one and only one solution of the initial value problem*

$$y' + p(t)\, y = f(t)\,, \quad y(t_0) = y_0\,.$$

EXAMPLE 3.8. *Solve the initial value problem $y' + 3ty = te^{t^2}$, $y(2) = 5$.*

SOLUTION First solve the homogeneous equation: $y' + 3ty = 0$. Since $P(t) = 3t^2/2$ is an anti-derivative of $3t$, the function $y_h(t) = e^{-3t^2/2}$ is a solution of the homogeneous equation. Next set $y_p = h(t)y_1(t) = h(t)e^{-3t^2/2}$ and plug into the nonhomogeneous differential equation to get

$$h'(t)e^{-3t^2/2} = te^{t^2} \text{ or } h'(t) = te^{5t^2/2}$$

Integration gives $h(t) = \frac{1}{5}e^{5t^2/2}$, so the general solution is

$$y(t) = \left(\frac{1}{5}e^{5t^2/2} + C\right) e^{-3t^2/2} = \frac{1}{5}e^{t^2} + Ce^{-3t^2/2}\,.$$

The initial condition $y(2) = 5$ then determines C: $\dfrac{1}{5}e^{(2)^2} + Ce^{-3(2)^2/2} = 5$. Thus,

$$C = e^{3(2)^2/2}\left(5 - \frac{1}{5}e^{(2)^2}\right) = 5e^6 - \frac{e^{10}}{5} \approx -2388 \text{ and } y(t) \approx \frac{1}{5}e^{t^2} - 2388e^{-3t^2/2}\,.$$

EXERCISE 3.2. Solve each of the following first order differential equations and initial value problems.

(a) $\dfrac{dy}{dx} + y = 5e^{-3x}$
(b) $(1+t^2)\dfrac{dy}{dt} + 2ty = t(1+t^2)\,, \quad y(1) = 1$
(c) $w' - 2tw = e^{t^2}$
(d) $\dfrac{dz}{dx} + 3z = \sin(6x)$.
(e) $y' - 3y = e^{2t}$.
(f) $y' + 3y = e^{2t}$.
(g) $y' + 3y = t^2$ (**Hint:** *Let $y_p(t) = at^2 + bt + c$ and solve for a,b, and c.*)
(h) (i) Find the general solution of the differential equation $2y' + y = 6$.
(ii) Find the solutions that satisfy each of the following initial conditions:
$$y(0) = 0,\ y(0) = 6, \text{ and } y(0) = 10.$$
(iii) Sketch the direction field for the differential equation along with the graphs of the three solutions you found in part (b).

CHAPTER 4

Modeling with First Order Differential Equations

We now have a geometric picture of first order differential equations (the direction field), and we have techniques for solving separable and linear differential equations. In this chapter, we study a few applications of the theory.

4.1. Linear Models

We begin with applications of first order *linear* differential equations.

4.1.1. Radioactive decay. Our first application is to the decay of radioactive elements, such as Carbon-14 or Strontium-90.

Each radioactive element decays (into lighter elements) at a rate proportional to the quantity remaining. Letting $k > 0$ denote the constant of proportionality. Assume that we start with a quantity Q_0 (measured in grams, say) at time $t = 0$ years Then the quantity $Q(t)$ after t years is then the solution of the initial value problem

$$\frac{dQ}{dt} = -kQ\,, \quad Q(0) = Q_0\,.$$

This is a linear differential equation with solution

$$Q(t) = Q_0 e^{-kt}\,.$$

Remark 4.1. Rather than specifying k directly, it is traditional to express it indirectly in term of the *half-life* t_h, which is a more intuitive measure of the rate of decay than the parameter k: The half-life of a radioactive element is the time required for half of the element to decay into other (lighter) elements.

The decay rate k can be computed from the half-life as follows. Let Q_0 be the amount of radioactive material at some time, set to $t = 0$, and let $Q(t)$ denote the amount of a radioactive material remaining t years later. Then, by definition, $Q(t_h)/Q_0 = 1/2$. On the other hand, $Q(t_h) = Q_0 e^{-kt_h}$.

Therefore, $Q_0 e^{-kt_h} = \frac{1}{2} Q_0$. Canceling the term Q_0, taking the natural logarithm, and solving for k yields the formula

$$k = \frac{\ln(2)}{t_h}\,. \tag{4.1}$$

Example 4.1. *Suppose that a certain quantity of an unknown radioactive substance is placed in a container and that after 10 years the amount has decreased by 0.01%. What percent of the original quantity will remain after 25 years? What is the half-life of this radioactive substances?*

Solution The decay rate k can be determined from the amount of radioactive substance remaining after 10 years:

$$Q(10) = Q_0 e^{-k10} = (1 - 0.0001)Q_0 = 0.9999\, Q_0 \implies e^{-k10} = 0.9999\,.$$

Taking the natural logarithm of both sides gives

$$-k(10) = \ln(0.9999) \implies k = -\frac{1}{10}\ln(0.9999) \approx 0.00001\,.$$

Substituting this value of k into the formula for $Q(t)$ yields the formula $Q(t) = Q_0 e^{-0.00001t}$. It remains only to compute $Q(25)$ as a percent of Q_0:

$$\frac{Q(25)}{Q_0} \times 100\% = 100e^{-(0.00001)25} \approx 99.975\%\,.$$

The half-life of the substance can be found by solving Equation (4.1) for t_h:

$$t_h = \frac{\ln(2)}{k} = \frac{\ln(2)}{0.00001} = 69315 \text{ years.}$$

Example 4.2. *You find a frozen animal and you determine by experiment that the concentration of C^{14} in it is 22% of the amount found in live animals. How old is the animal?*

Solution The concentration of C^{14} in live animals depends on the concentration of C^{14} in the atmosphere, and is roughly independent[10] of time. When an animal dies, it stops adsorbing carbon from its environment, and so the concentration of C^{14} decreases as the C^{14} decays into Nitrogen 14. This is the key fact behind C^{14} dating.

To compute the age of the animal, let $Q(t)$ denote the amount of C^{14} in the animal t years after its death and let $t = T$ denote the value of t when you determined the concentration of C^{14}. Solving the differential equation $Q' = -kQ$, gives

$$Q(t) = Q_0 e^{-kt}\,,$$

where Q_0 is the amount of C^{14} in the animal at the time of death. The half-life of C^{14} is 5568 years, so

$$k = \frac{\ln(2)}{5568} \approx 0.0001245\,.$$

Because $Q(T) = 0.22Q_0$ and $Q(t) = Q_0 e^{-kt}$, it follows that

$$0.22Q_0 = Q_0 e^{-kT}\,,$$

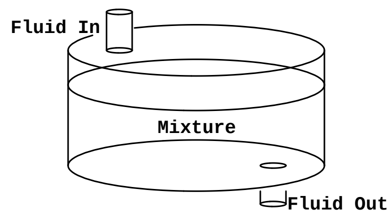
which can be solved for T:

$$T = -\frac{\ln(0.22)}{k} = -\frac{\ln(0.22)}{\ln(2)/5568} \approx 12,163 \quad \text{years.}$$

[10]In fact, to obtain more accurate estimates of the age of artifacts, archaeologists make allowance for changes in the concentration of C^{14} in the atmosphere over time by incorporating tree-ring data into their calculation of Q_0.

4.1.2. Mixing Problems. A *mixing problem* involves a quantity of a substance dissolved in a fluid (a liquid or a gas) in a container. Fluid with a known concentration of the substance enters the container at a certain rate, and fluid exits the container a another rate. The goal is to find an expression for either the quantity or the concentration of substance in the container as a function of time by modeling the process as a first order initial value problem.

EXAMPLE 4.3. *Suppose that a tank with a capacity of 300 gallons initially contains 100 gallons of pure water. A salt solution containing 3 pounds of salt per gallon is allowed to run into the tank at a rate of 8 gal/min, and the mixture is then removed at a rate of 6 gal/min, as shown in the figure below. The process is continued until the tank is filled. Determine the concentration of the salt solution in the tank at the end of the process.*



SOLUTION To make any progress, we need to label the various quantities that we need to keep track of. Let $Q(t)$ denote the quantity (in pounds) of salt in the tank at time t, measured in minutes, with $t = 0$ at the beginning of the process. Let $V(t)$ be the volume of solution in the tank at time t. Then the concentration of salt in the container at time t is $C(t) = \dfrac{Q(t)}{V(t)}$.

Salt both enters and leaves the tank are certain rates. Let $R_{in}(t)$ denote the rate that salt enters the tank and let $R_{out}(t)$ denote the rate that salt leaves the tank. Then the net rate that salt accumulates in the tank is the difference. Since the tank is initially full of pure water, $Q(0) = 0$. We can now model the process as an initial value problem:

$$\frac{dQ}{dt} = R_{in}(t) - R_{out}(t)\,, \quad Q(0) = 0\,.$$

We need to figure out formulas for $R_{in}(t)$ and $R_{out}(t)$. The salt solution enters the tank at 8 gallons a minute at a concentration of 3 pounds per gallon, so

$$R_{in}(t) = 8\frac{gal}{min} \times 3\frac{pounds}{gal} = 24\,\frac{pounds}{min}\,.$$

The solution exits the tank at a rate of 6 gallons per minute. So if $C(t)$ is the concentration of salt in the tank

$$R_{out}(t) = 6\,C(t) = 6\,\frac{Q(t)}{V(t)}\,,$$

provided the salt is mixed uniformly in the tank. This is the so-called *well-mixing assumption* , which we will make. Thus, $Q(t)$ is the solution of the initial value problem

$$\frac{dQ}{dt} = 24 - 6\,\frac{Q}{V(t)}\,, \quad Q(0) = 0\,.$$

It's nicer to write this in the form

$$\frac{dQ}{dt} + \frac{6}{V(t)}\,Q = 24\,, \quad (Q(0) = 0\,,$$

which makes it clearer that the differential equation is linear.

To solve this initial value problem, we have to find the formula for $V(t)$. But fluid enters the tank at 8 gal/min and leaves at 6 gal/min and $V(0) = 100$ gals, so V satisfies another initial value problem:

$$\frac{dV}{dt} = 8 - 6 = 2 \text{ and } V(0) = 100\,.$$

So $V(t) = 2t + 100$, and we now know that Q satisfies the initial value problem

$$\frac{dQ}{dt} + \frac{6}{2t+100}\,Q = 24\,, \quad (Q(0) = 0\,,$$

which we can now solve to find the formula for $Q(t)$.

The integrating factor for the differential equation is

$$\mu(t) = \exp\left(\int^t \frac{6ds}{100+2s}\right) = \exp(3\ln(50+t)) = (50+t)^3\,.$$

Thus,

$$(50+t)^3 Q(t) = \int 24(50+t)^3 dt = 6(50+t)^4 + c\,.$$

Using initial condition $Q(0) = 0$ (initially, the water is pure), gives $c = -6(50)^4$. Solving for Q, yields the formula

$$Q(t) = 6(50+t) - 300\left(\frac{50}{50+t}\right)^3 \quad \text{lbs.}$$

To find the time when the tank is full, solve the equation

$$V(t) = 100 + 2t = 300$$

for t to find that $t = (300 - 100)/2 = 100$ min. The concentration of the salt solution at end of the filling procedure is, therefore,

$$C(100) = \frac{Q(100)}{V(100)} = \frac{6(150) - 300(50/150)^3}{300}\,\text{lb/gal} = (80/27)\,\text{lb/gal} \approx 2.963\,\text{lb/gal}.$$

4.1.3. Falling bodies. The case of a falling body where air resistance is taken into account is more complicated than the simple case discussed in the introduction where the forces due to air resistance were ignored. For slowly moving bodies, the force caused by moving through air (called *drag*) is proportional to the speed of the object and points in the direction <u>opposite</u> the motion:

$$\text{drag } = -kv\,, k > 0\,,$$

where k is a positive constant called the *drag coefficient*. In this situation, Newton's second law of motion assumes the form

$$m\frac{dv}{dt} = -mg - kv$$

or better

$$\frac{dv}{dt} + \frac{k}{m}v = -g\,,$$

a linear differential equation, which we can solve by the method of undetermined coefficients:

$$v(t) = -\frac{mg}{k} + C\,e^{-(k/m)t}\,.$$

The initial condition $v(0) = v_0$ determines C:

$$-\frac{mg}{k} + C = v_0 \quad \Longrightarrow \quad C = v_0 + \frac{mg}{k},,$$

so

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-(k/m)t}\,.$$

As the speed increases, so does the drag. At a certain velocity the force of gravity will exactly cancel with the drag. This velocity is called the *terminal velocity*:

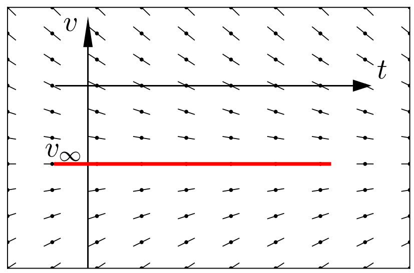
$$v_\infty = \lim_{t\to\infty} v(t)\,.$$

To find v_∞, solve the equation $-mg - kv_\infty = 0$ for v_∞ to obtain the formula

$$v_\infty = -\frac{mg}{k}\,.$$

We can gain further insight into this by sketching the direction field (shown below for the differential equation

$$\frac{dv}{dt} = -g - (k/m)v\,.$$



Integrating the formula for $v(t)$ yields an expression for $y(t)$:

$$y(t) = -\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-kt/m} + C\,.$$

The initial condition $y(0) = y_0$ determines C:

$$C = \frac{m}{k}\left(\frac{gm}{k} + v_0\right) + y_0\,,$$

giving rise to the formula

$$y(t) = -\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-(k/m)t} + \frac{m}{k}\left(\frac{mg}{k} + v_0\right) + y_0\,,$$

which simplifies to

$$y(t) = y_0 - \frac{mg}{k}t + \left(v_0\frac{m}{k} + \frac{gm^2}{k^2}\right)\left(1 - e^{-(k/m)t}\right)\,,$$

where y_0 and v_0 are the position and velocity, respectively, of the particle at time $t = 0$.

Remark 4.2. In the special case $v_0 = 0$, the formulas for $v(t)$ and $y(t)$ simplify further to

$$v(t) = -\frac{mg}{k}\left(1 - e^{-(k/m)t}\right) \text{ and } y(t) = y_0 + \frac{m^2 g}{k^2}\left(1 - e^{-(k/m)t}\right)$$

This can be thought of as a modification of the formulas

$$v(t) = -gt \text{ and } y(t) = y_0 - \frac{1}{2}gt^2$$

for free fall without air resistance. This can be seen by substituting the approximation,

$$e^{-\frac{k}{m}t} \approx 1 - \frac{k}{m}t + \frac{k^2}{2m^2}t^2 - \frac{k^3}{6m^3}t^3$$

(which is valid for $(k/m)t$ small) into the formulas for $v(t)$ and $y(t)$ when we included air resistance. After some algebra, we obtain the approximate formulas

$$v(t) \approx -gt + \frac{kg}{2m}t^2 \text{ and } y(t) \approx y_0 - \frac{1}{2}gt^2 + \frac{kg}{6m}t^3.$$

For $k = 0$, these formulas reduce to the formulas for $v(t)$ and $y(t)$ with no air resistance.

Example 4.4. *Suppose that a bag weighing 120 lb and having a coefficient of air resistance of 1 lb-sec/ft falls out of an airplane. How close to the terminal velocity will it be after 30 seconds? How many feet will the bag have fallen at that time?*

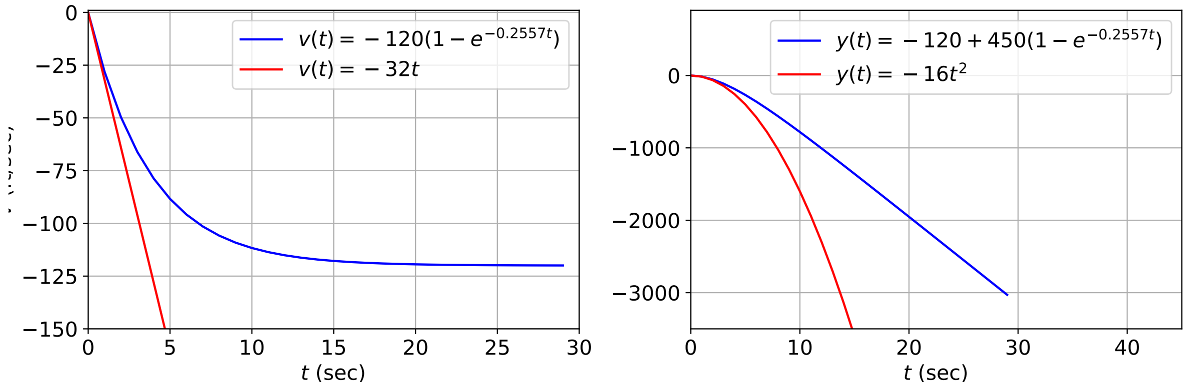


Figure 4.1. Free fall with air resistance taken into account. The velocity asymptotically approaches a constant $v(t) \approx -120$ ft/sec, and the height approaches a linear function $y(t) \approx -120t + 450$.

Solution For convenience, set $y = 0$ at the height of the airplane, so that $-y(t)$ is the distance the bag has fallen. The mass of the bag is

$$m = \frac{120}{32} \text{ slugs} = 3.75 \text{ slugs}$$

The terminal velocity is, therefore,

$$v_\infty = -\frac{mg}{k} = -120 \ \text{ft/sec}\,.$$

Substitution of these numerical values into the formulas for $v(t)$ and $y(t)$ gives

$$v(t) = -120(1 - e^{-0.2667t}) \text{ and } y(t) = -\,(120)\,t + 450(1 - e^{-(0.2667t})$$

The following table shows the difference between free fall with and without air resistance:

t (sec)	no air v (ft/sec)	air v (ft/sec)	no air y (ft)	air y (ft)
0	0.0	0.0	0.0	0.0
1	-32.0	-28.1	-16.0	-14.67
2	-64	-49.6	-144.0	-112.2
5	-160.0	-88.4	-400.0	-268.6
10	-320.0	-111.7	-1600.0	-781.3
20	-640.0	-119.4	-6400.0	-1952.2
30	-960.0	-119.96	-14,400.0	-3150.2

After 30 seconds, the velocity will be -119.96 feet/sec, almost indistinguishable from the terminal velocity; and the bag will have fallen 3150.2 feet.

4.1.4. Newton's Law of Cooling. Newton's law of cooling states that if an object is brought into an environment, then the rate of cooling is proportional to the difference between the temperature of the object and the temperature of the environment (the *ambient temperature*). This law can be reformulated as an initial value problem as follows.

Let
$$\begin{aligned} T(t) &= \text{temperature of the object at time } t,\\ T(0) = T_0 &= \text{ initial temperature of the object},\\ T_A(t) &= \text{ambient temperature}. \end{aligned}$$
Then Newton's law of cooling takes the form:
$$\frac{dT}{dt} = -k\left(T - T_A(t)\right), \quad T(0) = T_0\,, \tag{4.2}$$
where $k > 0$ is the constant of proportionality, which depends on the thermal properties of the object.

The differential equation (4.2) can we written in the form
$$\frac{dT}{dt} + kT = kT_A(t)$$
of a linear differential equation. Therefore, it can be solved by the method of integrating factors.

REMARK 4.3. Newton's law of cooling is only a rough model of how objects cool. It assumes that the temperature of the object is the same at all points in the object. In most cases, this is not the case and a more complex model, involving partial differential equations is needed.

EXAMPLE 4.5. *A cup of coffee at 200° F is brought out into a room that is kept at 60° F. Two minutes later you measure the coffee's temperature to be 180°. Find a formula for the coffee's temperature at any time t.*

SOLUTION Substituting $T(0) = 200$ and $T_A = 60$ into Equation (**??**) gives rise to the equation
$$\ln\left(\frac{T-60}{200-60}\right) = -kt\,.$$
To find k, substitute the values $t = 2$ and $T = 180$ in this formula to find
$$\ln\left(\frac{180-60}{200-60}\right) = \ln\left(\frac{120}{140}\right) = -2k \implies k = -\frac{1}{2}\ln(6/7) = \frac{1}{2}\ln(7/6) \approx 0.077$$
Consequently,
$$T(t) \approx 60 + 140e^{-0.077t}\,.$$

Let's consider a more complicated application of Newton's Law of Cooling, where separation of variables does not apply.

EXAMPLE 4.6. Let t be time in hours (with $t = 0$ at noon on Jan 1. Suppose further that during a particularly cold month of January, the outside temperature in Seattle varies between -10°C and 10°C according to the formula $T_A(t) = 10\cos\left(\frac{2\pi}{24}t\right)$. Let $T(t)$ be the temperature (in degrees C) inside a container that was left outside. Then according to Newton's law of cooling, the temperature inside the container satisfies the differential equation[11]
$$\frac{dT}{dt} = -k\left(T - T_A(t)\right) \text{ or } \frac{dT}{dt} + kT = kT_A(t)\,,$$

[11]Because $T_A(t)$ is the temperature around the container, it's the *ambient temperature* or the temperature of the environment around the container. That's why I chose to use the symbol T_A—"A" for "ambient."

where k is a measure of how well the container is insulated. Suppose further that $T(0) = 0$. Find a formula for $T(t)$. Next, find the largest value of k so that for large values of t, $T(t)$ will stay between $-2°$C and $2°$C.

Solution To simplify the computations, set $\omega = \pi/12$ and write the differential equation in the form

$$T' + kT = 10k\cos(\omega t)\ .$$

The function $T(t) = A\cos(\omega t) + B\sin(\omega t) + Ce^{-kt}$ is a solution for appropriate values of A,B, and C. To see this, substitute $T(t)$ into the differential equation and compute as follows:

$$\begin{aligned} T'(t) + kT(t) &= \omega\left\{-A\sin(\omega t) + B\cos(\omega t)\right\} + k\left\{A\cos(\omega t) + B\sin(\omega t)\right\} \\ &= (kB - \omega A)\sin(\omega t) + (\omega B + kA)\cos(\omega t) = 10k\cos(\omega t) \end{aligned}$$

For equality to hold, A and B must satisfy the equations

$$kB - \omega A = 0 \text{ and } \omega B + kA = 10k\,.$$

Solving for A and B gives

$$A = \frac{10k^2}{k^2+\omega^2} \text{ and } B = \frac{10\omega k}{k^2+\omega^2}\,.$$

Hence,

$$T(t) = \frac{10k^2}{k^2+\omega^2}\cos(\omega t) + \frac{10\omega k}{k^2+\omega^2}\sin(\omega t) + Ce^{-kt}\,.$$

Since $T(0) = 0$, it follows that that $\frac{10k}{k^2+\omega^2} + C = 0$, so

$$C = -\frac{10k^2}{k^2+\omega^2}\,.$$

But it isn't clear what the solution looks like! To obtain a better formula for $T(t)$, employ the "phase-shift" formula from Appendix A: Notice that

$$\frac{10k^2}{k^2+\omega^2}\cos(\omega t) + \frac{10\omega k}{k^2+\omega^2}\sin(\omega t) = \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\left(\frac{k}{\sqrt{k^2+\omega^2}}\cos(\omega t) + \frac{\omega}{\sqrt{k^2+\omega^2}}\sin(\omega t)\right)$$

Setting $\phi = -\arctan(B/A) = \arctan(\omega/k) = -\arctan\left(\frac{\pi}{12k}\right)$ gives the formula

$$T(t) = \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\cos(\omega t + \phi) - \frac{10k^2}{k^2+\omega^2}e^{-kt}\,.$$

To determine the value of k, notice that "in the long run" (i.e. for t large) $Ce^{-kt} \approx 0$. Ignoring the exponential term gives the approximation

$$T(t) \approx \left\{\frac{10k}{\sqrt{k^2+\omega^2}}\right\}\cos(\omega t + \phi)\,.$$

Because the exponential term vanishes quickly, it is called a *transient*; and remaining periodic term is called the the *steady state solution* of the differential equation. Thus, for t large, $T(t)$ is approximately a shifted "sine-wave" of amplitude $\dfrac{10k}{\sqrt{k^2+\omega^2}}$, and $T(t)$ will stay within $2°$C of $0°$C provided that we choose k to satisfy the inequality

$$\frac{10k}{\sqrt{k^2+\omega^2}} \le 2\,.$$

Clearly, the maximum value of k is a solution of the equation $\frac{10k}{\sqrt{k^2+\omega^2}} = 2$. Squaring and clearing denominators gives

$$100k^2 = 4(k^2+\omega^2) \quad\Longrightarrow\quad 96k^2 = 4\omega^2 \quad\Longrightarrow\quad k = \sqrt{\frac{4}{96}}\omega = \frac{\pi}{24\sqrt{6}} \approx 0.0534$$

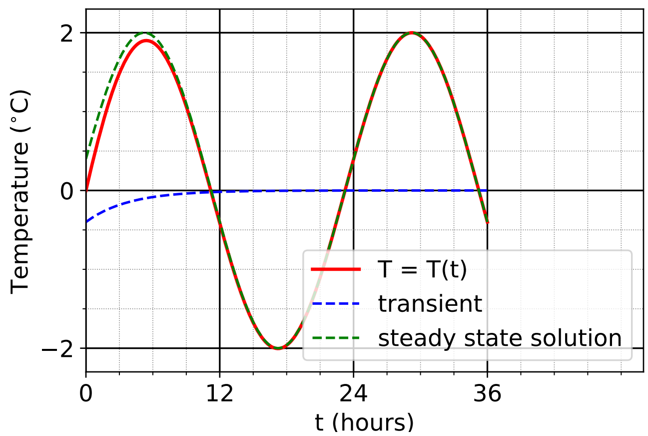


FIGURE 4.2. The graphs of the actual solution $T(t)$ when $k = 0.0534$, together with the transient Ce^{-kt} and the steady state solution.

4.2. Stability Analysis of Autonomous First Order Differential Equations

A differential equation of the form

$$\frac{dy}{dt} = F(y) \quad (F \text{ is independent of } t) \tag{4.3}$$

is called an *autonomous* differential equation. As illustrated in Figure 4.3, the direction elements of autonomous differential equations have constant slope along horizontal lines.

A point $y = y_e$ with $F(y_e) = 0$ is called an *equilibrium point* of the differential equation 4.3 (Such points are also called *critical points* or *fixed points*.) If y_e is a fixed point then the constant function $y(t) = y_e$ is a solution of the differential equation. The differential equation whose direction field is pictured in Figure 4.3 has three fixed points at $y = a$, $y = b$, and $y = c$..

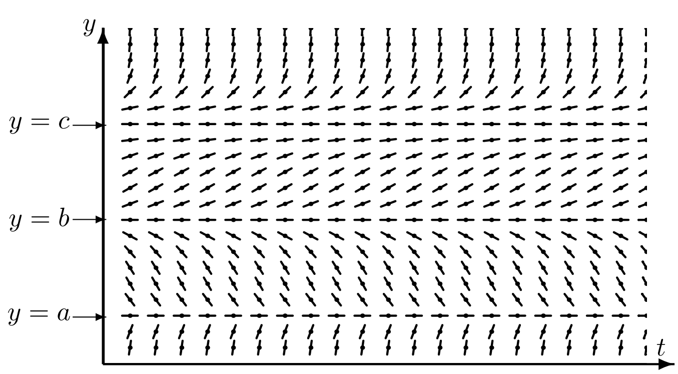


FIGURE 4.3. The direction field of an autonomous differential equation with three fixed points $y = a$ (stable), $y = b$ (unstable) and $y = c$ (semi-stable).

A fixed point of an autonomous differential equation that acts like an attractor is said to be a *stable equilibrium point*. That is, y_e is a stable equilibrium point if the solution of the initial value problem approaches y_e whenever the initial condition is sufficiently close to y_e. The equilibrium point y_e is said to be semi-stable if y_e is not a stable equilibrium point, but for some $\epsilon > 0$ the function $F(y)$ is negative for y in the interval $(y_e, y_e+\epsilon)$ or $F(y)$ is positive for all y in the interval $(y_e-\epsilon, y_e)$. If the equilibrium point y_e is neither stable nor semi-stable, then it is said to be unstable.

There is a simple way to test the stability of fixed points:

> **Criterion for stability:** *If for some $\epsilon > 0$, the function $F(y)$ is negative for y in the interval $(y_e, y_e+\epsilon)$ and positive for all y in the interval $(y_e-\epsilon, y_e)$, then y_e is a stable equilibrium point.*

Figure 4.3 illustrates the three types of equilibrium points. You should verify for yourself that $y = a$ satisfies the criterion for stability.

EXAMPLE 4.7. (a) The fixed points of the differential equation $y' = (1-y^2)$ are the points $y = 1$ and $y = -1$. Because y' is negative for $y < -1$, positive for $-1 < y < +1$, and negative for $y > 1$, it follows that $y = -1$ is an unstable fixed point and $y = +1$ is a stable fixed point.

(b) The fixed points of the differential equation $y' = \sin(\pi y)$ are the integers $y = 0, \pm 1, \pm 2, \ldots$. Do you see why the fixed points are unstable for y even and stable for y odd?

4.3. The Logistic Equation

The *logistic differential equation* is the autonomous differential equation

$$\frac{dy}{dt} = a\left(1-\frac{y}{b}\right)y\,, \tag{4.4}$$

where a and b are positive constants. The logistic equation has applications to several disciplines, including ecology, epidemiology, and medicine; and it's solution occurs numerous other areas, including machine learning, physics, and chemistry. After discussing its stability properties, we compute its solution and applications to epidemiology and population growth (where the logistic equation first appeared).

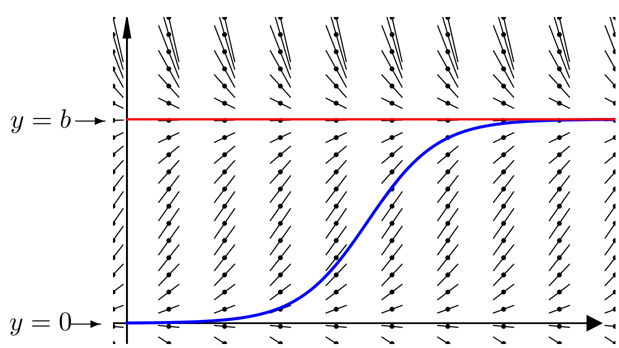


FIGURE 4.4. The direction field of the Logistic Equation differential equation $\dfrac{dy}{dt} = a\left(1-\dfrac{y}{b}\right)y$. The fixed point $y = 0$ is unstable; the fixed point $y = b$ is stable. The graph of the solution of the logistic equation is shown in blue.

As illustrated in Figure 4.4, the logistic equation has exactly two fixed points: an unstable fixed point at $y = 0$ and a stable fixed point at $y = b$. It follows that for all $y_0 > 0$, no matter how small, the solution of the initial value problem

$$\frac{dy}{dt} = a\left(1 - \frac{y}{b}\right)y\,, \quad y(0) = y_0$$

has limiting value $\lim_{t\to\infty} y(t) = b$.

We can obtain a formula for the solution of the logistic equation using separation of variables:

$$\int \frac{dy}{(1-y/b)y} = \int a dt\,.$$

The left hand side can be integrated by partial fractions:

$$\frac{1}{(1-y/b)y} = \left(\frac{1}{b-y} + \frac{1}{y}\right) \implies \int \frac{dy}{y(1-y/b)} = \int \frac{dy}{b-y} + \int \frac{dy}{y}$$

So

$$\int \frac{dy}{y(1-y/b)} = \int \frac{dy}{b-y} + \int \frac{dy}{y} = -\ln|b-y| + \ln|y| + C = \ln\left|\frac{y}{b-y}\right| + C$$

Thus $\ln\left|\frac{y}{b-y}\right| = at + C \implies \left|\frac{y}{b-y}\right| = e^{at+C} \implies \frac{y}{b-y} = Ae^{at}$. where $A = \pm e^C$. Solving for y yields an explicit formula for $y(t)$:

$$y(t) = \frac{b}{1 + e^{-at}/A}\,. \tag{4.5}$$

This function is called the *logistic function*. Its graph is the "S"-shaped curve sketched in Figure 4.4, called the *logistic curve*.

Example 4.8. (Epidemics) As a first application of the logistic equation, we return to Example 1.3 modeling epidemics, which you should review before continuing.

The model the population is divided into two sub-populations:

$$\begin{aligned} x &= \text{the proportion susceptible to infection ("well")} \\ y &= \text{the proportion infected ("sick")} \end{aligned}$$

so $x + y = 1$. The only meaningful values of x and y are those between 0 and 1, so only the region $0 \le y \le 1$ in Figure **??** has any meaning.

If $y(0) = y_0$ is the fraction of the total population that is initially infected, the spread of the epidemic was modeled by the initial value problem

$$\frac{dy}{dt} = \alpha\,(1-y)y\,, \quad y(0) = y_0\,.$$

This is the logistic equation with $a = \alpha$ and $b = 1$, which we have just solved. We now have an explicit formula to the portion of infected people at time t days after the start of the epidemic:

$$y(t) = \frac{1}{1 - e^{-\alpha t}/A}\,.$$

From $y(0) = y_0$ we find that $y_0 = \frac{1}{1-1/A}$, or $A = \frac{y_0}{y_0 - 1}$. Substituting this into the formula for $y(t)$ gives

$$y(t) = \frac{1}{1 - \frac{(y_0-1)e^{-\alpha t}}{y_0}} = \frac{y_0}{y_0 - (y_0-1)e^{-\alpha t}}\,.$$

We can explicitly compute $\lim_{t\to\infty} y(t)$ as follows:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{y_0}{y_0 - (y_0-1)e^{-\alpha t}} - \frac{y_0}{y_0} = 1\,,$$

which we already knew based on our stability analysis. Thus, (under the assumptions of the model) everyone will eventually become infected, even if initially only a small proportion of the population is infected.

Example 4.9. (The Logistic Equation for Population Growth) The logistic equation first appeared in 1838 in a paper by Pierre Verhulst, as a model of growth of population with limited resources. The most naive model of population growth is given by the differential equation

$$\frac{dP}{dt} = rP\,,$$

where P denotes the size of a population of some organism, usually measured either by total biomass or (for large numbers) by number of individuals. The solution

$$P(t) = P_0 e^{rt}$$

is clearly unrealistic because it predicts arbitrarily large values of P as t gets large. Verhulst introduced the logistic equation as the simplest model that takes limited resources into account.

In the late 1920's Raymond Pearl analyzed data collected by to determine how well the logistic equation predicted the population growth of yeast.[12] In Carlson's experiments, a small number of yeast cells were placed into a jar containing sugar, and the approximate number of yeast cells were counted each hour. Here is some of the data from the experiment:

$t =$	0	1	2	3	4	5	6	7	8	9
$Y(t) =$	10	18	19	47	71	119	175	257	351	441
$t =$	10	11	12	13	14	15	16	17	18	
$Y(t) =$	513	560	595	629	641	651	656	660	661	

Pearl conjectured that the number of yeast $Y(t)$ after t hours obeyed a differential equation of the form

$$\frac{1}{Y}\frac{dY}{dt} = R(Y)\,,$$

where $R(Y)$ is a function involving only the number of yeast. The left hand side is called the "fractional rate of growth" (or the *logarithmic growth rate*). The right hand side is called the *reproduction function*. The object of Pearl's analysis was to determine $R(Y)$.

At any t, the derivative can be approximated by a "difference quotient" $\Delta Y/\Delta t$. A better estimate can be obtained by averaging successive quotients. For example, $Y'(5)$ can be estimated by

$$Y'(5) \approx \frac{1}{2}\left(\frac{Y(6)-Y(5)}{1} + \frac{Y(5)-Y(4)}{1}\right) = \frac{Y(6)-Y(4)}{2}\,.$$

The logarithmic growth rate is then

$$\frac{1}{Y(5)}\frac{dY(5)}{dt} \approx \frac{175-71}{2\cdot 119} = 0.437\,.$$

Applying this approach to the data in the table above, and fitting a straight line to the data (see figure below) gives the formula

$$R(Y) = 0.53 - 0.00079Y = 0.53\left(1 - \frac{Y}{671}\right)$$

[12] "The growth of populations" , Raymond Pearl, Quarterly Review of Biology, 2 (1927) 532–548.

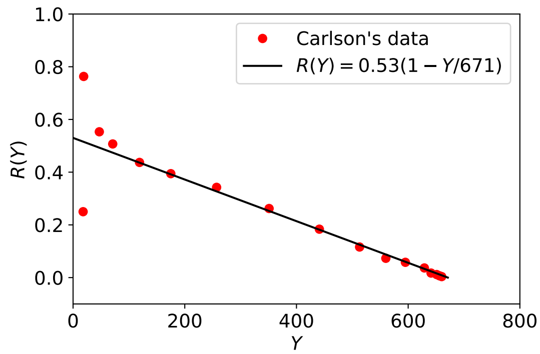


FIGURE 4.5. Carlson's data (horizontal axis: Y, vertical axis: approximation of $\frac{1}{Y}\frac{dY}{dt}$), together with best fitting line.

Given the initial population $Y(0) = 10$ yeast buds, the yeast population for $t > 0$, can be predicted by solving the initial value problem

$$\frac{dY}{dt} = 0.53\left(1 - \frac{Y}{671}\right)Y\,,\ Y(0) = 10\,,$$

which is a special case of the logistic equation with $a = 0.53$ and $b = 671$. Figure 4.6 illustrates the excellent agreement between Carslon's data and the solution of the initial value problem.

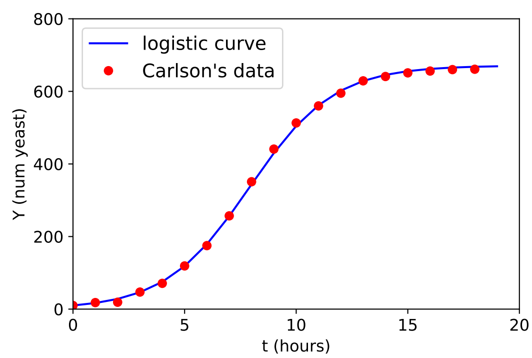


FIGURE 4.6. Logistic curve fitted to the growth of the yeast. The data is from a 1927 experiment by G.F. Gause.

A number of conclusions can be drawn from this model:

(i) For Y is small, $R(Y) \approx r = 0.53$ (*intrinsic rate of growth*) $Y(t) \approx Y_0 e^{rt}$ for Y small.

(ii) For $Y > 671$, $R(Y)$ is negative and the population decreases with time. The parameter $b = 671$ is called the *carrying capacity* of the environment.

(iii) For any initial value of Y, $\lim_{t\to\infty} Y(t) = b \approx 671$.

The book by G. Evelyn Hutchinson, *An Introduction to Population Ecology*, New Haven and London, Yale University Press, 1978, pages 23–32, presents data suggesting that many animal and plant populations obey the logistic model of growth.

4.4. The Harmonic Oscillator and Conservation of Energy

The *harmonic oscillator* discussed in Example 1.2 consists of an object of mass m attached to a spring and free to move to the right and left without friction. If x denotes the amount the spring has stretched relative to its equilibrium position, then by Hooke's law (see Figure 1.1) the force on the object is $-kx$; and Newton's second law "$F = ma$" takes the form

$$m\frac{d^2x}{dt^2} = -kx\,. \tag{4.6}$$

This differential equation is among the most important ones in physics and engineering, and it will play a central role in this course. Some physicists would claim that it is the <u>most important</u> differential equation in all of physics!

In any case, it's worth studying. Equation (4.6) is a second order differential equation. Beginning in Chapter 6, we will introduce techniques to solving second order differential equations directly, and in Chapter 8 we will study the harmonic oscillator using those techniques. In this section, we view the harmonic oscillator from a different perspective, using the principle of *conservation of energy.*

The first step is rewrite (4.6) in terms of velocity. Remember that both x and v are functions of t (time), that is $x = x(t)$ and $v = v(t)$. (We can't say more since we don't yet know the formulae for $x(t)$ and $v(t)$.) Remember also that $v(t) = \dfrac{dx(t)}{dt}$. So $\dfrac{d^2x}{dt^2} = \dfrac{dv(t)}{dt}$, and we can therefore write (4.6) this way:

$$m\frac{dv(t)}{dt} + kx(t) = 0\,.$$

Now comes a trick: Multiply by $v(t) = \frac{dx(t)}{dt}$ to get

$$mv(t)\frac{dv(t)}{dt} + kx(t)\frac{dx(t)}{dt} = 0\,.$$

The terms on the left can we rewritten in a nicer form:

$$mv(t)\frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv(t)^2\right) \text{ and } mx(t)\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}kx(t)^2\right)\,.$$

Using this shows that $v(t)$ and $x(t)$ satisfy the identity

$$\frac{d}{dt}\left(\frac{1}{2}mv(t)^2 + \frac{1}{2}kx(t)^2\right) = 0\,.$$

It follows that the quantity between the parentheses is constant:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E\,, \tag{4.7}$$

where E is some positive constant. If you took high school physics, you know that $\frac{1}{2}mv^2$ is the *kinetic energy* of the mass. And if you took Math 125 and studied "work" you know that $\dfrac{1}{2}kx^2$ is the energy stored in the spring, called the *potential energy.* So Equation (4.7) says that the sum of the kinetic energy and the potential energy is constant. We have discovered a special case of the law of *conservation of energy*!

Figure 4.7) gives a nice way to visualize how $(x(t), v(t))$are related: they trace out an ellipse in the (x, v)-plane.

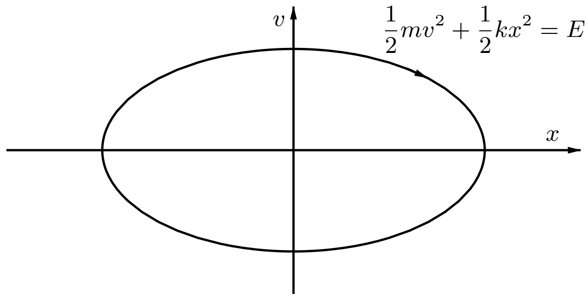


FIGURE 4.7. Conservation of energy states that curve $(x(t), v(t))$ in the (x, v)-plane is an ellipse.

But more is true! Remember that $v = \dfrac{dx}{dt}$. So we can write (4.7) this way:

$$m\left(\frac{dx}{dt}\right)^2 + kx^2 = 2E\,.$$

This is a first order differential equation that is satisfied by $x = x(t)$. We have succeeded in turning a second order differential equation into a first order differential equation, which we can try to solve by separation of variables:

$$\sqrt{m}\,\frac{dx}{dt} = \pm\sqrt{2E - kx^2} = \pm\sqrt{k}\sqrt{2E/k - x^2} \implies \frac{1}{\sqrt{2E/k - x^2}}\,\frac{dx}{dt} = \pm\sqrt{\frac{k}{m}}\,.$$

Integration yields

$$\sin^{-1}\left(\sqrt{\frac{k}{2E}}x\right) = \pm\sqrt{\frac{k}{m}}t + \phi\,,$$

where ϕ is a constant of integration. Solving for x yields the equation

$$x(t) = \sqrt{\frac{2E}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t + \phi\right)\,.$$

So $x(t)$ is a sine function with frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$.

REMARK 4.4. At this point, you should compare what we just found with Example 1.2 in the Introduction.

Exercise 4.1. Suppose that the population of a certain species grows at the instantaneous rate of 2% per year (i.e., its instantaneous rate of increase in number of individuals per year is 2% of the population at the moment). Let $y(t)$ stand for the population after t years.

(a) Write a differential equation for $y(t)$.
(b) Write the general solution to the differential equation you found in part (a).
(c) If the present population is 1,000,000, what will the population be in 1 year? In 20 years? How long will the population take to double?

Exercise 4.2. Assume that the population of the Earth changes at an instantaneous rate proportional to the population. Assume further that at time $t = 0$ (1650 CE) its population was 250 million and at time $t = 300$ (1950 CE) its population was 2.5 billion. Find an expression giving the population of the Earth at any time. If the greatest population that the Earth can support is 25 billion when will this limit be reached?

Exercise 4.3. At time $t = 0$ you buy a house, using a fixed-rate, fixed payment mortgage to pay for most of it. Let $y(t)$ be the amount you owe after t years. Thus, $y(0) = y_0$ is the cost of the house minus the down-payment.

(a) Write the differential equation for the amount $y(t)$ you owe after t years.
(b) Solve the differential equation to find a formula for $y(t)$
(c) Suppose that the bank will give you a 30-year mortgage at 4% interest, but they only consider you an acceptable credit risk if your monthly payments do not exceed one fourth of your $2,500 monthly salary. Compute the maximum mortgage they'll give you.

Exercise 4.4. Newton's law of cooling also applies when a colder object heats up in a warmer environment. Suppose water at 55°F is pumped into a swimming pool on a 90° summer day. After 2 hours the temperature of the water is 60°. In how many hours (assuming the outside temperature remains 90°) will the water reach a comfortable swimming temperature of 70°?

Exercise 4.5. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 gal of a dye solution with a concentration of 1 gm/gal. To prepare for the next experiment, the tank is to be rinsed with <u>fresh</u> water flowing in at a rate of 2 gal/min, the well- stirred solution flowing out at the same rate.

(a) Write a differential equation for y, the amount of dye in the tank after t minutes.
(b) Find $y(t)$.
(c) What is the "half-life of $y(t)$?
(d) How much time elapses before the concentration of dye in the tank reaches 1% of its original value?

Exercise 4.6. Suppose that a room containing 1200 cubic feet of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke containing 4 percent carbon monoxide is introduced into the room at a rate of 0.1ft^3/min and the well-circulated mixture is allowed to leave the room at the same rate.

(a) Find an expression for the concentration of carbon monoxide in the room at any time $t > 0$.
(b) Extended exposure to a carbon monoxide concentration as low as 0.00012 (i.e. 0.012%) is harmful to the human body. Find the time at which this concentration is reached.

Exercise 4.7. Unlike the case of an object moving at very low speeds, at higher speeds, the drag on an object traveling in the atmosphere is proportional to the *square* of the speed. Suppose that such a projectile is initially falling at 500 miles per hour and that 30 seconds later it is traveling at 200 miles per hour.

(a) Write down the appropriate initial value problem and solve it.
(b) How fast is the projectile traveling after one minute?
(c) How far will it travel in the first minute?

EXERCISE 4.8. Consider the logistic population model $\dfrac{dP}{dt} = 2\left(1 - \dfrac{P}{1000}\right)P$ for a species of fish in a lake, were t is measured in years and $P(t)$ is the number of fish in the lake at time t. Suppose that it is decided that fishing will be allowed in the lake, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 5 fish per year.

(a) What is the largest number of licenses that can be issued if the fish are to have a chance to survive in the lake?
(b) Suppose the number of fishing licenses in part (a) are issued. What will happen to the fish population—that is, how does the behavior of the population depend on the initial population?
(c) The simple population model above can be thought of as a model of an ideal fish population that is not subject to many of the environmental problems of an actual lake. For the actual fish population, there will be occasional changes in the population that were not considered in the building of the model. If the water level were high because of a heavy rainstorm, a few extra fish might be able to swim down a usually dry stream bed to reach the lake; or the extra water might wash toxic waste into the lake, killing a few fish. Given the possibility of unexpected perturbation of the population, not included in the model, what do you think will happen to the actual fish population if we fix the fishing level at the one determined in part (b)?

EXERCISE 4.9. Exercise 1.1 involved a boater and a motor boat, which together weighed 640 lbs. The thrust of the motor was equal to a constant force of 20 lb. in the direction of motion, and the resistance of the water to the motion was assumed to be equal numerically to twice the speed in feet per second. The boat was initially at rest. (a) Under those assumptions, solve the differential equation you found in 1.1 to find the formula for the velocity $v(t)$ of the boat at time t. (b) What is the limiting velocity?

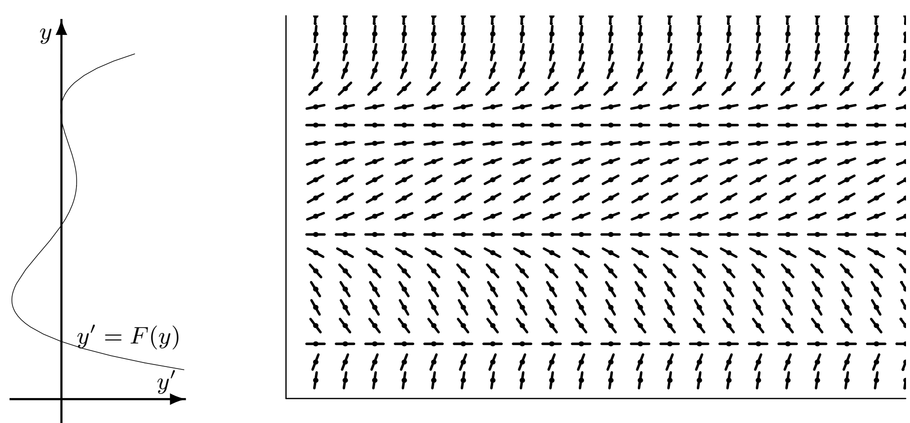


FIGURE 4.8. Flipping the graph of $F(y)$ along the diagonal and aligning it with the direction field illustrates the relation between the graph of $F(y)$ and the direction field of the differential equation.

EXERCISE 4.10. Consider the differential equation $y' = F(y)$ where the graph of $F(y)$ is shown in Figure 4.8 *Locate the equilibrium points in Figure 4.8 and determine which ones are stable and which ones are not stable. It may help to sketch a few solution curves.*

Part 2

Second Order Differential Equations

CHAPTER 5

Introduction to Second Order Differential Equations

A *second order ordinary differential equation* is an equation of the form

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right), \tag{5.1}$$

where $F(t, y, y')$ is a function of t, y and y'. A *solution* is a function $y = y(t)$ satisfying the equality

$$y''(t) = F\left(t, y(t), y'(t)\right)$$

for all t in some interval.

As was the case for the general first order differential equation, second order differential equations have many solutions. As mentioned in the introduction, differential equations have many solutions; additional information is needed to determine a unique solution. Usually, this is in the form of *initial conditions* of the form

$$y(t_0) = y_0 \text{ and } y'(t_0) = y_0',$$

specifying the value of the solution and its derivative at a fixed time $t = t_0$. A function $y = y(t)$ satisfying the differential equation together with the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ is a solution of the *initial value problem*

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

The remainder of these notes is devoted to the development of techniques for solving second order differential equations and initial value problems.

5.1. Special Cases

There are two cases where a second order differential equation can be solved by reducing it's solution to solving a sequence of two first order differential equations. In the first case, the second derivative is independent of t, and the differential equation is of the form

$$\frac{d^2y}{dt^2} = F\left(y, \frac{dy}{dt}\right). \tag{5.2a}$$

In the second case, the second derivative is independent of y and the differential equation is of the form

$$\frac{d^2y}{dt^2} = F\left(t, \frac{dy}{dt}\right). \tag{5.2b}$$

To solve (5.2a), let $v = dy/dt$, and use the chain rule to write the second derivative in the form $\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$. Equation (5.2a) can now be rewritten as the first order differential equation in the independent variable y, rather than t:

$$v\frac{dv}{dy} = F(y, v).$$

Suppose that $v = V(y, C_1)$ is the solution. Then

$$\frac{dy}{dt} = v = V(y, C_1)\,.$$

This is another first order differential equation that can be solved by separation of variables:

$$\int \frac{dy}{V(y, C_1)} = t + C_2\,.$$

EXAMPLE 5.1. Consider the equation $\dfrac{d^2y}{dt^2} = -ky$. Let $v = y'$. Then, $v\dfrac{dv}{dy} = -ky$, which we integrate to get $\dfrac{1}{2}v^2 = -\dfrac{k}{2}y^2 + C_1$. To make the computations simpler, let $C_1 = \dfrac{k}{2}A^2$, where $A > 0$ is another constant. Then $v^2 = k(A^2 - y^2)$. Solving for v gives

$$\frac{dt}{dt} = \sqrt{k}\sqrt{A^2 - y^2}\,,$$

which we can solve using separation of variables:

$$y(t) = A \sin\left(\sqrt{k}(t + \phi)\right)\,,$$

where A and ϕ are constants. We used a variant of this technique in Section 4.4 to study the harmonic oscillator using the principle of conservation of energy.

The approach to solving (5.2b) is similar. We again set $v = \frac{dy}{dt}$. The equation now becomes

$$\frac{dv}{dt} = F(t, v)\,.$$

This is a first order differential equation. If $v = V(t, C_1)$ is a solution, then

$$\frac{dy}{dt} = v = V(t, C_1)\,,$$

which can be solved directly by integration:

$$y(t) = \int V(t, C_1)\, dt\,.$$

EXAMPLE 5.2. To solve the differential equation $y'' + t(y')^2 = 0$, let $v = y'$. Then $v' + tv^2 = 0$, which can be solved by separation of variables:

$$\int \frac{dv}{v^2} = -\int t dt \quad\implies\quad -\frac{1}{v} = -\frac{t^2}{2} + C_1 \quad\implies\quad y' == \frac{1}{t^2/2 + C_1} = \frac{2}{t^2 + 2C_1}\,.$$

Consequently,

$$y(t) = 2\int \frac{1}{t^2 + 2C_1}\, dt = \frac{2}{\sqrt{2C_1}} \arctan\left(\frac{t}{\sqrt{2C_1}}\right) + C_2$$

EXERCISE 5.1. Consider a flat metal washer described in polar coordinates by $1 \le r \le 2$. If the inner boundary of the washer is held at temperature $T = 50°C$ and the outer boundary at $T = 100°C$ for a long time so that the washer reaches thermal equilibrium, the temperature T of at a point in the washer will depend only on the distance r from the center and it can be shown that it satisfies the second order differential equation

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0\,.$$

Find the temperature distribution $T = T(r)$. **Hint:** *Let $y(r) = T'(r)$ and solve a first order differential equation for $y(r)$.*

5.2. Linear Second Order Differential Equations

The most common class of differential equations occurring in applications is the class of *linear* second order differential equations. These are differential equations that can be written in the form

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y = f(t)\,,$$

where the functions $p(t)$, $q(t)$ and $f(t)$ are usually assumed to be continuous or piecewise continuous on an interval $a < t < b$. The function $f(t)$ is called a *forcing function.*When $f(t) = 0$, the differential equation becomes

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y = 0$$

and the differential equation is called a *homogeneous* linear differential equation. When the forcing function $f(t)$ is not zero the equation is said to be *nonhomogeneous.*.

Notice the similarity between the form of a linear second order differential equation and the form of a linear first order differential equation:

$$\frac{dy}{dt} + p(t)\,y = f(t)\,.$$

This suggests that some of the techniques for solving first order linear differential equations might be useful for solving second order differential equations.

(NOTATION) Rather than writing out the full (and rather long) expression

$$\frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y\,,$$

it is often convenient to use the shorthand notation $L[y]$ for the left-hand side:

$$L[y] = \frac{d^2y}{dt^2} + p(t)\,\frac{dy}{dt} + q(t)\,y.$$

For instance, if $L[y] = y'' + 4y$. Then we write

$$L[\sin(2t)] = 0$$

rather than

$$(\sin(2t))'' + 4(\sin(2t))\,.$$

The object L is called a *linear operator* because it satisfies the identity

$$L[C_1y_1(t) + C_2y_2(t)] = C_1L[y_1(t)] + C_2L[y_2(t)] \tag{5.3}$$

for any two functions $y_1(t)$ and $y_2(t)$ and any constants C_1 and C_2.

This property of L is called the *superposition principle*. The superposition principle states that if $y_1(t)$ and $y_2(t)$ are two solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then so is the linear combination $C_1y_1(t) + C_2y_2(t)$. This follows from Equation (5.3). For suppose $L[y_1(t)] = 0$ and $L[y_2(t)] = 0$. Then

$$L[C_1y_1(t) + C_2y_2(t)] = C_1L[y_1(t)] + C_2L[y_2(t)] = C_1 \cdot (0) + C_2 \cdot (0) = 0\,.$$

The superposition principle shows that by taking linear combinations we can build complicated solutions out of simple solutions. This fact underlies many of the computations done in physics and engineering.

We can verify Equation (5.3) by computing:

$$\begin{aligned} L[C_1y_1(t)+C_2y_2(t)] &= (C_1y_1(t)+C_2y_2(t))'' + p(t)(C_1y_1(t)+C_2y_2(t))' + q(t)(C_1y_1(t)+C_2y_2(t)) \\ &= (C_1y_1''(t)+C_2y_2''(t)) + p(t)(C_1y_1'(t)+C_2y_2'(t)) + q(t)(C_1y_1(t)+C_2y_2(t)) \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} &= C_1(y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)) + C_2(y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)) \\ &= C_1L[y_1(t)] + C_2L[y_2(t)] \end{aligned}$$

The following theorem is the main theoretical result about second order linear differential equations:

THEOREM 2. *Suppose that $p(t)$, $q(t)$ and $f(t)$ are continuous on the interval $a < t < b$. Suppose further that t_0 is between a and b. Then the initial value problem*

$$L[y] = y'' + p(t)y' + q(t)y = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

has one and only one solution defined on the entire interval $a < t < b$.

Some bad news. This theorem does NOT tell us how to find the solution—it only confirms that a solution exists and that it is unique. Even worse, unlike the first order linear differential equation, there is no general formula for the solution of the most general second order initial value problem.

Some good news. There are general techniques for finding a solution in the most commonly occurring case where $p(t)$ and $q(t)$ are constants and the differential equation assumes the special form

$$L[y] = y'' + by' + cy = f(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,, \tag{5.4}$$

where b and c are constants.

The remainder of these notes is devoted to the study of initial value problems of the special form (5.4).

CHAPTER 6

Complex Numbers

Complex numbers are routinely used in electrical, mechanical, and aeronautical engineering applications, as well as in physical chemistry, and almost all areas of physics. In particular, the techniques we are going to introduce for solving linear second order differential equations depend on a good understanding of complex numbers. In this chapter, we present only the relevant parts of complex arithmetic and complex calculus that we need.

6.1. Complex Numbers

A *complex number* z is given by a pair of real numbers x and y and is written in the form[13] $z = x + yi$, where i satisfies $i^2 = -1$. [14] If $z = x + yi$, then the term x is called the *real part* of z and written $x = \operatorname{Re} z$. The term y is called the *imaginary part* of z and written $y = \operatorname{Im} z$. Thus,

$$\operatorname{Re}(4+5i) = 4 \text{ and } \operatorname{Im}(4+5i) = 5\,.$$

Remember: $\operatorname{Im} z$ is a *real* number!

Complex numbers are added in a natural way: If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \tag{6.1}$$

For example, $(4 + i) + (2 + 3i) = (6 + 4i)$. Complex numbers are also multiplied in a natural way:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \tag{6.2}$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2+i)(-2+4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i\,.$$

There is only one way to satisfy the equality $z_1 = z_2$, namely, if $x_1 = x_2$ <u>and</u> $y_1 = y_2$. An equivalent statement (one that is important to keep in mind) is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If a is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what one would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$).

Division is more complicated. To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1. The rule for finding the reciprocal of $z = x + yi$ is given by:

$$\frac{1}{x+yi} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x-yi}{(x+yi)(x-yi)} = \frac{x-yi}{x^2+y^2} \tag{6.3}$$

[13] At times, it is more convenient to write $z = x + iy$, rather than $z = x + yi$. Both forms are used in these notes.

[14] Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

For instance,

$$\frac{1}{3+4i} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i\,.$$

Using the formula for the product of complex numbers gives

$$(3+4i)\left(\frac{3}{25} - \frac{4}{25}i\right) = \frac{9+16}{25} + \frac{(3)(-4)+4(3)}{25} = 1 + 0i = 1\,,$$

as one would expect!

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the *complex conjugate* of $z = x + iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3+4i} = 3-4i$, as illustrated in Figure 6.1(a). Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$.

Another important quantity associated with a given complex number z is its *modulus*

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2+y^2} = \left((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\right)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$.

The modulus of a complex number is a generalization of the notion of the absolute value of a real number, as the following example illustrates:

$$|-3| = |(-3)+0i| = ((-3)^2 + (0)^2)^{1/2} = (9)^{1/2} = 3\,.$$

6.2. The Complex Plane

The complex numbers, as well as various operations involving complex numbers have elegant geometric descriptions. The complex numbers may be represented as points in the plane (sometimes called the Argand diagram) or as vectors. The real number 1 is represented by the point $(1,0)$, and the complex number i is represented by the point $(0,1)$. The x-axis is called the "real axis", and the y-axis is called the "imaginary axis".

Complex conjugation is given by reflection about the real axis, as illustrated in Figure 6.1(a). Addition of complex numbers is given by the *parallelogram rule*, as illustrated in Figure 6.1(b).

The geometric description of multiplication involves both a rotation and a stretch. As illustrated in Figure 6.2, to visualize the product $z_1 z_2$, construct a triangle with vertices 0, 1 and z_1 (red triangle at left of figure). Then construct a similar triangle where the "base" edge from 0 to 1 is replaced by the segment from 0 to z_2 (red triangle at right of figure). Then the vertex opposite the base is the product $z_1 z_2$. By high school geometry, one can show that the coordinates of the product are given by Equation (6.2).

Exercise 6.1.

(1) Compute the product $(x+iy)\left(\dfrac{x-iy}{x^2+y^2}\right)$,.

(2) Write each of the following in the form $a + bi$:

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i)$ (b) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

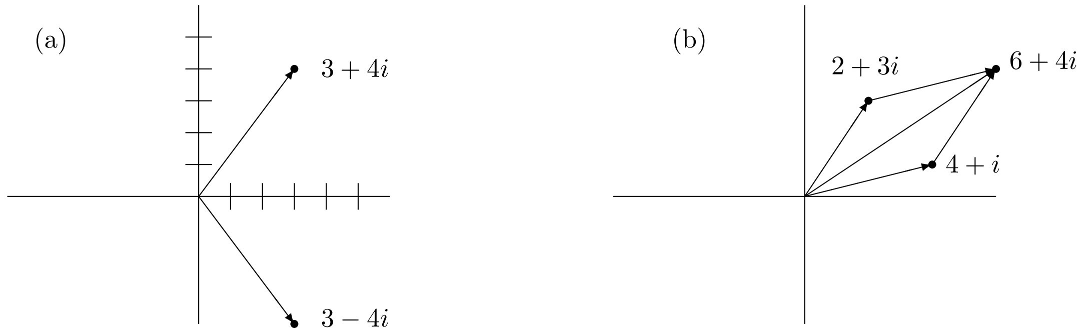


FIGURE 6.1. (a) The complex numbers $3-4i$ and $3+4i$ are complex conjugates of one another. (b) The complex number $6+4i$ is the sum of $2+3i$ and $4+i$.

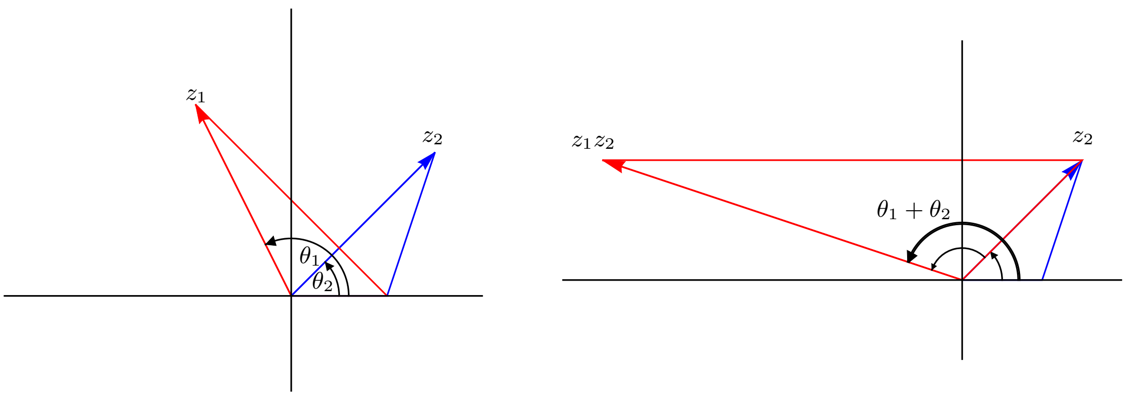


FIGURE 6.2. Geometric description of complex multiplication. The red triangles in the figure are similar, with "bases" of lengths 1 and $|z_2|$, respectively. By high school geometry, one can show that $|z_1z_2| = |z_1||z_2|$. The angle that the product z_1z_2 makes with the positive real axis is the sum of the angles that z_1 and z_2 make with the positive real axis.

(c) $\dfrac{5}{(1-i)(2-i)(3-i)}$ (d) $(1-i)^4$

(3) Prove the following:
  (a) $z + \overline{z} = 2\mathrm{Re}\, z$ and z is a real number if and only if $\overline{z} = z$.
  (b) $\overline{z_1z_2} = \overline{z}_1\overline{z}_2$.
(4) Prove that $|z_1z_2| = |z_1||z_2|$ (Hint: Use (3b).)
(5) Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

6.3. Polar Representation of Complex Numbers

Points in the plane can be represented by both rectangular and polar coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) is

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan(\theta) = \frac{y}{x}$$

Thus, (See Figure 6.3) the complex number $z = x + iy$ can we written in the form:

$$z = r\,(\cos(\theta) + i\sin(\theta)) = re^{i\theta}\,, \quad (\textit{Polar Representation}), \tag{6.4}$$

where

$$r = \sqrt{x^2 + y^2} = |z| \text{ and } \tan(\theta) = \frac{y}{x}\,. \tag{6.5}$$

The angle θ is called the *argument* of the complex number z. It is often denoted by $\arg(z)$.

Example 6.1. The complex number $z = 8 + 6i$ may also be written as $re^{i\theta}$, where $r = \sqrt{8^2 + 6^2} = 10$ and $\theta = \arg(8 + 6i) = \arctan(6/8) \approx 0.64$ radians.

Remark 6.1. In formula (6.4), we are defining $e^{i\theta}$ to be $\cos(\theta) + i\sin(\theta)$. We justify this definition later in these notes.

Remark 6.2. (Caution) There is ambiguity in equation (6.5) about the inverse tangent, which can (and must) be resolved by looking at the signs of x and y, respectively, in order to determine the quadrant in which θ lies. If $x > 0$, then $\theta = \arctan(y/x)$. If $x < 0$, then $\theta = \arctan(y/x) \mp \pi$, depending on the sign of y. When $x = 0$, then $\theta = \pm\pi/2$, depending on the sign of y. (If $z = 0$, then $r = 0$ and θ can be anything.)

If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

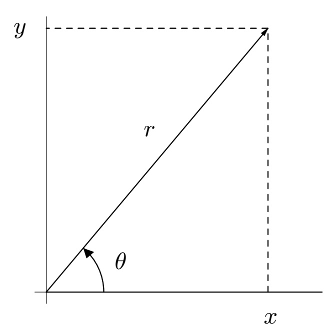


Figure 6.3. The Polar Representation: $x + yi = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = y/x$.

Remark 6.3. The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k,\ k = 0, \pm 1, \pm 2, \ldots$.

Example 6.2. For instance, $i = e^{i\pi/2} = e^{i5\pi/2}\,, \quad -1 = e^{\pi i} = e^{3\pi i}\,,$ and $+1 = e^{0i} = e^{(0+2\pi)i} = e^{2\pi i}$.

Example 6.3. If $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$, therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle that differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus, $-4 + 4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$.

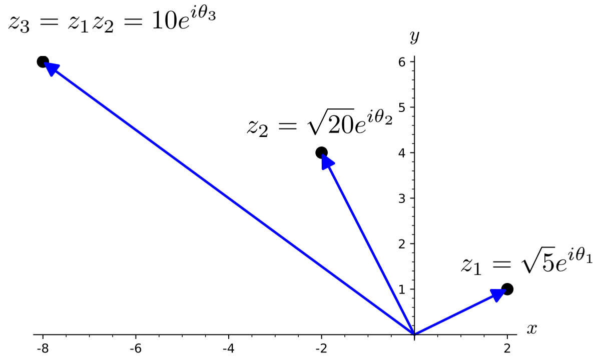


FIGURE 6.4. Complex multiplication in polar form.

As illustrated in Figure 6.2, complex multiplication involves both a stretch and a rotation. The polar representation gives another way to describe of complex multiplication:

$$\text{if } z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}, \quad \text{then} \quad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}\,. \tag{6.6}$$

For example, let

$$z_1 = 2 + i = \sqrt{5}e^{i\theta_1},\ \theta_1 \approx 0.464 \quad \text{and} \quad z_2 = -2 + 4i = \sqrt{20}e^{i\theta_2},\ \theta_2 \approx 2.034$$

Then $z_3 = z_1 z_2$, where $z_3 = -8 + 6i = \sqrt{100}e^{i\theta_3}$ $\theta_3 \approx 2.498$. (see Figure 6.4)

EXERCISE 6.2.

(1) Let $z_1 = 3i$ and $z_2 = 2 - 2i$
  (a) Plot the points z_1, z_2, $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.
  (b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
  (c) Express z_1 and z_2 in polar form.
(2) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, $z_1 z_2$ and z_1/z_2.
(3) Let $z = re^{i\theta}$. Show that $\overline{z} = re^{-i\theta}$ and $z^{-1} = r^{-1}e^{-i\theta}$.

6.4. Complex-valued Functions

Now suppose that $w = w(t)$ is a complex-valued function of the real variable t. That is

$$w(t) = u(t) + iv(t)\,,$$

where $u(t)$ and $v(t)$ are real-valued functions. A complex-valued function can be thought of a defining a curve in the complex plane. The derivative of $w(t)$ with respect to t is *defined* to be the function

$$w'(t) = u'(t) + iv'(t) = \frac{dw(t)}{dt}$$

(This is just like the definition of the derivative of a vector-valued function—just differentiate the components.) The derivative $w'(t)$ can be thought of as the tangent to that curve $w(t)$.

It is easily checked (just expand the left and right hand sides of each identity) that, just as in the case of real-valued functions, the following formulas hold for complex-valued functions $z = z(t)$ and $w = w(t)$:

$$C' = 0\,, \text{ where } C = \text{constant} \tag{6.7a}$$

$$(Cz)' = Cz'\,, \text{ where } C = \text{constant} \tag{6.7b}$$

$$(z + w)' = z' + w' \qquad \text{(the sum rule)} \tag{6.7c}$$

$$(zw)' = z'w + zw' \qquad \text{(the product rule)} \tag{6.7d}$$

$$\left(\frac{z}{w}\right)' = \frac{z'w - zw'}{w^2} \qquad \text{(the quotient rule)} \tag{6.7e}$$

In other words, the derivatives of complex-valued functions behave the same as the derivatives of real valued functions.

Example 6.4. The complex-valued function

$$f(t) = \cos(t) + i\sin(t)$$

is of particular interest. When viewed as a curve in the complex plane, it defines a circle. It has two important properties:

$$\text{(i)} f(t)f(s) = f(t + s)$$
$$\text{(ii)} f'(t) = if(t)\,.$$

To verify (i), compute as follows using the sum of angle formulas from trigonometry:

$$\begin{aligned} f(t)f(s) &= (\cos(t) + i\sin(t))(\cos(s) + i\sin(s)) \\ &= (\cos(t)\cos(s) - \sin(t)\sin(s)) + (\cos(t)\sin(s) + \sin(t)\cos(s))i \\ &= \cos(t + s) + \sin(t + s)i = f(t + s)\,. \end{aligned}$$

To verify (ii), compute as follows from the definition of the derivative:

$$z'(t) = -\sin(t) + i\cos(t) = i(\cos(t) + i\sin(t)) = iz(t)\,.$$

Because (i) and (ii) are also satisfied by the exponential function: e^{rt}:

$$e^{rt}e^{st} = e^{(r+s)t} \text{ and } (e^{rt})' = re^{rt}\,,$$

the same notation is used here:

$$e^{it} = \cos(t) + i\sin(t)\,. \tag{6.8}$$

This is called *Euler's Formula.*. With this notation (i) and (ii) assume the forms

$$e^{it}e^{is} = e^{i(t+s)} \text{ and } \left(e^{it}\right)' = i\,e^{it}\,. \tag{6.9}$$

6.5. The complex exponential function

One function is of particular interest to us: the *complex exponential function*. It is defined as follows:

$$e^{(\rho+i\omega)t} = e^{\rho t}e^{i\omega t} = e^{\rho t}(\cos(\omega t) + i\sin(\omega t)) = e^{\rho t}\cos(\omega t) + ie^{\rho t}\sin(\omega t)\,. \tag{6.10}$$

Thought of as a curve in the complex plane, the complex exponential is the formula for a spiral curve (Figure 6.5). The quantity ω is the angular velocity of the spiral ($\omega > 0$ corresponds to a counterclockwise spiral, $\omega < 0$ to a clockwise one). The quantity ρ measures the rate at which the spiral expands outward ($\rho > 0$) or contracts inward ($\rho < 0$).

As the following examples illustrate, functions of the form

$$f(t) = C_1e^{\rho t}\cos(\omega t) + C_2e^{\rho t}\sin(\omega t)$$

can be rewritten in terms of the complex exponential function.

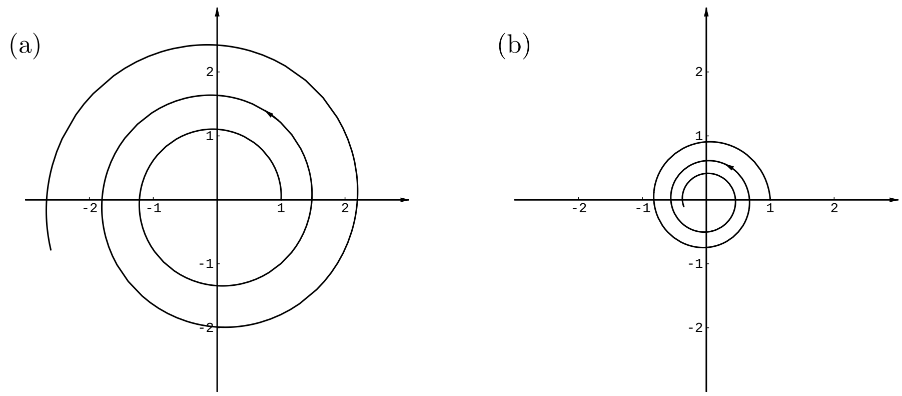


FIGURE 6.5. (a) $z(t) = e^{(\rho+i\omega)t}$, $\rho > 0$, $\omega > 0$. (b) $z(t) = e^{(\rho+i\omega)t}$, $\rho < 0$, $\omega > 0$.

EXAMPLE 6.5. Show that $5e^{-4t}\cos(3t) + 3e^{-4t}\sin(3t) = \text{Re}\left((5-3i)e^{(-4+3i)t}\right)$.

SOLUTION By definition,

$$\begin{aligned}(5-3i)e^{(-4+3i)t} &= (5-3i)e^{-4t}(\cos(3t) + i\sin(3t)) \\ &= e^{-4t}\left((5\cos(3t) + 3\sin(3t)) + i(5\sin(3t) + 3\cos(3t))\right) .\end{aligned}$$

Hence, $\text{Re}\left((5-3i)e^{(-4+3i)t}\right) = e^{-4t}\left(5\cos(3t) + 3\sin(3t)\right)$.

REMARK 6.4. In polar form $5 + 3i = \sqrt{34}\exp(\arctan(3/5)i)$. Hence, we can compute as follows:

$$\begin{aligned}\text{Re}\left((5-3i)e^{(-4+3i)t}\right) &= \text{Re}\left(\overline{(5+3i)}e^{(-4+3i)t}\right) \\ &= \sqrt{34}e^{-4t}\text{Re}\left(\sqrt{34}e^{-\arctan(3/5)i}e^{3i)t}\right) \\ &= \sqrt{34}e^{-4t}\text{Re}\left(\exp((3t - \arctan(3/5))i)\right) \\ &= \sqrt{34}\,e^{-4t}\cos\left(3t - \arctan(3/5)\right) .\end{aligned}$$

EXAMPLE 6.6. Express $\text{Re}\left(\dfrac{1}{3+3i}e^{(6+4i)t}\right)$ in the form $Ae^{\rho t}\cos(\omega t + \phi)$.

SOLUTION Since $3 + 3i = 3\sqrt{2}e^{(\pi/4)i}$, it follows that

$$\text{Re}\left(\frac{1}{3+3i}e^{(6+4i)t}\right) = \text{Re}\left(\frac{1}{3\sqrt{2}e^{(\pi/4)i}}e^{6t}e^{4it}\right)\ \frac{e^{6t}}{3\sqrt{2}}\,\text{Re}\left(e^{(4t-\pi/4)i}\right) = \frac{1}{3\sqrt{2}}e^{6t}\cos\left(4t - \pi/4\right)$$

To find the derivative of the complex exponential function, compute the derivatives of the real and imaginary parts and collecting terms to obtain the formula

$$\left(e^{(\rho+i\omega)t}\right)' = (\rho + i\omega)e^{(\rho+i\omega)t} .$$

In other words, *even for $r = \rho + i\omega$, the formula*

$$\frac{d}{dt}e^{rt} = re^{rt} \tag{6.11}$$

holds!

More generally, if $z(t) = x(t) + iy(t) = Ce^{(\rho+i\omega)t}$, where $C = C_1 + iC_2$, then clearly

$$z'(t) = C \cdot (\rho + i\omega)e^{(\rho+i\omega)t} \text{ and } z''(t) = C \cdot (\rho + i\omega)^2 e^{(\rho+i\omega)t}\,..$$

On the other hand, from the definition of the derivative

$$z'(t) = x'(t) + iy'(t)\,,$$

gives a simple way to compute derivatives of

$$x(t) = \mathrm{Re}\,(z(t)) = (C_1 \cos(\omega t) - C_2 \sin(\omega t))e^{\rho t} \tag{6.12a}$$

and

$$y(t) = \mathrm{Im}\,(z(t)) = (C_1 \sin(\omega t) + C_2 \cos(\omega t))e^{\rho t} \tag{6.12b}$$

the real and imaginary parts of $z(t)$:

$$x'(t) = \mathrm{Re}\,\left(C \cdot (\rho + i\omega)e^{(\rho+i\omega)t}\right) \text{ and } y'(t) = \mathrm{Im}\,\left(C \cdot (\rho + i\omega)e^{(\rho+i\omega)t}\right)\,. \tag{6.12c}$$

$$x''(t) = \mathrm{Re}\,\left(C \cdot (\rho + \omega i)^2 e^{(\rho+i\omega)t}\right) \text{ and } y''(t) = \mathrm{Im}\,\left(C \cdot (\rho + \omega i)^2 e^{(\rho+i\omega)t}\right)\,. \tag{6.12d}$$

EXAMPLE 6.7. *Consider the function*

$$x(t) = (5\cos(2t) + 4\sin(2t))e^{-t/5}\,.$$

Graph $x(t)$, then compute its first and second derivatives.

SOLUTION Observe that $x(t) = \mathrm{Re}\,(z(t))$ with $z(t) = (5 - 4i)e^{(-1/5+2i)t}$. To graph $x(t)$, write $z(t)$ in polar form: $(5 - 4i) = Ae^{i\phi}$, with $A = \sqrt{5^2 + 4^2} = \sqrt{41} \approx 6.40$ and $\phi = -\arctan(4/5) \approx -0.67$. Hence,

$$z(t) = Ae^{i\phi}e^{((1/5)+i2)t} = Ae^{-t/5}e^{(2t\phi)i}$$

From this, it follows that

$$x(t) = Ae^{-t/5}\cos(2t + \phi) = Ae^{-t/5}\cos(2(t + \phi/2)) \approx 6.40e^{-t/5}\cos(2(t - 0.38))\,,$$

from which one can more easily visualize the graph (see Example A.1 in Appendix A).

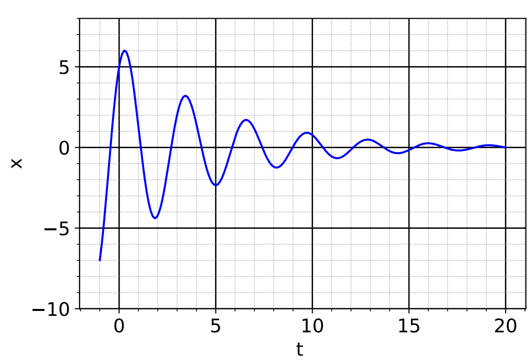


FIGURE 6.6. The graph of $x(t) = Ae^{-t/5}\cos(2t + \phi)$.

One could, of course, compute the first and second derivatives of $x(t)$ directly from the original formula, but that would be tedious. It's easier, however, to first compute the derivatives of $z(t)$ and then to take the real part to obtain the derivatives of $x(t)$. Here's the computation:

Since

$$z'(t) = (5-4i)(-1/5+2i)e^{(-1/5+2i)t} = \left(7 + \frac{54}{5}i\right) e^{(-1/5+2i)t},$$
$$x'(t) = \left(7\cos(2t) - \frac{54}{5}\sin(2t)\right) e^{-t/t}.$$

Since

$$z''(t) = (5-4i)(-1/5+2i)^2 e^{(-1/5+2i)t} = \left(-23 + \frac{296}{25}i\right) e^{(-1/5+2i)t},$$
$$x''(t) = -\left(23\cos(2t) + \frac{296}{25}\sin(2t)\right) e^{-t/t}.$$

EXAMPLE 6.8. Evaluate the definite integral $\int_0^1 (3\cos(2t) - 4\sin(2t))e^{5t}\, dt$.

SOLUTION Observe that $(3\cos(2t) - 4\sin(2t))e^{5t} = \mathrm{Re}\left((3+4i)e^{(5+2i)t}\right)$. We can now compute as follows:

$$\begin{aligned}
\int_0^1 (3\cos(2t) - 4\sin(2t))e^{5t}\, dt &= \mathrm{Re}\left(\int_0^1 (3+4i)e^{(5+2i)t}\, dt\right) \\
&= \mathrm{Re}\left(\frac{3+4i}{5+2i} e^{(5+2i)t}\Big]_0^1\right) = \mathrm{Re}\left(\frac{3+4i}{5+2i}(e^{(5+2i)} - 1)\right) \\
&= \mathrm{Re}\left(\left(\frac{23}{29} + \frac{14}{29}i\right)\left(e^5(\cos(2) + i\sin(2)) - 1\right)\right) \\
&= \mathrm{Re}\left(\left(\frac{23}{29} + \frac{14}{29}i\right)\left((e^5\cos(2) - 1) + ie^5\sin(2)\right)\right) \\
&= \frac{23}{29}(e^5\cos(2) - 1) - \frac{14}{29}e^5\sin(2) \approx -114.9
\end{aligned}$$

EXERCISE 6.3.

(1) Sketch the graph of the curve $z(t) = (2+2i)e^{\left(\frac{1}{2}+\pi\, i\right)t}$ for $0 \le t \le 3$ in the complex plane.
(2) Write the function $x(t) = 3e^{-2t}\cos(4t) + 5e^{-2t}\sin(4t)$ in each of the forms $x(t) = \mathrm{Re}\left(Ce^{rt}\right)$ and $x(t) = Ae^{\rho\, t}\cos(\omega\, t + \phi)$, where A, ω and ϕ are real numbers and C and r are complex numbers.
(3) Using the complex exponential function, compute the second derivative of the function $x(t) = (2\cos(4t) - 3\sin(4t))e^{-t}$ Check your answer by also computing the second derivative directly.
(4) Evaluate the definite integral $\int_0^{\pi} e^{t/\pi}\sin(t)\, dt$.

CHAPTER 7

Solving Homogeneous Linear Differential Equations

The goal of this chapter is to understand how to solve initial value problems of the form

$$L[y] = ay'' + by' + cy = 0\,, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'\,. \tag{7.1}$$

Suppose that we have succeeded in finding two solutions of the equation $L[y] = 0$, say $y_1(t)$ and $y_2(t)$. Then, by the *superposition principal* (see Equation (5.3)) the "linear combination"

$$y = C_1y_1(t) + C_2y_2(t)$$

is also a solution of $L[y] = 0$. Let's verify that this is the case:

$$\begin{aligned}
L[y] &= L[C_1y_1 + C_2y_2] \\
&= a(C_1y_1 + C_2y_2)'' + b(C_1y_1 + C_2y_2)' + c(C_1y_1 + C_2y_2) \\
&= a(C_1y_1'' + C_2y_2'') + b(C_1y_1' + C_2y_2') + c(C_1y_1 + C_2y_2) \\
&= C_1(ay_1'' + by_1' + cy_1) + C_2(ay_2'' + by_2' + cy_2) \\
&= C_1L[y_1] + C_2L[y_2] \\
&= C_1 \cdot 0 + C_2 \cdot 0 = 0
\end{aligned}$$

Therefore, once we have found two solutions $y_1(t)$ and $y_2(t)$, we can construct lots of solutions by taking linear combinations. This fact is the basis for the following general strategy for solving <u>any</u> initial value problem of the form

$$L[y] = 0\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,.$$

(i) First find two <u>independent</u> solutions of the differential equation

$$L[y] = ay'' + by' + cy = 0\,.$$

Call them $y_1(t)$ and $y_2(t)$.

(ii) Form the *general solution* $y(t) = C_1y_1(t) + C_2y_2(t)$

(iii) Solve the system of equations

$$\left\{\begin{array}{l} C_1y_1(t_0) + C_2y_2(t_0) = y_0 \\ C_1y_1'(t_0) + C_2y_2'(t_0) = y_0' \end{array}\right. \tag{7.2}$$

for the unknowns C_1 and C_2.

In examples, it is usually easy to solve for C_1 and C_2 by hand. However, the formulas:

$$C_1 = \frac{y_0\, y_2'(t_0) - y_0'\, y_2(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)} \qquad C_2 = \frac{y_0'\, y_1(t_0) - y_0\, y_1(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)}$$

for C_1 and C_2 are also occasionally useful. The denominator is called the *Wronskian* of $y_1(t)$ and $y_2(t)$.

REMARK 7.1. The functions $y_1(t)$ and $y_2(t)$ found in step (i) cannot be scalar multiples of one another! That is what we meant by the condition that the solutions $y_1(t)$ and $y_2(t)$ are *independent solutions*. The pair $y_1(t)$, $y_2(t)$ is called a *fundamental basis* of solutions.

The fundamental basis is not unique! *There are MANY fundamental bases of solutions for a given homogeneous linear differential equation.*

EXAMPLE 7.1. Consider the differential equation

$$y'' - y = 0\,.$$

The functions e^t and e^{-t} form a fundamental basis of solutions. But so do the functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \text{ and } \cosh(t) = \frac{e^t + e^{-t}}{2}\,.$$

Yet another fundamental basis is the pair of functions

$$\sinh(t-3) \text{ and } \cosh(t-3)\,.$$

Choosing the right fundamental system can often simplify the solution of initial value problems. For instance, consider the IVP

$$y'' - y = 0 \quad y(3) = 11 \quad y'(3) = 13\,.$$

The function $y(t) = C_1 \cosh(t-3) + C_2 \sinh(t-3)$ is the general solution of $y'' - y = 0$. From this, it is easy to determine C_1 and C_2:

$$y(3) = C_1 \cosh(0) + C_2 \sinh(0) = C_1 = 11$$

and

$$y'(3) = C_1 \sinh(0) + C_2 \cosh(0) = C_2 = 13\,.$$

Hence,

$$y(t) = 11 \cosh(t-3) + 13 \sinh(t-3)\,.$$

Of course, the function $y(t)$ can also be expressed in terms of the fundamental system e^t and e^{-t} as can be seen by expanding as follows:

$$\begin{aligned} y(t) &= 11 \cosh(t-3) + 13 \sinh(t-3) \\ &= \frac{11}{2}(e^{t-3} + e^{-t+3}) + \frac{13}{2}(e^{t-3} - e^{-t+3}) = 12e^{-3}e^t - e^3 e^{-t}\,. \end{aligned}$$

7.1. The Characteristic Polynomial

To solve the homogeneous differential equation

$$L[y] = ay'' + by' + cy = 0$$

we need to find two independent solutions. To do this, we will look for solutions of the special form $y = e^{rt}$. Let's compute $L[e^{rt}]$:

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) = (ar^2 + br + c)\, e^{rt}\,.$$

This shows that e^{rt} is a solution of the differential equation if r is a solution of the quadratic equation

$$ar^2 + br + c = 0\,. \tag{7.3}$$

We have reduced the problem of solving the differential equation to a problem in algebra. The polynomial $ar^2 + br + c$ is called the *characteristic polynomial* of the differential equation, and the equation (7.3) is called the *characteristic equation*.

Let r_1 and r_2 be the roots of the characteristic polynomial. There are three cases to consider:

(i) r_1 and r_2 are both real and $r_1 \neq r_2$ ($b^2 > 4ac$).
(ii) $r_1 = r_2$ ($b^2 = 4ac$).
(iii) r_1 and r_2 are complex ($b^2 < 4ac$).

In case (i), the two functions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ clearly form a fundamental system of solutions. But this fails in case (ii), where the characteristic polynomial has only one root. In case (iii), the roots are complex. We consider each case separately.

7.2. Distinct Real Roots of the Characteristic Polynomial

This is the easiest case: the functions $e^{r_1 t}$ and $e^{r_2 t}$ are independent; and, therefore,

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{7.4}$$

is the general solution of the differential equation.

EXAMPLE 7.2. Solve the initial value problem

$$L[y] = y'' - 3y' + 2y = 0 \text{ and } y(0) = 0\,, \quad y'(0) = 1\,.$$

SOLUTION Substituting $y = e^{rt}$ into the equation $L[y] = 0$ gives

$$(r^2 - 3r + 2) \cdot e^{rt} = 0\,,$$

This implies that r satisfies the quadratic equation

$$r^2 - 3r + 2 = (r-2)(r-1) = 0\,.$$

Therefore, $y = e^t$ and $y = e^{2t}$ are two independent solutions of the differential equation and

$$y(t) = C_1 e^t + C_2 e^{2t}$$

is the general solution. The initial conditions are

$$y(0) = C_1 + C_2 = 0$$

and

$$y'(0) = C_1 + 2C_2 = 1\,.$$

It follows that $C_1 = -1$ and $C_2 = 1$. Therefore,

$$y(t) = -e^t + e^{2t}\,.$$

EXAMPLE 7.3. Solve the initial value problem

$$L[y] = y'' - 4y = 0\,, \quad y(0) = 7\,, \quad y'(0) = 8\,.$$

SOLUTION First find the general solution of the differential equation by looking for solutions of the form $y = e^{rt}$. Since $L[e^{rt}] = (r^2 - 4) \cdot e^{rt}$, $r = \pm 2$ and the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-2t}\,.$$

The initial conditions give

$$C_1 + C_2 = 7 \text{ and } 2C_1 - 2C_2 = 8\,,$$

which are easily solved to give

$$C_1 = \frac{11}{2} \text{ and } C_2 = \frac{3}{2}\,.$$

Hence, $y(t) = \dfrac{11}{2} e^{2t} + \dfrac{3}{2} e^{-2t}$.

EXAMPLE 7.4. Solve the initial value problem:

$$y'' - y = 0 \quad y(3) = 11, \quad y'(3) = 13\,.$$

SOLUTION The roots of the characteristic polynomial $r^2 - 1$ are ± 1, hence the general solution is

$$y = C_1 e^t + C_2 e^{-t}\,.$$

The initial conditions give

$$y(3) = C_1 e^3 + C_2 e^{-3} = 11 \quad y'(3) = C_1 e^3 - C_2 e^{-3} = 13\,.$$

Solving for C_1 and C_2 gives:

$$C_1 = 12e^{-3} \text{ and } C_2 = -e^3\,.$$

The solution of the initial value problem is, therefore,

$$y = (12e^{-3})\,e^t - e^3\,e^{-t} = 12e^{(t-3)} - e^{-(t-3)}\,.$$

EXERCISE 7.1. Solve each of the following differential equations and initial value problems.

(1) $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 1$.
(2) $y'' - 4y' + 3y = 0$, $y(1) = 0$, $y'(1) = 1$.
(3) $y'' + 4y' + 3y = 0$.
(4) $y'' - 3y = 0$.

7.3. Repeated Roots of the Characteristic Polynomial

EXAMPLE 7.5. Consider the differential equation

$$y'' + 2y' + y = 0\,.$$

The characteristic polynomial is

$$r^2 + 2r + 1 = (r+1)^2\,,$$

which has only one root $r = -1$. Therefore, the function e^{-t} is a the only solution of the differential equation of the form e^{rt}. We need a second independent solution.

Fortunately, one can check directly that te^{-t} is also a solution:

$$\begin{aligned} L[te^{-t}] &= (te^{-t})'' + 2(te^{-t})' + (te^{-t}) \\ &(t-2)e^{-t} + 2(1-t)e^{-t} + te^{-t} \\ &(t - 2t + t)e^{-t} + (-2+2)e^{-t} = 0\,. \end{aligned}$$

Since te^{-t} is not a constant multiple of e^{-t}, the general solution is

$$y(t) = C_1\,e^{-t} + C_2\,te^{-t} = (C_1 + C_2\,t)e^{-t}\,.$$

This idea works in all cases where the characteristic polynomial has a double root. For suppose that the characteristic polynomial factors has the double root r_0. Then

$$ar^2 + br + c = a(r - r_0)^2 = a(r^2 - 2r_0 r + r_0^2)\,.$$

One solution of the differential equation is $e^{r_0 t}$. To see that $te^{r_0 t}$ is another, compute as follows:

$$\begin{aligned} L[te^{r_0 t}] &= a\left\{(te^{r_0 t})'' - 2r_0(te^{r_0 t})' + r_0^2(te^{r_0 t}\right\} \\ &= a\left\{(2r_0 + r_0^2 t)e^{r_0 t} - a2r_0(e^{r_0 t} + r_0 te^{r_0 t}) + r_0^2 te^{r_0 t}\right\} \\ &= a\left\{(r_0^2 - 2r_0^2 + r_0^2)te^{r_0 t} + (2r_0 - 2r_0)e^{r_0 t}\right\} = 0 \end{aligned}$$

Thus, the general solution is

$$y = C_1\,e^{r_0 t} + C_2\,te^{r_0 t} = (C_1 + C_2\,t)\,e^{r_0 t}\,. \tag{7.5}$$

EXAMPLE 7.6. Find the solution of the initial value problem.

$$y'' - 6y' + 9y = 0\,, \quad y(2) = 3\,, \quad y'(2) = 0\,.$$

SOLUTION The characteristic polynomial factors as

$$r^2 - 6r + 9 = (r-3)^2\,.$$

So the general solution is $y = (C_1 + C_2 t)e^{3t}$. The initial conditions give

$$(C_1 + 2C_2)e^6 = 3\,, \quad (3C_1 + 7C_2)e^6 = 0$$

Solving for C_1 and C_2 gives $C_1 = 21e^{-6}$, $C_2 = -9e^{-6}$. Hence,

$$y(t) = (21 - 9t)e^{3t-6}\,.$$

EXERCISE 7.2. Solve each of the following differential equations and initial value problems.

(1) $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 1$.
(2) $y'' - 4y' + 4y = 0$, $y(1) = 0$, $y'(1) = 1$.
(3) $y'' + 4y' + 4y = 0$.
(4) $y'' - 2y' + y = 0$.

7.4. Complex Roots of the Characteristic Polynomial

It remains to consider case (iii) where the characteristic polynomial of the differential equation

$$L[y] = ay'' + by' + cy = 0$$

has complex roots. Specifically, suppose $b^2 < 4ac$, then the two complex roots are

$$\rho \pm \omega i = \left(-\frac{b}{2a}\right) \pm i\left(\frac{\sqrt{4ac - b^2}}{2a}\right)\,.$$

Therefore, the pair of functions

$$e^{(\rho+\omega i)t} = e^{\rho t}\cos(\omega t) + ie^{\rho t}\sin(\omega t)\,, \quad e^{(\rho-\omega i)t} = e^{\rho t}\cos(\omega t) - ie^{\rho t}\sin(\omega t)$$

forms a fundamental basis of solutions. The pair of functions

$$e^{\rho t}\cos(\omega t) = \frac{1}{2}e^{(\rho+\omega i)t} + \frac{1}{2}e^{(\rho-\omega i)t}\,, \quad e^{\rho t}\sin(\omega t) = \frac{1}{2i}e^{(\rho+\omega i)t} - \frac{1}{2i}e^{(\rho-\omega i)t}$$

also forms a fundamental basis. This implies that the general solution is

$$y(t) = e^{\rho t}(C_1\cos(\omega t) + C_2\sin(\omega t)) = \mathrm{Re}\left((C_1 - C_2 i)e^{(\rho+\omega i)t}\right)\,.$$

EXAMPLE 7.7. *Find the solution of the initial value problem*

$$y'' - 4y' + 13y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 4\,.$$

SOLUTION The characteristic polynomial is (completing the square)

$$r^2 - 4r + 13 = (r-2)^2 - 4 + 13 = (r-2)^2 + 9\,,$$

whose roots are $2 \pm 3i$. Therefore, the general solution is

$$y(t) = \mathrm{Re}\left((C_1 - C_2 i)e^{(2+3i)t}\right) = e^{2t}(C_1\cos(3t) + C_2\sin(3t))\,.$$

Since

$$y'(t) = \mathrm{Re}\left((C_1 - C_2 i)(2+3i)e^{(2+i3)t}\right)$$

the initial conditions are

$$\begin{aligned} y(0) &= \mathrm{Re}\ (C_1 - C_2 i) = C_1 = 1 \\ y'(0) &= \mathrm{Re}\ ((C_1 - C_2 i)(2+3i)) = 2C_1 + 3C_2 = 4\,. \end{aligned}$$

Hence, $C_1 = 1$ and $C_2 = \dfrac{4-2C_1}{3} = \dfrac{2}{3}$. Therefore, the solution to the initial value problem is

$$y(t) = \mathrm{Re}\ \left((1 - \frac{2}{3} i) e^{(2+3i)t} \right) = \left(\cos(3t) + \frac{2}{3}\sin(3t) \right) e^{2t}\,.$$

EXERCISE 7.3. Write the solution of the initial value problem

$$y'' + 25y = 0 \qquad y(0) = 1 \quad y'(0) = 2\,.$$

in the form $y(t) = \mathrm{Re}\,\left(Ce^{(\rho+i\omega)t}\right)$. By converting C to polar form write the solution in the form

$$y(t) = Ae^{\rho t}\cos(\omega t + \phi)\,,$$

where A and ϕ are real numbers determined by the initial conditions. Graph the solution.

EXERCISE 7.4. The function

$$y(t) = 2e^{-t}\cos(5t) + 3e^{-t}\sin(5t)$$

is the solution of the initial value problem

$$y'' + 2y' + 26 = 0 \quad y(0) = 2\,,\; y'(0) = 13\,.$$

In the form above, it is difficult to graph. Rewrite it in each of the two forms

$$y(t) = \mathrm{Re}\,(Ce^{rt}) \text{ and } y(t) = Ae^{-at}\cos(\omega t + \phi)\,,$$

where C and r are complex numbers that you have to determine, and A, ω and ϕ are real numbers that you also have to determine. Sketch the solution.

EXERCISE 7.5. Solve each of the following differential equations and initial value problems.

(1) $y'' + 2y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$.
(2) $y'' - 4y' + 7y = 0$, $y(1) = 0$, $y'(1) = 1$.
(3) $y'' + 4y' + 8y = 0$.
(4) $y'' + 6y' + 13y = 0$.

CHAPTER 8

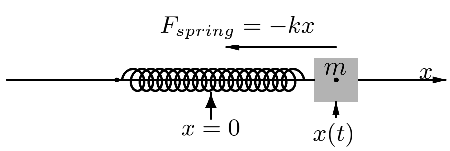
The Harmonic Oscillator

The harmonic oscillator (see Example 1.2) is the mechanical system consisting of an object of mass m attached to a spring with spring constant k. If x denotes the amount that the spring is stretched relative to its equilibrium position, then by Newton's second law of motion, the function $x = x(t)$, t = time, is a solution of the linear differential equation

$$mx'' + kx = 0\,.$$

Because m and k are both positive,

$$x'' + \omega_0^2\, x = 0\,, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}\,. \tag{8.1}$$



Since $\pm\omega_0\, i$ are the roots of the characteristic polynomial $r^2 + \omega_0^2$, the general solution is

$$x(t) = \mathrm{Re}\left(Ce^{i\omega_0 t}\right),\ \text{where } \omega_0 = \sqrt{\frac{k}{m}}\,.$$

If we write the complex number C in the polar form $C = Ae^{i\phi}$, the solution wan be written in the form

$$x(t) = \mathrm{Re}\left(Ae^{i(\omega_0 t + \phi)}\right) = A\cos(\omega_0 t + \phi) = A\cos(\omega_0(t - t_0))\,.\ \text{where } t_0 = -\frac{\phi}{\omega_0}\,, \tag{8.2}$$

in which it is easier to visualize the graph of $x(t)$. (See Figure 8.1.) In particular, we can see that the spring-mass system oscillates at frequency ω_0 and period $T = \dfrac{2\pi}{\omega_0}$.

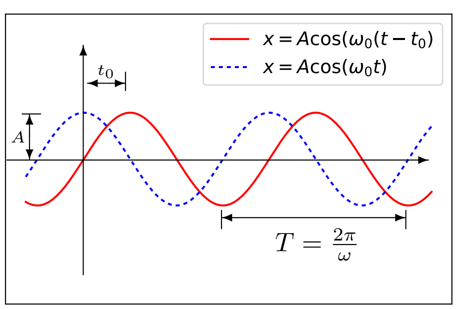
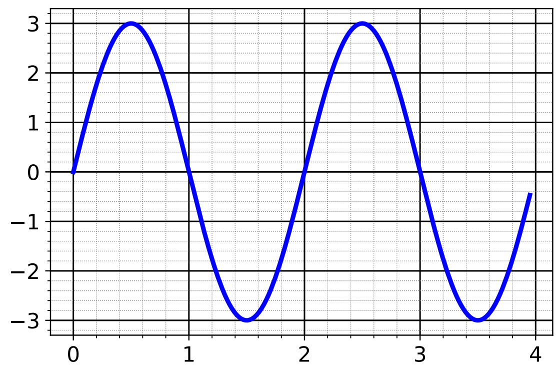


FIGURE 8.1. The graph of $x(t) = A\cos(\omega_0(t - t_0))$, $t_0 > 0$.

EXERCISE 8.1. The graph below shows the motion of an harmonic oscillator. Determine the values of A, ω_0, and ϕ. Choose $-\pi < \phi \le \pi$.



Recall the law of conservation of energy, which we introduced in Section 4.4 to solve the harmonic oscillator. It gives additional insight into the dynamics of the mass-spring system. The Energy of the system is split between the kinetic energy of the object and the potential energy of the spring. In Section 4.4, we showed that the sum of kinetic energy and potential energy

$$\text{K.E.} + \text{P.E.} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

is constant. We can check this directly:

$$\begin{aligned} \text{K.E} &= \frac{1}{2}mx'(t)^2 = \frac{1}{2}m\left(-A\omega_0 \sin(\omega_0 t + \phi)\right)^2 = \frac{A^2}{2}k^2 \sin^2(\omega_0 t + \phi) \\ \text{P.E} &= \frac{1}{2}kx(t)^2 = \frac{1}{2}k\left(A\cos(\omega_0 t + \phi)\right)^2 = \frac{A^2}{2}k^2 \cos^2(\omega_0 t + \phi) \end{aligned}$$

So,

$$\begin{aligned} \text{K.E.} + \text{P.E.} &= \frac{A^2}{2}k^2 \sin^2(\omega_0 t + \phi) + \frac{A^2}{2}k^2 \cos^2(\omega_0 t + \phi) \\ &= \frac{1}{2}A^2k^2(\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)) = \frac{1}{2}A^2k^2\,, \end{aligned}$$

a constant. As the speed of the object increases, so does its kinetic energy; and conservation of energy forces the potential energy of the spring to decrease; and as the speed of the object decreases, the potential energy of the spring increases. Energy is therefore transferred back and forth between the object and the spring, as shown in Figure 8.2.

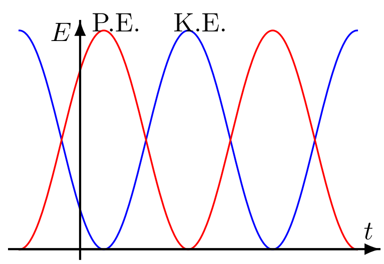


FIGURE 8.2. Conservation of energy for the Harmonic Oscillator. The potential energy of the spring is the red curve. The kinetic energy of the object is the blue curve.

8.1. LC-circuits

The mass-spring system above is only one of many systems modeled on the differential equation (8.2). The electrical circuit described here is another. The exercises at the end of this section present others.

Figure 8.3) shows an electrical circuit with two components: an inductor and a capacitor. As current moves through the circuit, an electric charge q accumulates on the plates of the capacitor and forms an electric field between the plates. At the same time, the current I in inductor, which is a coil of wire wrapped around a metal core (see Figure 1.2) forms a magnetic field.

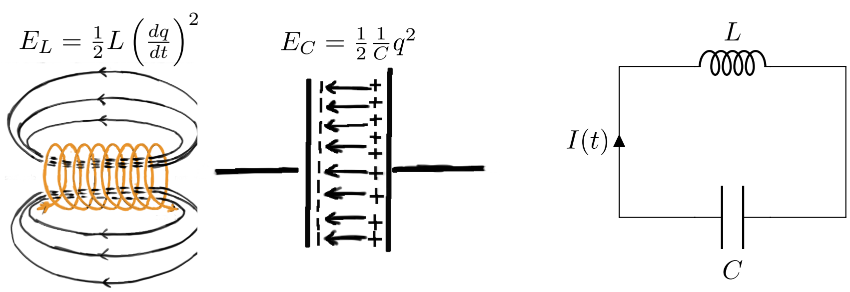


FIGURE 8.3. A current $I = q'$ flowing in the circuit shown on the right flows through the coil of the inductor (shown on the left) and generates a magnetic field. A charge q accumulates on the plates of the capacitor, generating an electric field. The circuit is called an *LC-circuit*.

Recall that Kirchhoff's law states that the sum of the voltage drops around a closed circuit sums to zero, so applying the formulas (1.7) for the voltage drops across the inductor and the capacitor gives the differential equation

$$Lq'' + \frac{1}{C}q = 0\,, \tag{8.3}$$

which governs the behavior of the charge $q(t)$ on the capacitor. Apart from a change of symbols, the differential equation (8.3) is the same as the differential equation (8.1). Therefore, the charge on the capacitor is also modeled by a function of the form

$$q(t) = A\cos(\omega_0 t + \phi)\,,$$

where in this case $\omega_0 = \sqrt{1/LC}$. Consequently, in an LC-circuit, the charge on the capacitor and the voltage drop across it both oscillate with frequency $\omega_0 = \sqrt{1/LC}$ and period $T = 2\pi\sqrt{LC}$.

As in the mass-spring system, energy is stored both in the electric field and in the magnetic field. The energy stored in the electric field of the capacitor is given by the formula

$$E_C = \frac{1}{2}\frac{1}{C}q^2$$

where q is the charge on the plates of the capacitor; and the energy storied in the magnetic field of the inductor is given by the formula

$$E_L = \frac{1}{2}LI^2 = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2$$

where L is the inductance of the inductor and $I = q'$ is the current. As in the mass-spring system, energy flows back and forth between the magnetic field and the electric field, with the sum of energies

$$E_L + E_C = L\left(\frac{dq}{dt}\right) + \frac{1}{2}\frac{1}{C}q^2$$

remaining constant.

EXAMPLE 8.1. Suppose you want to design an LC-circuit in which the current oscillates at 60 Hertz (cycles per second). Suppose further that you have only one inductor with inductance of $9100\,\mu\,\mathrm{H}$. (μH = micro Henries) What capacitance should you choose for the capacitor?

SOLUTION If the frequency is 60 Hertz, then $\omega_0 = 2\pi \cdot 60 = 120\pi\,\mathrm{sec}^{-1}$. Since $\omega_0 = 1/\sqrt{LC}$,

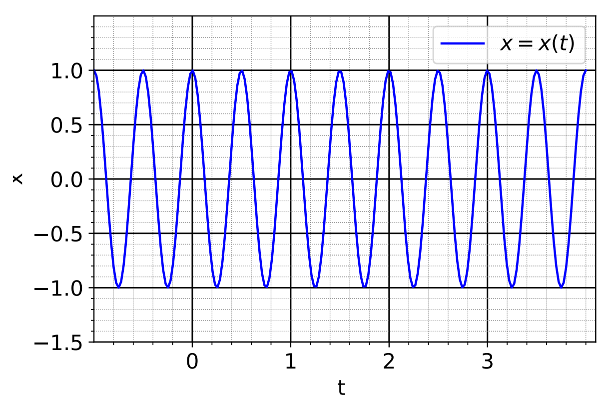
$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(120\pi)^2(9100 \times 10^{-6})}\,\mathrm{F} \approx 773.2\mu\mathrm{F}\,,$$

(μ F = micro Farads)

EXERCISE 8.2. A weight of mass $m = 5$ kg is suspended from a spring with unknown spring constant k. The weight is free to move up and down. Ignoring friction, its position relative to its equilibrium position satisfies a differential equation of the form

$$mu'' + ku = 0\,,$$

where t denotes time measured in seconds. To find the spring constant, the spring is set in motion and the graph of $u(t)$ plotted. The result is shown in the following figure. The horizontal axis is t (measured in seconds), vertical axis is u (measured in meters):



(a) The function $u = u(t)$ is the solution of an initial value problem. What are the initial conditions? That is, what are the values of $u(0)$ and $u'(0)$?
(b) What is the period T measured in seconds?
(c) What is the spring constant k? (*Your answer can most easily be expressed in terms of* π.)

EXERCISE 8.3. A cylindrical log of radius 1/10 meter, 5 meters in length, and with a mass of 50 kilograms is placed vertically in a lake so that it is free to bob up and down. Assume that there is no water resistance. A weight of 50 kilograms of negligible volume is attached to the bottom of the log so that it remains vertical (so the total mass of the log and weight together is 100 kilograms). The mass density of water is 1000 kilograms per cubic meter. (For convenience, assume that the acceleration due to gravity is $g = 10$ meters per sec^2 (*It is actually closer to* 9.81.)

There are two forces acting on the log: gravity and the buoyant force of the water. The buoyant force can be computed from Archimedes' principle:

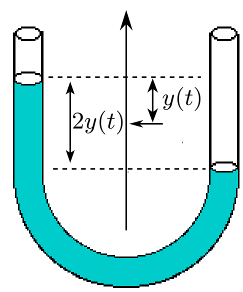
> *An object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.*

Let t be time in seconds and let $d(t)$ denote the depth (in meters) of the bottom of the log .

(a) Compute the depth d_{eq} of the log in its equilibrium position, i.e. when the magnitude of the buoyant force is exactly equal the combined weight (in Newtons) of the log plus the mass. **Hint:** Draw a good picture!
(b) Write down a differential equation for $d(t)$.
(c) Now let $y(t) = d(t) - d_{eq}$, the displacement of the log from its equilibrium position. Assuming that $y(0) = 1$ meters and $y'(0) = 0$ meters/sec, write down an initial value problem for y.
(d) Solve the initial value problem you wrote down in part (c).

Exercise 8.4. Consider a "U"-shaped tube filled with liquid Mercury as shown in the figure below. The radius of the tube is 1 centimeter (so its diameter is 2 centimeters). There are 500 grams of Mercury in the tube. Liquid Mercury has a mass density of 13.5 grams per cubic centimeter. The mercury in the tube will oscillate with a certain period T, measured in seconds. Your task is to compute T by completing the following steps:

(a) Let $y(t)$ be the height above its equilibrium position of the liquid surface at the left vertical segment of the tube. (At equilibrium, both surfaces are at the same height above sea level and $y = 0$. When $y(t) < 0$ the right surface is higher than the left surface.) The only force acting on the mass of fluid in the tube is due to gravity. Compute the total force on the fluid (vertical component only).
(b) Use formula you found in part (a) to find a linear, homogeneous, constant coefficient, second order differential equation for $y = y(t)$.
(c) Solve the differential equation you found in part (a).
(d) What is the period T? Hints: Treat the fluid as a single rigid body. The net force acting on the fluid is twice the weight of the fluid above the equilibrium level (all other forces cancel).



Exercise 8.5. Consider a mechanical system consisting of a spring with spring constant $k = 10$ lb/ft and a 100 pound weight that is free to move up and down. Assume that at time $t = 0$ sec the spring is unstretched but taut and the velocity of the weight is 0 ft/sec. At time $t = 0$ sec the net force on the weight is the 100 pound downward force due to gravity. Describe the subsequent motion of the weight, particularly the period and frequency of the resulting periodic behavior and the amplitude of the oscillations.

8.2. The Damped Harmonic Oscillator

We have ignored frictional forces in our discussion of the mass-spring system. Even under ideal conditions, in actual mechanical systems (small) frictional forces dissipate energy, and cause the amplitude of the oscillations to decay exponentially. In this section, we develop a more realistic model that takes this into account.

The *damped harmonic oscillator*, is a mathematical model of a mechanical system consisting of an object of mass m attached to a spring with spring constant k, and also to a damping mechanism. If x is the amount by which the spring is stretched, then the motion of the damped harmonic oscillator is governed by the differential equation

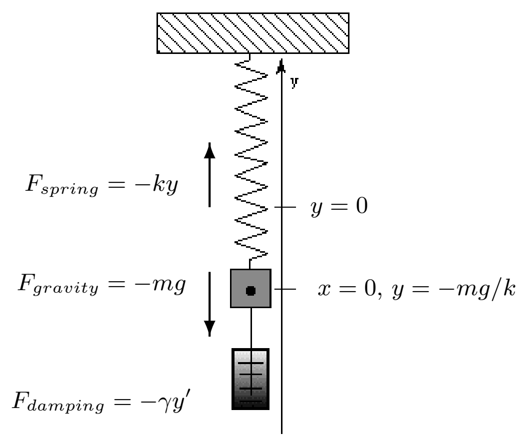
$$m\,x'' + \gamma\,x' + k\,x = 0\,. \tag{8.4}$$

The term $\gamma\,x'$ models frictional forces.

Example 8.2. As a concrete example (pictured on the left) consider a weight of mass m hanging from the end of a spring and under the influence of gravity. A rod connects the weight to a plunger which moves in a thick fluid exerting a drag force on the weight.

Let y denote the position of the weight measured in meters along a vertical coordinate axis pointing up Align the y-axis so that $y = 0$ at the bottom end of the spring when no mass is attached. There are three forces acting on the weight:

- $F_{gravity} = -gm$, $g = 9.8\text{m/sec}^2$.
- $F_{spring} = -ky$, the spring force, where k is the *spring constant* and where we have set $y = 0$ at the rest position of the spring. Hence, for $y > 0$ the spring is compressed and the force spring exerts on the weight is negative (pointing down); and for $y < 0$, the spring is stretched and the force is positive (pointing up).
- $F_{damping} = -\gamma\frac{dy}{dt}$, the damping force, where γ is the *damping coefficient*.



Applying Newton's second law of motion ("$F = ma$") results in the differential equation $m\,y'' = -g\,m - \gamma\,y' - k\,y$, which is usually written in the form

$$m\,y'' + \gamma\,y' + ky = -g\,m\,. \tag{8.5}$$

The weight is in equilibrium when it has velocity zero ($y' = 0$) and the upward force of the spring cancels with the downward force of gravity. At equilibrium

$$-ky - mg = 0 \text{ or } y = \frac{mg}{k}\,.$$

Let $y_{eq} = -mg/k$ (the value of y at equilibrium configuration). This suggests measuring the position of the weight relative to its equilibrium position:

$$y = y_{eq} + x\,.$$

The quantity x is the *displacement* of the weight from its equilibrium position $y = y_{eq}$. The effect of expressing the position of the weight in terms of its displacement from equilibrium is to turn the nonhomogeneous equation (8.5) into (the equivalent) homogeneous equation (8.4).

We have shown that the motion of the weight under the influence of gravity is equivalent to its motion without gravity—that is, centering the origin of the coordinate system at the equilibrium position has the effect of eliminating gravity from the equation of motion.

The behavior of the damped harmonic oscillator depends on the numerical values of the parameters m, γ, and k. As the damping constant increases, the rate of exponential decay of the oscillations increases until at a certain *critical value* they disappear altogether. More precisely, the behavior depends on the number and type of the roots of the characteristic polynomial $mr^2 + \gamma r + k$. There are four cases:

(i) $\gamma = 0$ (Undamped) The roots are pure imaginary.
(ii) $\gamma^2 - 4mk < 0$ (Under damped) The roots are complex conjugates of one another.
(iii) $\gamma^2 - 4mk = 0$ (Critically damped) There is a single double root.
(iv) $\gamma^2 - 4mk > 0$ (Over damped) The roots are both negative real numbers.

Figure 8.4 illustrates typical behaviors of the damped harmonic oscillator in each of these four cases.

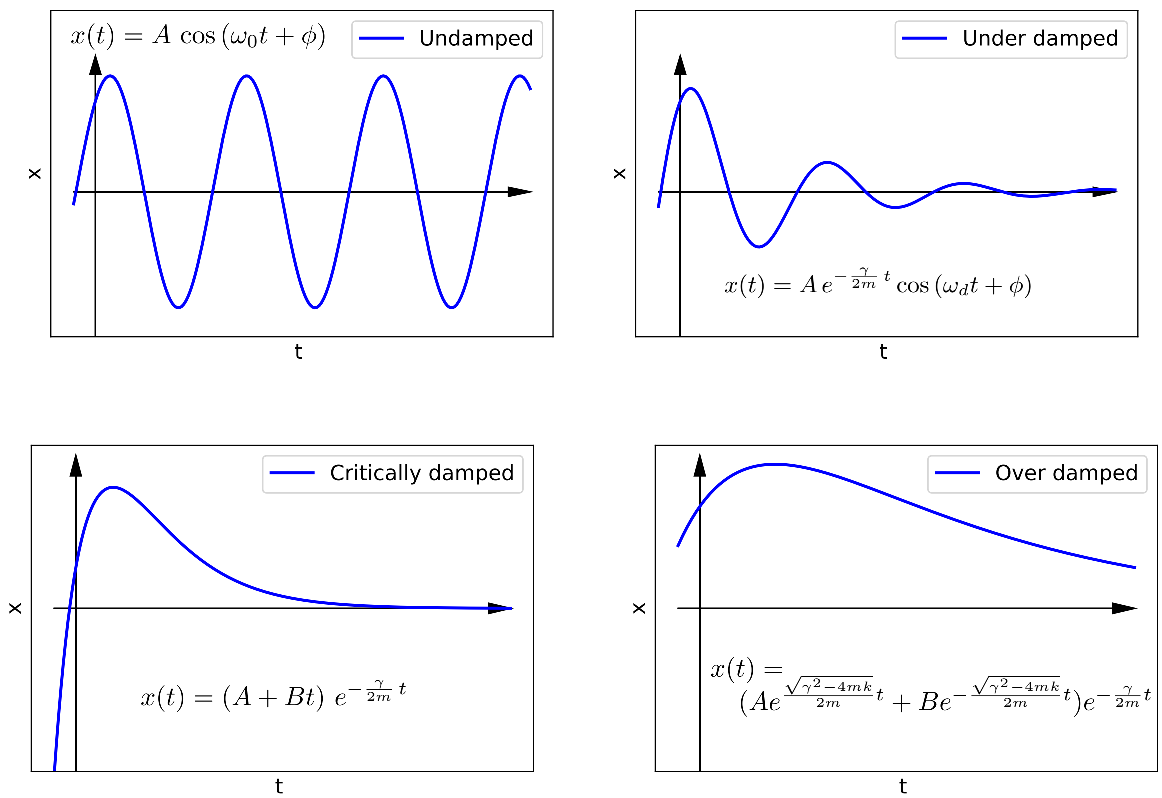


FIGURE 8.4. The four behaviors of the damped harmonic oscillator. The frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$ is called the *natural frequency* and $\omega_d = \sqrt{\omega_0^2 - \left(\dfrac{\gamma}{2m}\right)^2}$ is called the *quasi-frequency.*

EXERCISE 8.6. Suppose $m = 0.4$ kg, $\gamma = 2.0$ kg/sec, and $k = 4.0$ kg/sec^2, the initial displacement is 2.0 meters and the initial velocity is -3.0 meters/sec. Find the solution of (8.4) that satisfies the initial condition.

EXERCISE 8.7. A 1 kilogram mass is suspended from the end of a spring with a spring constant of 1 N/n (Newtons per meter). The mass is free to move up and down, y is the amount (measured in meters) that the spring is stretched and there is no gravity. In addition, there is a damping mechanism that exerts a force of $-\gamma\, y'$ Newtons, where γ is a constant.
(a) What value of γ will make the system *critically damped*?
(b) If at time $t = 0$ the spring is not stretched and the mass is moving at a rate of 0.5 m/sec (meters per second), what is the formula for $y(t)$. (Use the value of γ obtained in part a).
(c) What is the maximum amount by which the spring will be stretched?

A Note on Units. Because fundamental properties of the system, such as the criterion for critical damping, do not depend on units, it is worthwhile to express properties in terms of dimensionless quantities. Since $\omega_0 t$ is dimensionless, the dimensions of ω_0 are $(time)^{-1}$. Because the dimensions of $(\gamma/m)x'$ are the dimensions of x'', the dimensions of γ/m must agree with the dimensions of x''/x', i.e. $(time)^{-1}$. It follows that the quantity

$$\zeta = \frac{\gamma/m}{2\omega_0}$$

is dimensionless (that is, it is independent of units). In terms of ζ, the differential equation (8.4) can be written in the form

$$x'' + 2\zeta\omega_0\, x' + \omega_0^2\, x = 0\,, \tag{8.6}$$

and the characteristic polynomial can we written (completing the square)

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = (r - \zeta\omega_0)^2 - \omega_0^2(\zeta^2 - 1)\,.$$

Since the roots are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \omega_0^2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_0\,,$$

the four cases above can be expressed in the following dimensionless form:

$$\zeta = 0 \text{ (undamped)}\,, 0 < \zeta < 1 \text{ (under damped)}\,, \zeta = 1 \text{ (critically damped)}\,, \zeta > 1 \text{ (over damped)}.$$

Estimating quantities from a graph. Engineers often describe under damped harmonic motion with the formula

$$x(t) = A\exp(-\zeta\omega_0 t)\cos(\omega_d t + \phi)\,, \quad \text{where } \omega_d = \sqrt{\zeta - 1}\,\omega_0,$$

because both ζ and ω_d can be measured in a straightforward way from two points on the graph of $x(t)$. To see this, suppose you measure the times and displacements, (t_1, x_1) and (t_2, x_2) at two consecutive

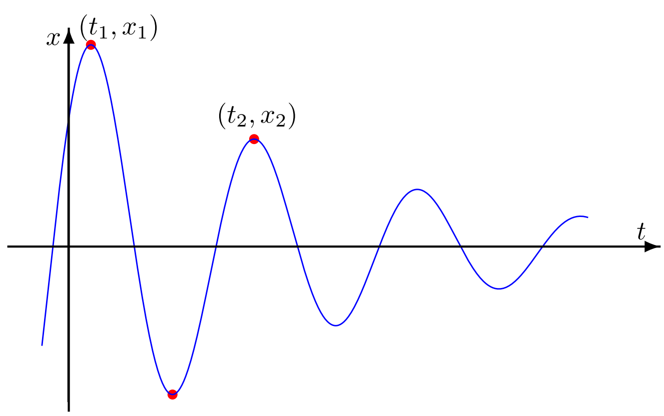


FIGURE 8.5.

peaks of the graph of $x(t)$. (See Figure 8.5). We can estimate ω_d because of the approximately periodic nature of $x(t)$:

$$t_2 - t_1 = \frac{2\pi}{\omega_d} \text{ or } \omega_d = \frac{2\pi}{t_2 - t_1}\,.$$

The time difference $T = t_2 - t_1$ is called the *quasi-period*, and the frequency $\omega_d = \dfrac{2\pi}{t_2 - t_1}$ is called the *quasi-frequency*. Using this, we can compute as follows:

$$\begin{aligned}\frac{x_1}{x_2} &= \frac{A\exp(-\zeta\omega_0 t_1)\cos(\omega_d t_1 + \phi)}{A\exp(-\zeta\omega_0 t_2)\cos(\omega_d t_2 + \phi)}\\ &= \frac{\exp(-\zeta\omega_0 t_1)\cos(\omega_d t_1 + \phi)}{\exp(-\zeta\omega_0 t_2)\cos(\omega_d t_1 + \phi + 2\pi)} = \frac{\exp(-\zeta\omega_0 t_1)}{\exp(-\zeta\omega_0 t_2)} = \exp(\zeta\omega_0(t_2 - t_1))\,.\end{aligned}$$

Taking the natural logarithm gives a formula for $\zeta\omega_0$ in terms of quantities that can be estimated from the graph of $x(t)$:

$$\Delta = \ln\left(\frac{x_1}{x_2}\right) = (t_2 - t_1)\zeta\omega_0 \text{ or } \zeta\omega_0 = \frac{1}{t_2 - t_1}\ln\left(\frac{x_1}{x_2}\right) = \frac{\omega_d}{2\pi}\Delta\,.$$

The quantity Δ is called the *logarithmic decrement*.

Substitute $\omega_d = \sqrt{1-\zeta^2}\omega_0$ into the last formula and compute as follows:

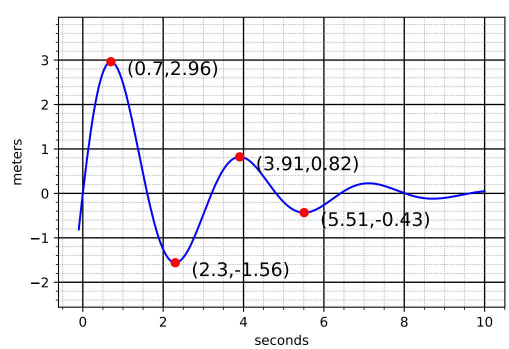
$$\zeta\omega_0 = \frac{\Delta}{2\pi}\sqrt{1-\zeta^2}\,\omega_0 \implies \zeta^2 = \frac{\Delta^2}{4\pi^2}(1-\zeta^2) \implies (\Delta^2 + 4\pi^2)\zeta^2 = \Delta^2\,.$$

Consequently,

$$\zeta = \frac{\Delta}{\sqrt{\Delta^2 + 4\pi^2}}\,.$$

The ratio $\zeta = \dfrac{\Delta}{\sqrt{\Delta^2 + 4\pi^2}} \simeq= \dfrac{\Delta}{2\pi}$ is called the *damping ratio*. The natural frequency ω_0 can be estimated by $\omega_0 = \dfrac{\omega_d}{\sqrt{1-\zeta^2}}$.

Exercise 8.8. The graph below shows the motion of an unforced damped harmonic oscillator, whose solution can be written in the form $x(t) = A\exp(-\zeta\omega_0 t)\cos(\omega_d t + \phi)$.



(a) Use the measured values from the graph to find: ω_d, ζ, and ω_0.
(b) Now write the differential equation that $x(t)$ satisfies in the form

$$x'' + (?)\,x' + (?)\,x = 0\,,$$

with the appropriate numbers replacing the question marks.
(c) It is obvious from the graph that the solution satisfies the initial condition $x(0) = 0$. Use the original formula, together with your measured values of (t_1, x_1) to estimate A, and the initial condition $x'(0)$.

EXERCISE 8.9. Suppose that you are designing a new shock absorber for an automobile. The car has a mass of 1000 kg (kilograms) and the combined effect of the springs in the suspension system is that of a spring constant of 20000 N/m.

(a) Before a damping mechanism is installed in the car, when the car hits a bump it will bounce up and down. How may bounces will a rider experience in the minute right after the car hits a bump?
(b) Your job is to design a damping mechanism that eliminates oscillations when the car hits a bump. What is the minimum value of the effective damping constant that can be used?
(c) Suppose that at time $t = 0$ the car hits a bump. Immediately before that time the car was not moving up and down and the effect of the bump is to add a vertical component to the speed of the car of 1.0 meter/sec. How high will the car rise above its equilibrium position if you design the system with the damping constant you found in part (b)?

CHAPTER 9

Solving Nonhomogeneous Differential Equations

The goal of this chapter is to develop a technique for finding the general solution to the nonhomogeneous differential equation

$$L[y] = ay'' + by' + cy = f(t)\,.$$

The technique is a generalization of the method of "undetermined coefficients" that we used in Chapter 3 to solve nonhomogeneous first order differential equations.

Before discussing the technique in general, it is instructive to work out an example.

Example 9.1. Consider the nonhomogeneous differential equation

$$L[y] = y'' + 4y' + 3y = 10e^{7t}\,.$$

We begin by looking for a particular solution of the form $y_p(t) = Ae^{7t}$. The computation

$$y_p''(t) + 4y_p'(t) + 3y_p(t) = 7^2Ae^{7t} + 4\cdot 7Ae^{7t} + 3Ae^{7t} = (49 + 28 + 3)Ae^{7t} = 80Ae^{7t} = 10e^{7t}$$

shows that $A = \frac{1}{8}$. So $y_p(t) = \frac{1}{8}e^{7t}$ is a solution.

On the other hand, $r^2 + 4r + 3 = (r + 3)(r + 1)$ is the characteristic polynomial of the homogeneous differential equation

$$L[y] = y'' + 4y' + 3y = 0\,.$$

Therefore,

$$y_h(t) = C_1e^{-t} + C_2e^{-3t}$$

is the general solution of the homogeneous differential equation. But by the *superposition principle* (see Equation (5.3))

$$L[y_p(t) + y_h(t)] = L[y_p(t)] + L[y_h(t)] = 10e^{7t} + 0 = 10e^{7t}\,.$$

Consequently, the function

$$y(t) = \frac{1}{8}e^{7t} + C_1e^{-t} + C_2e^{-3t}$$

is the general solution.

For instance, to solve the initial value problem

$$L[y] = 10e^{7t}\,, \quad y(0) = 2\,, \quad y'(0) = 3\,,$$

choose C_1 and C_2 to satisfy the equations

$$y(0) = \frac{1}{8} + C_1 + C_2 = 2 \text{ and } y'(0) = \frac{7}{8} - C_1 - 3C_2 = 3\,.$$

This gives $C_1 = \frac{31}{8}$ and $C_2 = -2$, so the function

$$y(t) = \frac{1}{8}e^{7t} + \frac{31}{8}e^{-t} - 2e^{-3t}$$

solves the initial value problem.

Computation similar to the ones we just did generalize. The *general solution* of the homogeneous differential equation

$$L[y] = ay'' + by' + cy = 0\,.$$

is of the form

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)\,,$$

where $y_1(t)$ and $y_2(t)$ form a fundamental basis of solutions.

Now suppose that we have found a particular solution, say $y_p(t)$, of the nonhomogeneous differential equation

$$L[y] = ay'' + by' + cy = f(t). \tag{9.1}$$

Then can then be written in the form

$$y(t) = y_p(t) + y_h(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)\,, \tag{9.2}$$

where $y_p(t)$ is a *particular solution* of the nonhomogeneous differential equation.

To show this, assume that $y(t)$ is any function with $L[y(t)] = f(t)$. Then the difference $y(t) - y_p(t)$ is a solution of the homogeneous differential equation as the following computation shows:

$$L[y(t) - y_p(t)] = L[y(t)] - L[y_p(t)] = f(t) - f(t) = 0\,.$$

But every solution of the homogeneous differential equation is of the form $y_h(t)$. Consequently,

$$y(t) - y_p(t) = C_1 y_1(t) + C_2 y_2(t)\,,$$

for some choice of C_1 and C_2. Hence, $y(t)$ satisfies (9.2).

Solving the initial value problem

$$L[y] = f(t)\,, \quad y(t_0) = y_0\,, \quad y'(t_0) = y_0'\,,$$

then reduces to solving the following pair of equations:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) + y_p(t_0) &= y_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) + y_p'(t_0) &= y_0'\,. \end{aligned}$$

The problem of finding $y_1(t)$ and $y_2(t)$ was addressed in the previous chapter. So we only need to find techniques for finding a *particular solution* $y = y_p(t)$ of the nonhomogeneous equation

$$L[y] = ay'' + by' + cy = f(t)\,.$$

Bu a *particular solution*, we mean a solution that does not involve any arbitrary constants like C_1 and C_2. This will become more clear as we work out more examples.

There are several approaches to finding a particular solution. Two will be addressed in these notes:

- *Undetermined Coefficients*
- *Laplace Transforms.*

Remark 9.1. There is a third approach, called *variation of parameters*. Because undetermined coefficients and Laplace transforms are sufficient in most cases, variation of parameters will not be included in these notes. The interested reader can find a number of explanations of this method on the web.

9.1. Undetermined Coefficients

When the forcing function $f(t)$ is of a special form, the method of *undetermined coefficients* reduces the problem of finding a particular solution to a problem algebra. This method applies whenever $f(t)$ is of one of the forms

$$f(t) = p(t)e^{rt}$$

or

$$p(t)\sin(\omega t)e^{rt} + q(t)\cos(\omega t)e^{rt}\,,$$

where $p(t)$ and $q(t)$ are polynomials or when $f(t)$ is a sum of terms like these.

Here are some examples of differential equations where the method applies:

(1) $y'' + 2y' - y = (3t+1)e^{2t}$
(2) $y'' + 4y = (1-t^3)\cos(3t)$
(3) $y'' - 2y' + y = (1+t+t^2)e^{3t}\cos(3t) + te^{3t}\sin(3t)$
(4) $y'' - y = (1+2t)e^{t} + (t^2\sin(3t) + (2-t+t^2)\cos(3t))$

Example 9.2. Find a particular solution of the nonhomogeneous differential equation

$$L[y] = y'' + 3y' + 2y = (t-2)\,e^{2t}\,.$$

Solution The function $f(t) = (t-2)\,e^{2t}$ is of the form $p(t)e^{rt}$, where $p(t) = t-2$, a polynomial of degree 1, and $r = 2$. Let

$$y_p(t) = (At+B)e^{2t}\,,$$

where A and B are to be determined. A direct computation gives:

$$L[y_p] = \{12A\,t + (7A+12B)\}\;e^{2t}\,.$$

Observe that y_p will satisfy the equation

$$L[y_p] = (t-2)\,e^{2t}\,,$$

provided that A and B satisfy the equation

$$\{12A\,t + (7A+12B)\}\;e^{2t} = (t-2)\,e^{2t}$$

for all t. Equating like terms results in two equations in two unknowns:

$$12A = 1 \text{ and } 7A + 12B = -2\,.$$

The first equation gives $A = 1/12$. Substituting this value into the second equation gives $B = -31/144$. We conclude that

$$y_p(t) = \left\{\frac{1}{12}t - \frac{31}{144}\right\}\,e^{2t}\,.$$

is a particular solution.

Example 9.3. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = 1 - 2t\,.$$

Solution Set

$$y_p(t) = (At+B)e^{0t} = (At+B)\,.$$

Comparing

$$L[y_p] = (3A+2B) + 2A\,t\,.$$

with the function $f(t) = 1-2t$ yields the two equations

$$3A + 2B = 1 \text{ and } 2A = -2\,.$$

Solving the second equation gives $A = -1$ and substituting that value into the first equation gives $B = 2$. Hence

$$y_p = 2 - t\,.$$

EXAMPLE 9.4. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = f(t) = (t^2 - 2)\, e^{2t}\,.$$

SOLUTION In this case $p(t) = t^2 - 2$, a polynomial of degree 2, so set

$$y_p(t) = \left(At^2 + Bt + C\right)\, e^{2t}$$

$$L[y_p] = \left\{12A\, t^2 + (14A + 12B)\, t + (2A + 7B + 12C)\right\}\, e^{2t}$$

and

$$L[y_p] = (t^2 - 2)\, e^{2t}\,.$$

Equating coefficients gives the three equations

$$\begin{cases} 12A = 1 \\ 14A + 12B = 0 \\ 2A + 7B + 12C = -2\,. \end{cases}$$

The first equation shows $A = 1/12$.
The second (together with $A = 1/12$) forces $B = -7/72$.
And the third (together with $A = 1/12$ and $B = -7/12$) forces $C = -107/864$.

Hence,

$$y_p(t) = \left\{\frac{1}{12}t^2 - \frac{7}{72}t - \frac{107}{864}\right\}\, e^{2t}\,.$$

EXAMPLE 9.5. Find a particular solution of

$$L[y] = y'' + 3y' + 2y = (t - 2)\, e^{-2t}\,.$$

SOLUTION In this case, the function $y_p(t) = (At + B)e^{-2t}$ cannot be a particular solution. Indeed, a simple computation gives

$$L[(At + B)\, e^{-2t}] = -Ae^{-2t}\,;$$

but

$$L[(At + B)\, e^{-2t}] = (t - 2)\, e^{-2t}\,.$$

Clearly, no choice of A and B can yield $(t - 2)e^{-2t}$. T The solution to this problem is to multiply by the original guess by t:

$$y_p(t) = t\ (At + B)\ e^{-2t}\,.$$

Then

$$L[t(At + B)\, e^{-2t}] = \{-2At + (2A - B)\}\ e^{-2t}\,.$$

The coefficients A and B can then be chosen to satisfy the equation

$$\{-2At + (2A - B)\}\ e^{-2t} = (t - 2)e^{-2t}\,.$$

Comparing terms as before yields two equations

$$-2A = 1 \text{ and } 2A - B = -2\,;$$

and solving for A and B gives $A = -1/2$ and $B = 1$. Hence,

$$y_p(t) = t(-t/2 + 1)e^{-2t} = (t - t^2/2)\, e^{-2t}\,.$$

EXAMPLE 9.6. Find a particular solution to the differential equation

$$L[y] = y'' - 6y' + 9y = (t-2)e^{3t}\,.$$

SOLUTION Because $r^2 - 6r + 9 = (r-3)^2$,

$$L[e^{3t}] = 0 \text{ and } L[te^{3t}] = 0\,.$$

Consequently, the function $y_p(t) = (At+B)e^{3t}$ cannot be a solution. Neither can $y_p(t) = t(At+B)e^{3t}$ because

$$L[t(At+B)e^{3t}] = 2Ae^{3t}$$

This suggests multiplying by t^2, and setting $y_p(t) = t^2(At+B)e^{3t}$. A short computation shows that

$$L[y_p] = (6At + 2B)e^{3t} = (t-2)e^{3t}\,,$$

Comparing coefficients shows that

$$y_p(t) = t^2(t/6 - 1)e^{3t}$$

is a particular solution.

EXAMPLE 9.7. Find a particular solution for

$$L[y] = y'' - 6y' + 13y = 5\cos(2t)e^{4t}$$

SOLUTION

The characteristic polynomial $r^2 - 6r + 13$ has roots $3 \pm 2\,i$. So the general solution of $L[y] = f(t)$ has the form

$$y(t) = y_p(t) + \{C_1\cos(2t) + C_2\sin(2t)\}\; e^{3t}\,.$$

Substituting

$$y_p(t) = (A\cos(2t) + B\sin(2t))\, e^{4t}$$

into the differential equation yields (after a lengthy computation)

$$\begin{aligned} L[y_p] &= \{(A+4B)\cos(2t) + (-4A+B)\sin(2t)\}\; e^{4t} \\ &= 5\cos(2t)\, e^{4t}\,. \end{aligned}$$

Equating coefficients gives the system of equations

$$A + 4B = 5 \text{ and } -4A + B = 0$$

whose solution is $A = 5/17$ and $B = 20/17$. Hence, the function

$$y_p(t) = \left\{\frac{5}{17}\cos(2t) + \frac{20}{17}\sin(2t)\right\}\; e^{4t}$$

is a particular solution.

EXAMPLE 9.8. Find the general solution of the differential equation

$$L[y] = y'' - 6y' + 13y = 5\cos(2t)e^{4t} - 2t\sin(2t)e^{4t}\,.$$

SOLUTION Let $y_p(t) = (A+Bt)e^{4t}\cos(2t) + (C+Dt)e^{4t}\sin(2t)$. A lengthy computation gives

$$\begin{aligned} L[y_p] =& \{(A + 2B + 4C + 4D) + (B + 4D)t\}\cos(2t)e^{4t} \\ &+ \{(-4A - 4B + C + 2D) + (-4B + D)t\}\sin(2t)e^{4t} \end{aligned}$$

Comparing coefficients leads to the following system of equations:

$$\begin{cases} A + 2B + 4C + 4D & = 5 \\ B + 4D & = 0 \\ -4A - 4B + C + 2D & = 0 \\ -4B + D & = -2\,. \end{cases}$$

One finds (after some computation) that

$$A = -\frac{67}{289}, \quad B = \frac{8}{17}, \quad C = \frac{344}{289}, \quad D = -\frac{2}{17}$$

and thus

$$y_p(t) = \left(-\frac{67}{289} + \frac{8}{17}t\right)\cos(2t)e^{4t} + \left(\frac{344}{289} - \frac{2}{17}t\right)\sin(2t)e^{4t}$$

The roots of the characteristic polynomial $r^2 - 6r + 13$ are $3 \pm 2i$. Therefore, the general solution is

$$\begin{aligned} y = &\left(-\frac{67}{289} + \frac{8}{17}t\right)\cos(2t)e^{4t} + \left(\frac{344}{289} - \frac{2}{17}t\right)\sin(2t)e^{4t} \\ &+ \{C_1\cos(2t) + C_2\sin(2t)\}\ e^{3t}. \end{aligned}$$

The general case. The approach presented in the above example can be distilled into a general algorithm, called the *method of undetermined coefficients*. Consider the differential equation

$$L[y] = ay'' + by' + cy = f(t),$$

where the forcing function $f(t)$ is one of the two forms $f(t) = \begin{cases} p(t)e^{r_0 t} \\ \{p(t)\cos(\omega t) + q(t)\sin(\omega t)\}\ e^{r_0 t} \end{cases},$

with

$$\begin{aligned} p(t) &= p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n \\ q(t) &= q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n. \end{aligned}$$

(1) Let r_1 and r_2 be the the roots of the characteristic polynomial $ar^2 + br + c$.

(2) If $f(t) = p(t)e^{r_0 t}$ then let

$$y_p(t) = \begin{cases} P(t)e^{r_0 t} & \text{if } r_0 \neq r_1, r_2 \\ tP(t)e^{r_0 t} & \text{if } r_0 = r_1 \text{ and } r_0 \neq r_2 \\ t^2P(t)e^{r_0 t} & \text{if } r_0 = r_1 = r_2 \text{ (double root),} \end{cases}$$

where $P(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$.

If $f(t) = \{p(t)\cos(\omega t) + q(t)\sin(\omega t)\}\ e^{r_0 t}$ then set

$$y_p(t) = \begin{cases} \{P(t)\cos(\omega t) + Q(t)\sin(\omega t)\}\, e^{r_0 t} & \text{if } r_0 + \omega\, i \neq r_1, r_2 \\ t\,\{P(t)\cos(\omega t) + Q(t)\sin(\omega t)\}\, e^{r_0 t} & \text{if } r_0 + \omega\, i = r_1 \text{ or } r_2 \end{cases}$$

where $\begin{cases} P(t) &= A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n \\ Q(t) &= B_0 + B_1 t + B_2 t^2 + \cdots + B_n t^n. \end{cases}$

(3) Equate coefficients of powers of t in the equation $L[y_p(t)] = f(t)$ to get a linear system of equations in the unknown coefficients A_i and B_i.

(4) Solve the system to get $P(t)$ (and $Q(t)$), and thus $y_p(t)$.

(5) If

$$L[y] = f(t) = f_1(t) + f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are both of the above form (but with different values of n, r_0 and/or ω), then set

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t),$$

where

$$L[y_{p_1}] = f_1(t) \text{ and } L[y_{p_2}] = f_2(t).$$

9.2. Undetermined Coefficients Using Complex-valued Functions

When $f(t)$ involves trig functions, the algebra involved in applying the method of undetermined coefficients can be messy. In such cases, using complex-valued functions often simplifies the computations.

Recall from Section 6 that the complex exponential function is the function

$$e^{(\rho+i\omega)t} = (\cos(\omega t) + i\sin(\omega t))e^{\rho t}\,,$$

where ρ and ω are real numbers. Its derivative satisfies the identity

$$\frac{de^{(\rho+i\omega)t}}{dt} = (\rho+\omega i)e^{(\rho+i\omega)t}\,.$$

Consequently, its second derivative can be easily evaluated:

$$\frac{d^2e^{(\rho+i\omega)t}}{dt^2} = (\rho+\omega i)^2e^{(\rho+i\omega)t}\,.$$

As already mentioned in Section sec:complex-functions, this fact greatly simplifies computations of derivatives of functions of the form

$$x(t) = a\cos(\omega t)e^{\rho t} + b\sin(\omega t)e^{\rho t}\,.$$

Example 9.9. Find a particular solution of the differential equation

$$y'' + y' + y = (\cos(t) - \sin(t))e^{2t}$$

Solution Since $(\cos(t) - \sin(t))e^{2t} = \mathrm{Re}\left((1+i)e^{(2+i)t}\right)$, there is a particular solution of the form $y_p(t) = \mathrm{Re}\,(z_p(t))$, where $z_p(t)$ is a particular solution solution of

$$z'' + z' + z = (1+i)e^{(2+i)t}\,.$$

Set $z_p(t) = Ae^{(2+i)t}$. Then

$$\begin{aligned} z_p''(t) + z_p'(t) + z_p(t) &= \left((2+i)^2 + (2+i) + 1\right)Ae^{(2+i)t} \\ &= (6+5i)Ae^{(2+i)t} = (1+i)e^{(2+i)t} \end{aligned}$$

Solving for A gives

$$A = \frac{1+i}{6+5i} = \frac{11}{61} + \frac{1}{61}i\,.$$

Hence,

$$z_p(t) = \left(\frac{11}{61} + \frac{1}{61}i\right)e^{(2+i)t}\,.$$

Taking the real part of $z_p(t)$ gives a particular solution of the original differential equation:

$$y_p(t) = \mathrm{Re}\,(z_p(t)) = \left(\frac{11}{61}\cos(t) - \frac{1}{61}\sin(t)\right)e^{2t}\,.$$

Example 9.10. Solve the initial value problem

$$u'' + \omega_0^2u = \sin(\omega t)\,, \quad u(0) = u'(0) = 0\,,$$

for $\omega \neq \omega_0$.

Solution The general solution of the corresponding homogeneous differential equation is

$$u_h(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\,,$$

so the general solution of the original differential equation is of the form

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + u_p(t)\,.$$

Since $\sin(\omega t) = \mathrm{Re}\left(-ie^{i\omega t}\right)$, let $u_p(t) = \mathrm{Re}\left(z_p(t)\right)$, where $z_p(t) = Ae^{i\omega t}$ is a solution of the complex differential equation

$$z'' + \omega_0 z = -i\, e^{i\omega t}\,.$$

To find A substitute $z_p(t)$ into the complex differential equation:

$$z_p''(t) + \omega_0^2 z_p(t) = (-\omega^2 + \omega_0^2)Ae^{i\omega\, t} = -ie^{i\omega t}$$

If follows that $A = \dfrac{-i}{\omega_0^2 - \omega^2}$ and $u_p(t) = \mathrm{Re}\left(z_p(t)\right) = \dfrac{1}{\omega_0^2 - \omega^2}\sin(\omega t)$. The general solution is, therefore,

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\sin(\omega t)\,.$$

The initial conditions

$$u(0) = C_1 = 0 \text{ and } u'(0) = \omega_0 C_2 + \frac{\omega}{\omega_0^2 - \omega^2} = 0$$

force $C_1 = 0$ and $C_2 = \dfrac{\omega/\omega_0}{(\omega^2 - \omega_0^2)}$. Consequently,

$$u(t) = \frac{\omega/\omega_0}{(\omega^2 - \omega_0^2)}\sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\sin(\omega t) = \frac{1}{(\omega^2 - \omega_0^2)}\left((\omega/\omega_0)\sin(\omega_0 t) - \sin(\omega t)\right)\,.$$

Example 9.11. Find a particular solution of the differential equation

$$u'' + \omega_0^2 u = \sin(\omega_0 t)\,.$$

Solution Proceeding as before, the particular solution will be the real part of a particular solution of the complex differential equation

$$z'' + \omega_0^2 z = -ie^{\omega_0 it}\,.$$

The function $z_p(t) = Ae^{\omega_0 it}$ won't work since $z_p''(t) + \omega_0^2 z_p(t) = 0$. So try the next best thing: $z_p(t) = Ate^{\omega_0 it}$:

$$z_p'(t) = A(1 + \omega_0 it)e^{\omega_0 it} \implies z_p''(t) = A\left((\omega_0 i) + (1 + \omega_o it)\omega_0 i\right)e^{\omega_0 it} = A\left(2\omega_0 i - {}_0^2 t\right)e^{\omega_0 it}\,,$$

Then

$$z_p''(t) + \omega_0^2 z_P(t) = A\left(2\omega_0 i - \omega_0^2 t + \omega_0^2 t\right)e^{\omega_0 it} = (2\omega_0 i)Ae^{\omega_0 it} = -ie^{\omega_p it}\,.$$

Solving for A gives $A = \frac{-i}{2\omega_0 i} = -\frac{1}{2\omega_0}$. Hence, the function

$$u_p(t) = \mathrm{Re}\left(-\frac{1}{2\omega_0}te^{\omega_0 it}\right) = -\frac{t}{2\omega_0}\cos(\omega_0 t)$$

is a particular solution.

Exercise 9.1. Solve each of the following differential equations and initial value problems.

(a) $y'' + 3y = t^2 + 1$.
(b) $y'' + y = 2\sin(t)$, $y(0) = 0$, $y'(0) = 0$.
(c) $y'' + y' + y = e^t \sin(t)$.
(d) $y'' - y' = e^t$
(e) $y'' - 4y = e^{2t}$, $y(0) = 1$, $y'(0) = 1$.
(f) $y'' - 4y' + 4y = e^{2t}$
(g) $y'' + 4y = 3\cos(t) + 4\sin(t)$, $y(0) = 0$, $y'(0) = 0$.
(h) $y'' + 4y = 12\cos(4t) - 12\sin(4t)$, $y(0) = 0$, $y'(0) = 0$.
(i) $y'' - 2y' + y = \sin(2t)e^{-t}$.

CHAPTER 10

The Driven Harmonic Oscillator

In an earlier chapter, we studied the undriven harmonic oscillator. If there are additional time-dependent ("external") forces on the object, mechanical system is then modeled by the nonhomogeneous differential equation

$$m\,x'' + \gamma\,x' + k\,x = F(t)\,, \tag{10.1}$$

where $F(t)$ denotes the external "*driving force*." The most important case is when $F(t)$ is of the form

$$F(t) = F_0\,\cos(\omega t)\,,$$

where the phenomenon of "resonance" occurs. Most of this chapter is devoted to understanding resonance and applications where it occurs.

10.1. Resonance

Consider the special case where there is no damping. The mechanical system is then modeled on the differential equation

$$x'' + \omega_0^2\,x = \frac{F_0}{m}\,\cos(\omega t)\,. \tag{10.2}$$

Figure 10.1 indicates how such an external force might be applied. As we are about to discover, when the frequency ω of the driving force equals ω_0 (the *natural frequency*), the external force adds energy to the system, increasing the amplitude of the oscillations (see Figure 10.2). This is the phenomenon called *resonance*.

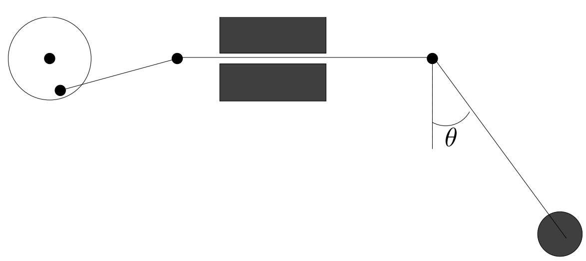


FIGURE 10.1. A simple pendulum with a driving force. For small angles, θ satisfies a differential equation of the form $\theta'' + \omega_0^2\theta = A\cos(\omega t)$.

Applying the techniques of the previous chapter, shows that the solution of (10.2) can we written in the form

$$x(t) = x_p(t) + x_h(t) = x_p(t) + A\cos(\omega_0 t + \phi),$$

where $x_p(t)$ is a particular solution. There are two cases consider: $\omega \neq \omega_0$ and $\omega = \omega_0$:

Case 1: $\omega \neq \omega_0$. Notice that $(F_0/m)\cos(\omega t) = \mathrm{Re}\left((F_0/m)e^{i\omega t}\right)$, so try $x_p(t) = \mathrm{Re}\left(z_p(t)\right)$ where $z_p(t)$ is a solution of

$$z_p'' + \omega_0^2 z_p = (F_0/m)e^{i\omega t}$$

Substitute $z_p(t) = Ce^{i\omega t}$ into the differential equation to get

$$z_p'' + \omega_0^2 z_p = (-\omega^2 + \omega_0^2)Ce^{i\omega t} = (F_0/m)e^{i\omega t}\,.$$

Hence, $C = \dfrac{F_0/m}{\omega_0^2 - \omega^2}$ and $x_p(t) = \mathrm{Re}\left(\dfrac{F_0/m}{\omega_0^2 - \omega^2}\, e^{i\omega t}\right) = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$.

Case 2: $\omega = \omega_0$. Try $z_p(t) = Cte^{i\omega_0 t}$ Then

$$z_p'' + \omega_0^2 z_p = \left((2\omega_0 i - \omega_0^2 t\right) Ce^{i\omega_0 t} + \omega_0^2 Cte^{i\omega_0 t} = 2\omega_0 iCe^{-\omega_0 t} = (F_0/m)e^{i\omega_0 t}$$

Therefore, $C = \dfrac{F_0/m}{2\omega_0 i}$ and $x_p(t) = \mathrm{Re}\left(\dfrac{F_0/m}{2\omega_0 i} te^{i\omega_0 t}\right) = \left(\dfrac{F_0/m}{2\omega_0}\right)\, t\,\sin(\omega_0 t)$.

This shows that the general solution of (10.2) is

$$x(t) = \begin{cases} \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t) + A\cos(\omega_0 + \phi) & \text{for } \omega \neq \omega_0 \\ \\ \left(\dfrac{F_0/m}{2\omega_0}\right)\, t\,\sin(\omega_0 t) + A\cos(\omega_0 + \phi) & \text{for } \omega = \omega_0. \end{cases} \tag{10.3}$$

This shows that regardless of the initial conditions, if the frequency ω of the forcing function equals the natural frequency ω_0 of the system, then the amplitude of the oscillations of the system increase without bound—this is he phenomenon on known as *resonance*

The special case where the system starts at rest at time $t = 0$ and a periodic force is applied is particularly instructive. In this case, $x(t)$ is the solution of the initial value problem:

$$x'' + \omega_0^2\, x = \frac{F_0}{m}\,\cos(\omega t)\,, \quad x(0) = 0\,, \quad x'(0) = 0$$

When $\omega = \omega_0$,

$$x(t) = \frac{F_0/m}{2\omega_0}\, t\,\sin(\omega_0 t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\,.$$

The initial condition $x(0) = 0$ implies $C_1 = 0$ and $x'(0) = 0$ implies $C_2 = 0$. Hence,

$$x(t) = \frac{F_0/m}{2\omega_0}\, t\,\sin(\omega_0 t)$$

When $\omega \neq \omega_0$, the general solution of the differential equation is then

$$x(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2}\right)\cos(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\,.$$

The initial conditions $x(0) = \dfrac{F_0/m}{\omega_0^2 - \omega^2} + C_1 = 0$ and $x'(0) = C_2 = 0$ together give

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\ \left(\cos(\omega t) - \cos(\omega_0 t)\right)\,.$$

Applying a trig identity from Appendix A leads to the formula

$$x(t) = \left\{\left(\frac{2F_0/m}{\omega_0^2 - \omega^2}\right)\sin\left(\frac{\omega_0 - \omega}{2}\, t\right)\right\}\sin\left(\frac{\omega_0 + \omega}{2}\, t\right)\,.$$

When ω is close to ω_0, the term in braces corresponds to a slowly varying amplitude, and the term $(\omega_0 + \omega)/2 \approx \omega_0$ corresponds to a high frequency. This leads to the phenomenon of *beats*, which is illustrated in Figure 10.2. As ω approaches ω_0 the frequency of the beats decreases, leading to the solution when $\omega = \omega_0$:

$$\lim_{\omega\to\omega_0} \left\{ \left(\frac{2F_0/m}{\omega_0^2 - \omega^2} \right) \sin\left(\frac{\omega_0 - \omega}{2}\, t \right) \right\} \sin\left(\frac{\omega_0 + \omega}{2}\, t \right) = \frac{F_0/m}{2\omega_0}\, t\, \sin(\omega_0 t)\,.$$

To see this, apply l'Hôpital's to the expression in braces:

$$\lim_{\omega\to\omega_0} \left\{ \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega_0 - \omega}{2} t \right) \right\} = \lim_{\omega\to\omega_0} \frac{(2F_0/m)\sin\left(\frac{(\omega_0-\omega)}{2} t\right)}{(\omega_0+\omega)(\omega_0-\omega)} = \lim_{\omega\to\omega_0} \frac{(F_0/m)t\cos\left(\frac{\omega_0-\omega}{2} t\right)}{(\omega_0+\omega)} = \frac{F_0/m}{2\omega_0} t$$

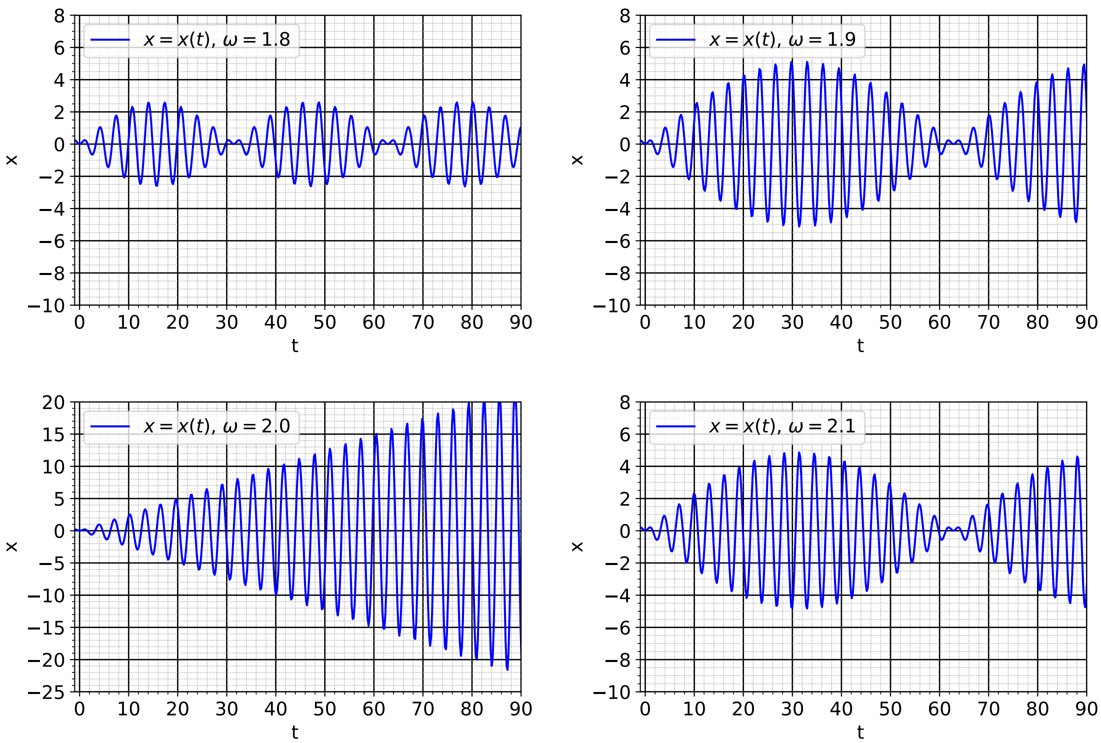


FIGURE 10.2. Graphs of solutions of $x'' + 4x = \cos(\omega t)$, $x(0) = x'(0) = 0$ for various values of ω. "Resonance" occurs when $\omega = 2.0$.

10.2. Forced Oscillations with Damping

The analysis of a forced, damped harmonic oscillator is similar to the analysis of the forced (undamped) harmonic oscillator. In this case, the system is modeled by the differential equation

$$m\, x'' + \gamma\, x' + k\, x = F_0 \cos(\omega\, t)$$

The general solution is of the form

$$x(t) = x_p(t) + C_1\, x_1(t) + C_2\, x_2(t)\,,$$

where $x_1(t)$ and $x_2(t)$ are solutions of the homogeneous differential equation and $x_p(t)$ is a particular solution of the nonhomogeneous equation.

For $\gamma > 0$ there are three cases, based on the number and type of the roots of the characteristic polynomial $mr^2 + \gamma r + k$:

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \omega_0^2}\,,$$

where $\omega_0 = \sqrt{k/m}$ (natural frequency). These case, however, only effect the form $x_h(t)$, In all three cases, the term

$$x_h(t) = C_1 x_1(t) + C_2 x_2(t)$$

has the property

$$\lim_{t\to\infty} x_h(t) = 0\,.$$

The function $x_h(t)$ is called a *transient* because when t is large it can be ignored. For t sufficiently large solution is approximately given by $x_p(t)$. That is

$$x(t) \approx x_p(t) \text{ for large } t$$

Using the method of undetermined coefficients allows us to write $x_p(t)$ in the form

$$x_p(t) = R\cos(\omega t + \phi)\,.$$

The function $x_p(t)$ is called the *steady state solution.* The amplitude R of the oscillations caused by the forcing function depends on ω, as does the phase shift ϕ. These two quantities characterize the response of the mechanical system to the forcing function.

The computation of $x_p(t)$ proceeds as follows: Set $x_p(t) = \mathrm{Re}\,(z_p(t)$, where $z_p(t) = Ce^{i\omega t}$. Substitute $z_p(t)$ into the equation $mz_p'' + \gamma z_p' + kz_p = F_0 e^{i\omega t}$ and note that $k = m\omega_0^2$ to get

$$(m(i\omega)^2 + \gamma(i\omega) + k)Ce^{i\omega t} = (m(\omega_0^2 - \omega^2) + i\gamma\omega)Ce^{i\omega t} = F_0 e^{i\omega t}\,.$$

Therefore,

$$(m(\omega_0^2 - \omega^2) + i\gamma\omega)C = F_0\,.$$

Putting the coefficient of C into polar form gives

$$\left(\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\, e^{i\delta}\right))\, C = F_0\,,$$

where $\tan(\delta) = \dfrac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$, $0 < \delta < \pi$ Hence,

$$C = \frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} e^{-i\delta}\,.$$

Consequently,

$$x_p(t) = \mathrm{Re}\left(\frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} e^{-i\delta} e^{i\omega t}\right) = \frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} \cos(\omega t - \delta) \tag{10.4a}$$

and

$$R = \frac{F_0}{\sqrt{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2)}} \text{ and } \phi = -\delta \tag{10.4b}$$

Resonant frequency. By analogy with the undamped case, it is useful to find the value of ω that results in the maximum amplitude of the oscillations in $x_p(t)$. In other words, we seek the value of ω that maximizes R. This is again called the *resonant frequency.*

To get a feel for what happens, we set $m = 1$, $k = 1$, and $F_0 = 1$, and graph R for a increasing values of the damping constant γ. As one would expect, the maximum value of R decreases as the damping constant increases.

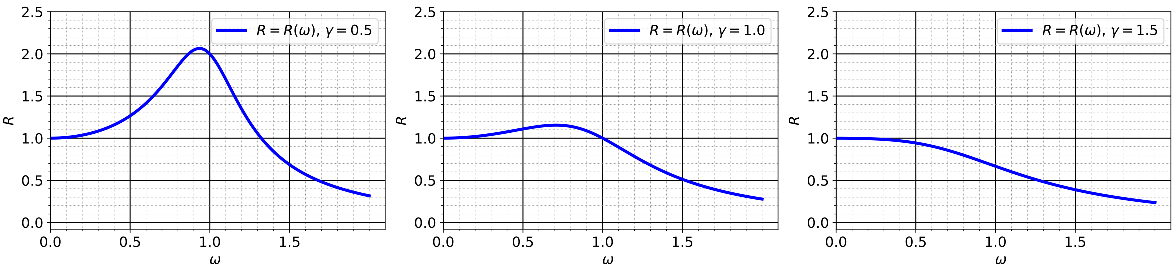


FIGURE 10.3. The amplitude R for three values of damping constant γ. As γ increases, the value of ω maximizing R decreases and eventually becomes 0, corresponding to a constant applied force.

The resonant frequency coincides with the value of ω at which the denominator

$$f(\omega) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2 \,.$$

of R achieves its minimum. Differentiating with respect to ω gives

$$f'(\omega) = 2\omega\,\left(\gamma^2 + 2m^2(\omega^2 - \omega_0^2)\right) = 4m^2\,\omega\,\left(\omega^2 - \left(\omega_0^2 - \frac{\gamma^2}{2m^2}\right)\right)\,.$$

The critical values of $f(\omega)$ are, therefore, $\omega = 0$ and $\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{2m^2}}$.

Conclusion: If $\omega_0^2 > \dfrac{\gamma^2}{2m^2}$ or (equivalently) if $\left(\dfrac{\gamma/m}{\omega_0}\right)^2 < 2$, the resonant frequency is

$$\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}} = \omega_0\,\sqrt{1 - \frac{1}{2}\left(\frac{\gamma/m}{\omega_0}\right)^2}\,.$$

Otherwise, the maximum amplitude is achieved for $\omega = 0$, corresponding to a constant driving force.

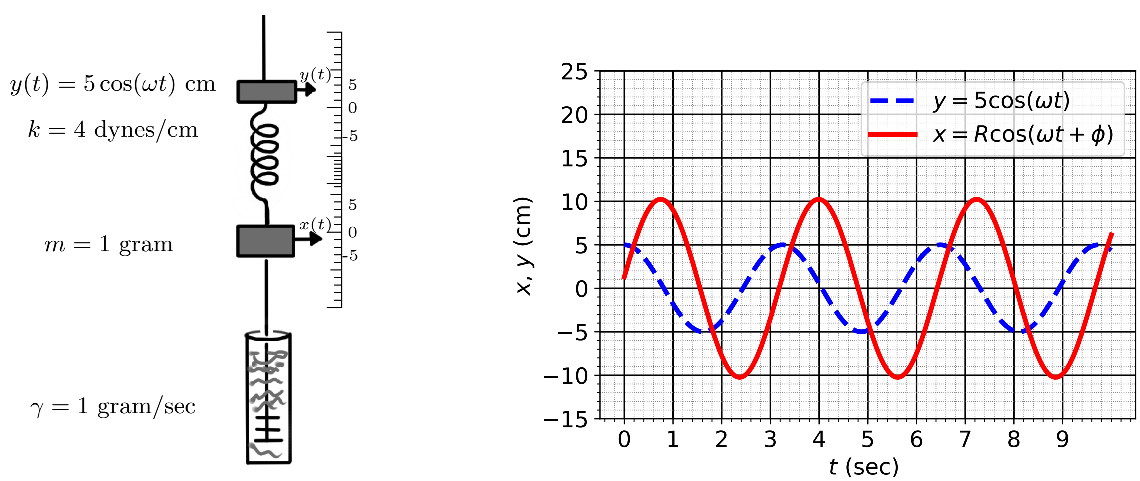


FIGURE 10.4. The oscillating plunger (dashed blue sine curve) causes the mass m to oscillate (red sine curve).

EXAMPLE 10.1. Consider the mechanical system pictured in Figure 10.4. Assume that in its rest configuration $x = 0$ and $y = 0$, where the forces exerted on the mass by gravity and the spring cancel. The net force exerted on the mass by gravity, the spring, and the damping mechanism is then

$$F = -k(x - y(t) - \gamma x' .$$

Applying Newton's second law of motion shows that the position of the mass $x = x(t)$ satisfies the differential equation

$$mx'' = -\gamma\, x' - k(x - y(t)) \text{ or } mx'' + \gamma\, x' + k\, x = ky(t)\,.$$

Assume for simplicity that $m = 1\,\text{gram}$, $k = 4.0\,\text{dynes/cm}$, and $\gamma = 1.0\,\text{grams/sec}$.

Finally assume that the plunger at the top of the figure moves up and down according to the rule $y(t) = 5\cos(\omega t)$ cm; causing the mass to also move up and down. Ignoring transients, $x(t)$ is of the form

$$x(t) = R\cos(\omega t + \phi)$$

where both R and ϕ depend on ω. Find the value of ω that maximizes R.

SOLUTION Set $x(t) = \operatorname{Re}(z(t))$, where $z(t)$ satisfies the differential equation

$$mz'' + \gamma\, z' + k\, z = k\left(5e^{i\omega t}\right)\,.$$

Using the values of m, γ, and k above this simplifies to

$$z'' + z' + 4z = 20\, e^{i\omega t}\,.$$

Substituting $z(t) = Ae^{i\omega t}$ into this equation gives

$$\{(4 - \omega^2) + i\omega\}\}A = 20\,,$$

whose polar form is

$$\sqrt{(4-\omega^2)^2 + \omega^2}\, e^{\alpha i} A = 20\,, \qquad \text{where } \tan(\alpha) = \frac{\omega}{4 - \omega^2}\,, \text{ and } 0 < \alpha < \pi\,.$$

It follows that

$$z(t) = \frac{20}{\sqrt{(4-\omega^2)^2 + \omega^2}} e^{-i\alpha} e^{i\omega t}$$

Consequently, setting $\phi = -\alpha$, gives

$$x(t) = R\cos(\omega t + \phi) = \frac{20}{\sqrt{(4-\omega^2)^2 + \omega^2}} \cos(\omega t - \alpha)\,.$$

To maximize R is suffices to minimize $f(\omega) = (4 - \omega^2)^2 + \omega^2$, which can be accomplished by solving

$$f'(\omega) = -4(4 - \omega^2)\omega + 2\omega = 0$$

for ω. We can ignore the spurious solutions $\omega = 0$, and $\omega = -\sqrt{7/2}$. (Why?) Hence,

$$\omega = \sqrt{\frac{7}{2}} \approx 1.87\text{sec}^{-1}\,, R = \frac{20}{\sqrt{(4 - 7/2)^2 + (7/2)}} \approx 10.33 \text{ and } \phi = -\arctan\left(\frac{\omega}{4-\omega^2}\right) \approx -1.31.$$

Thus, the amplitude of the oscillations in x is about twice the amplitude of the oscillations in y. (See Figure 10.4.)

EXAMPLE 10.2. (AUTOMOBILE STRUTS) A similar analysis can be done for the mechanical system modeling the struts on an automobile. Place the x-axis and the y-axis so that $x = y = 0$ at equilibrium, so the forces of gravity and the spring cancel—for this reason we make no mention of the force of gravity. Then, as shown in Figure 10.5, the spring force F_{spr} and the damping force F_{damp} both depend on the relative values of x and y. Newton's second law of motion then implies that the function $x = x(t)$ is a solution of the differential equation

$$mx'' = -\gamma(x' - y'(t)) - k(x - y(t)) \text{ or } m\, x'' + \gamma\, x' + k\, x = \gamma\, y'(y) + k\, y(t)\,.$$

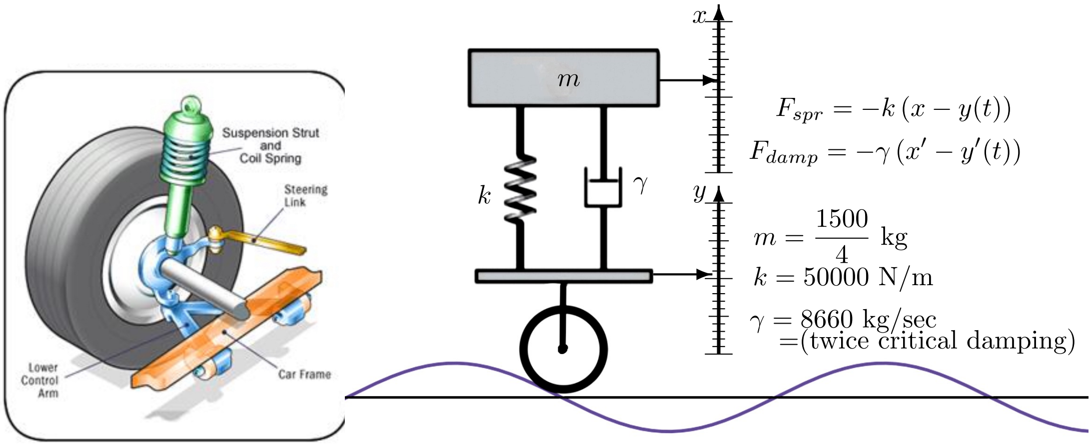


FIGURE 10.5. Left: sketch of strut on an automobile. Right: Simplified model of the system. The values of m, k, and γ in the figure are similar to those found in automobiles. The mass is divided by four because the weight of an automobile is distributed over four wheels.

Assume that the automobile is moving at a constant speed along a straight (<u>but not flat</u>!) road. For simplicity, also assume that the rise and fall of the road is given by the sine function

$$y(t) = a\cos(\omega t)\,,$$

where $a > 0$ is a constant and ω depends on the speed of the car. The steady-state solution is then

$$x(t) = aR\cos(\omega t + \phi)\,,$$

where both R and ϕ have yet to be determined. Since both R and ϕ are independent of a, there is no loss of generality in setting $a = 1$.

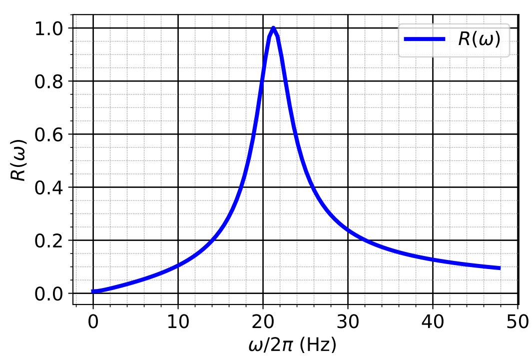


FIGURE 10.6. The maximum response is only slightly above 1. This implies that the amplitude of oscillations in the road is never amplified by the struts, and, in fact, it is reduced except at frequencies of around 21.5 Hz (cycles per second).

As in earlier examples, set $x(t) = \mathrm{Re}\,(z(t))$, where $z(t) = Ae^{i\omega t}$ is a solution of the complex differential equation

$$m\,z'' + \gamma\,z' + kz = k\,e^{i\omega t} + \gamma\,(e^{i\omega t})' = (k + i\gamma\omega)e^{i\omega t}$$

Proceeding as above we arrive at the equation $\left\{(-m\omega^2 + k) + \gamma\,\omega\,i\right\} Ae^{i\omega t} = (k + \gamma\,\omega i)e^{i\omega t}$. Hence,

$$z(t) = \frac{k + \gamma\,\omega\,i}{m(\omega_0^2 - \omega^2) + \gamma\omega i}e^{i\omega t} = \frac{\omega_0^2 + (\gamma/m)\,\omega\,i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i}e^{i\omega t}$$

and

$$R = \left| \frac{\omega_0^2 + (\gamma/m)\,\omega\, i}{(\omega_0^2 - \omega^2) + (\gamma/m)\omega i} \right| = \sqrt{\frac{\omega_0^4 + (\gamma/m)^2\omega^2}{(\omega_0^2-\omega^2)^2 + (\gamma/m)^2\omega^2}}\,.$$

Figure 10.6 shows the graph of R for the numerical values of m, γ, and k given in Figure 10.5.

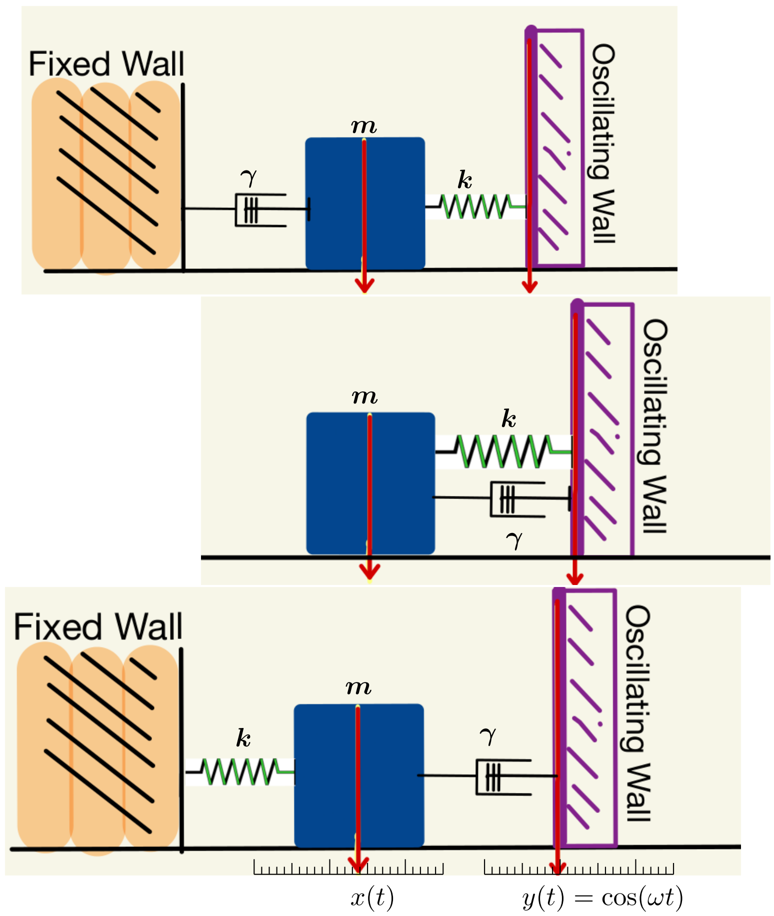


FIGURE 10.7. Three configurations of the driven damped oscillator. From top to bottom:

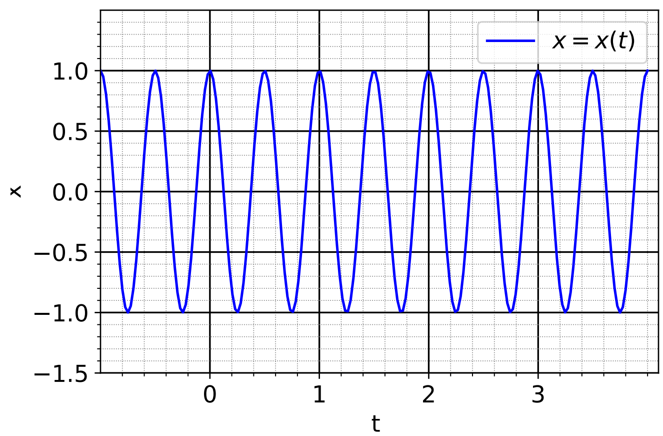
$$mx'' + \gamma x' + kx = f(t) = \begin{cases} ky(t) = k\cos(\omega t) \\ ky(t) + \gamma y'(t) = k\cos(\omega t) - \gamma\omega\sin(\omega t) \\ \gamma y'(t) = -\gamma\omega\sin(\omega t) \end{cases}$$

Example 10.1 above is an instance of the first configuration. Example 10.2 is an instance of the second configuration.

EXERCISE 10.1. A weight of mass $m = 5$ kg is suspended from a spring with unknown spring constant k. The weight is free to move up and down. Ignoring friction, its position relative to its equilibrium position satisfies a differential equation of the form

$$mx'' + kx = 0\,,$$

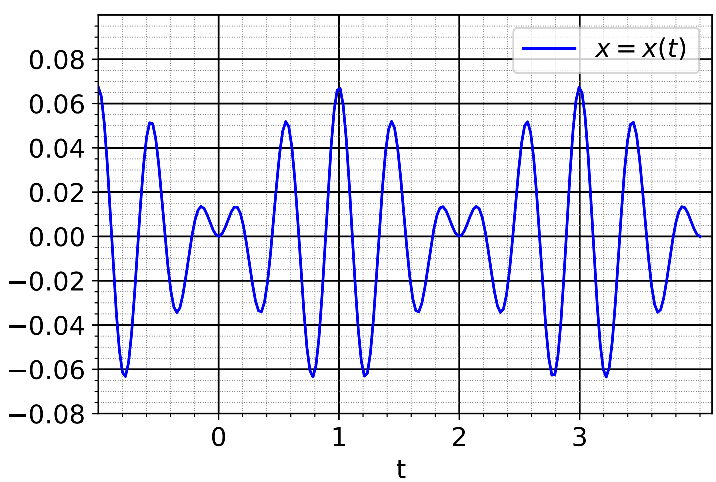
where t denotes time measured in seconds and x denotes its position measured in meters. To find the spring constant, the spring is set in motion and the graph of $u(t)$ plotted. The result is shown in the following figure (horizontal axis is t, vertical axis is x):



In a subsequent experiment, an external force of the form $F(t) = F_0\,\cos(\omega t)$ is applied to the mass so that the function $u(t)$ now obeys the differential equation

$$mx'' + kx = F_0\,\cos(\omega t)\,.$$

The graph of the position $x(t)$ in that experiment is shown in the graph below.

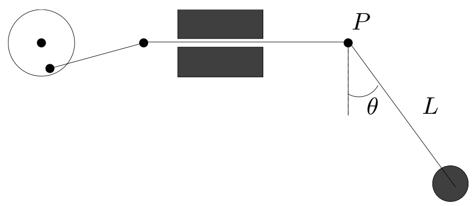


(a)Estimate the spring constant k. (*Your answer can again most easily be expressed in terms of* π.)
(b) Using your estimate of k, estimate as best you can the frequency ω of the applied force. (*Your answer can again most easily be expressed in terms of* π.)
(c) Estimate as best you can the amplitude F_0 of the applied force.

Exercise 10.2. A simple pendulum of mass m and length L is hinged at a point P (see figure). If the wheel at the left of the figure rotates at a rate of ω radians/second it forces the point P to move periodically back and forth. For small angle θ (where $\sin(\theta) \approx \theta$) the angle θ satisfies the differential equation

$$L\frac{d^2\theta}{dt^2} + g\theta = A\omega^2 \cos(\omega t)\,.$$

Assume, for simplicity that $L = 1$ meter, $A = 1$ meter/sec^2, $\omega = 1$ rad/sec and $g = 9.8$ meter/sec^2. Find the solution that satisfies the initial conditions $\theta(0) = \theta'(0) = 0$.



Exercise 10.3. A ball of mass 1 kilogram moves in a viscous fluid. The viscous force on the ball is given by $-v$, where v is the speed of the ball measured in meters per second, and $= 2$ newton-sec/meter. An external force is applied to the ball along a fixed axis and with magnitude

$$F(t) = 2\cos(t)\ \text{N}$$

(t is time measured in seconds.) Let $y(t)$ be the displacement of the ball along the axis of the external force and assume that at time $t = 0$ the ball is at rest and $y = 0$. Find $y(t)$. Ignore gravity.

Exercise 10.4. A spring-mass system has spring constant 3 N/m (i.e. 3 Newtons per meter). A weight of mass 2 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance (measured in Newtons) numerically equal to twice the magnitude of the instantaneous velocity (measured in meters per second).

Let u denote the displacement of the weight from its equilibrium position. If the system is driven by an external force of $3\cos(3t) - 2\sin(3t)$ N, determine the formula for $u(t)$ ignoring all "transients," i.e. $u(t)$ is the steady-state solution. Express your answer in the form $u(t) = A\cos(\omega t + \phi)$.

Exercise 10.5. Find the steady state solution $x(t) = R\cos(\omega t + \phi)$ for the bottom configuration shown in Figure 10.7.

Part 3

Laplace Transforms

CHAPTER 11

Laplace Transforms

This chapter is an introduction to *Laplace transforms*, which provide an alternate way to solve initial value problems of the form

$$\begin{cases} L[y] = ay'' + by' + cy = f(t) & , a,\, b,\, c \text{ constant} \\ y(0) = y_0\,, y'(0) = y_0' \end{cases} \tag{11.1}$$

that is particularly useful when the forcing function $f(t)$ has discontinuities. The idea is to transform the initial value problem into a algebraic equation, solve the algebraic equation for the transformed of the solution, and then inverse transform to obtain the solution of the initial value problem.

More precisely, the Laplace transform turns the initial value problem (11.1) into the equation

$$(as^2 + bs + c)Y(s) = a\, y_0\, s + (b\, y_0 + a\, y_0') + F(s)\,, \tag{11.2}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $F(s)$ is the Laplace transform of $f(t)$. Solving Equation (11.2) for $Y(s)$ gives

$$Y(s) = \frac{a\, y_0\, s + (b\, y_0 + a\, y_0')}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}\,, \tag{11.3}$$

an explicit formula for the Laplace transform of the solution. Computing the *inverse Laplace transform* then solves the initial value problem.

Putting this idea into practice requires knowing how to compute $F(s)$ from $f(t)$ and how to compute $y(t)$ from $Y(s)$. In much the same way that derivatives and integrals are computed from a few basic properties (e.g. the product rule and integration by parts) together with a table of integral, so can Laplace transforms and inverse Laplace transforms be computed from a few basic rules, together with a table of Laplace transforms (see Appendix C).

11.1. Computing Laplace transforms

Suppose $f(t)$ is a function defined for all t with $0 \le t < \infty$. Its *Laplace transform* is the function

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt \tag{11.4}$$

provided this integral converges.[15]

Remark 11.1. The values of $f(t)$ for $t < 0$ have no effect on its Laplace transform. When we use Laplace transforms, we are only interested in solving the initial value problem (11.1) for $t \ge 0$, that is, in the future. We get no information about the past. For all practical purposes, we might as well assume that $f(t) = 0$ for $t < 0$.

[15] Convergence is only briefly discussed in these notes. For virtually all functions encountered in practice, the integral converges when s is sufficiently large.

NOTATION. The notation $\mathcal{L}\{f\}$ is awkward. It is often more convenient to denote the Laplace transform of $f(t)$ by $F(s)$. Similarly, we write $Y(s) = \mathcal{L}\{f\}$, $G(s) = \mathcal{L}\{g\}(s)$, etc. For instance, $\mathcal{L}\{\cos(t)\}$ and $\mathcal{L}\{\cos\}$ both denote the Laplace transform of the function cos.

EXAMPLE 11.1. Here are three cases where the Laplace transform can be directly computed from the definition.

(a) If $f(t) = 1$, then
$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt = \lim_{A\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^A = \lim_{A\to\infty}\left(-e^{-st}+1\right)\frac{1}{s} = \frac{1}{s}\,.$$
Therefore,
$$\mathcal{L}\{1\} = \frac{1}{s}$$
The integral converges to $1/s$ for $s > 0$ and diverges for $s < 0$.

(b) If $f(t) = e^{at}$, then
$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{(-(s-a)t}\,dt \\ &= \lim_{A\to\infty}\int_0^A e^{-(s-a)t}\,dt = \lim_{A\to\infty}\left.\frac{-e^{-(s-a)t}}{(s-a)}\right|_0^A = \lim_{A\to\infty}\left(-e^{-(s-a)t}+1\right)\frac{1}{s-a} = \frac{1}{s-a} \end{aligned}$$
Therefore,
$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$
The integral converges to $1/(s-a)$ only for $s > a$ and diverges for $s < a$.

(c) Finally,
$$\mathcal{L}\{t\} = \frac{1}{s^2}\,.$$
Check this yourself. (Hint: use integration by parts.)

EXAMPLE 11.2. Sometimes, using complex valued functions simplifies the computation of the Laplace transform. Consider the following two cases:
$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt \text{ and } \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt\,.$$
Both integrals could be evaluated directly, but the computations are messy, involving integration by parts twice. It's easier to use complex-valued functions as follows:

Since $e^{iat} = \cos(at) + i\sin(at)$,
$$\mathcal{L}\left\{e^{iat}\right\} = \mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\}\,.$$
Consequently,
$$\begin{aligned} \mathcal{L}\left\{e^{iat}\right\} &= \int_0^\infty e^{-st}e^{iat}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt \\ &= \left.\frac{-e^{-(s-ia)t}}{s-ia}\right|_0^\infty \\ &= \frac{1}{s-ia} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \end{aligned}$$
Therefore,
$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \text{ and } \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \tag{11.5}$$

A similar computation shows that

$$\mathcal{L}\left\{e^{(a+ib)t}\right\} = \frac{1}{s-(a+ib)} \tag{11.6}$$

To see this, compute as before:

$$\begin{aligned}
\mathcal{L}\left\{e^{(a+bi)t}\right\} &= \int_0^\infty e^{-st}e^{(a+bi)t}\,dt \\
&= \int_0^\infty e^{-(s-(a+bi))t}\,dt \\
&= \lim_{A\to\infty} \left.\frac{-e^{-(s-(a+bi))t}}{s-(a+bi)}\right|_0^A \\
&= \frac{1}{s-(a+bi)}\,.
\end{aligned}$$

Note that the last step is only valid for $s > a$.

Because $e^{(a+ib)t} = e^{at}\cos(bt) + ie^{at}\sin(bt)$ and $\dfrac{1}{s-(a+ib)} = \dfrac{s-a}{(s-a)^2+b^2} + i\dfrac{b}{(s-a)^2+b^2}$, it follows that

$$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2} \text{ and } \mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}\,.$$

11.2. Properties of the Laplace transform

Rather than continuing to derive Laplace transforms of specific functions, it is more efficient to find general properties of the Laplace transform.

The Laplace transform is a *linear operator*. This means that if $f(t)$ and $g(t)$ are functions and a and b are numbers, then

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = aF(s) + bG(s)\,, \tag{11.7}$$

where $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively. Linearity follows immediately from linearity of the definite integral:

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = \int_0^\infty e^{-st}(af(t) + bg(t))\,dt = a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt\,.$$

Because of linearity, we can decompose the Laplace transform of a sum of functions as a sum of the Laplace transform of each of the summands.

Example 11.3. By linearity and the table of Laplace transforms,

$$\mathcal{L}\left\{5e^{-2t} - 3\sin(4t)\right\} = 5\mathcal{L}\left\{e^{-2t}\right\} - 3\mathcal{L}\left\{\sin(4t)\right\} = 5\frac{1}{s-(-2)} - 3\frac{4}{s^2+16} = \frac{5}{s+2} - \frac{12}{s^2+16}\,.$$

The next theorem shows that the Laplace transform of the derivatives of a function can be expressed in terms of the Laplace transform of the function, itself.

Theorem 3. *Suppose $g(t)$ is a continuously differentiable with Laplace transform $G(s)$, then*

$$\mathcal{L}\left\{g'\right\} = sG(s) - g(0).$$

If $g(t)$ is has continuous second derivatives, then

$$\mathcal{L}\left\{g''\right\} = s^2G(s) - sg(0) - g'(0)\,.$$

PROOF OF THEOREM. In order for $\mathcal{L}\{g\}$ to exist, $g(t)$ must be *piecewise continuous* (required for the integral to exist) and it must be of *exponential order*: there are constants M and c so that $y(t) \le Me^{ct}$. This implies that when $s > c$,

$$\lim_{a\to\infty} g(a)e^{-sa} = 0.$$

To compute

$$\mathcal{L}\{g'\} = \int_0^\infty e^{-st} g'(t)\,dt,$$

we use integration by parts: $u = e^{-st}$, $dv = g'(t)dt$, so $du = -se^{-st}\,dt$ and $v = g(t)$, so

$$\begin{aligned}
\mathcal{L}\{g'\} &= \int_0^\infty e^{-st} g'(t)\,dt \\
&= e^{-st} g(t)\big|_0^\infty + \int_0^\infty s e^{-st} g(t)\,dt \\
&= -g(0) + s\int_0^\infty e^{-st} g(t)\,dt \\
&= sG(s) - g(0),
\end{aligned}$$

as desired.

If, in addition, $g(t)$ has continuous second derivatives, apply the first part of the theorem, twice as follows:

$$\mathcal{L}\{g''(t)\} = s\mathcal{L}\{g'(t)\} - g'(0) = s(sG(s) - g(0)) - g'(0) = s^2G(s) - sg(0) - g'(0)\,.$$

□

Linearity and Theorem 3 are key ingredients for solving initial value problems. For suppose we have a linear constant coefficient differential equation

$$ay'' + by' + cy = f(t)\,,$$

together with the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$. By linearity, applying $\mathcal{L}\{-\}$ to the differential equation gives

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{f\} = F(s).$$

By Theorem 3 applying $\mathcal{L}\{-\}$ to the terms $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ gives

$$a(s^2Y(s) - sy_0 - y_0') + b(sY(s) - y_0) + cY(s) = F(s)\,,$$

which simplifies to

$$(as^2 + bs + c)Y(s) - (ay_0 s + ay_0' + by_0) = F(s)\,. \tag{11.8}$$

This equation can be solved for $Y(s)$:

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{ay_0 s + ay_0' + by_0}{as^2 + bs + c}\,. \tag{11.9}$$

EXAMPLE 11.4. Consider the initial value problem $y'' - 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$. Applying the Laplace operator $\mathcal{L}\{-\}$, the equation becomes

$$(s^2Y(s) - 2s - 1) - 3(sY(s) - 2) + 2Y(s) = 0,$$

or

$$(s^2 - 3s + 2)Y(s) = 2s - 5.$$

Therefore,

$$Y(s) = \frac{2s-5}{s^2-3s+2} = \frac{2s-5}{(s-1)(s-2)} = \frac{3}{s-1} + \frac{-1}{s-2}\,.$$

We write $\mathcal{L}^{-1}\{-\}$ for the operator that undoes the Laplace transform: if $Y(s) = \mathcal{L}\{y\}$, then $\mathcal{L}^{-1}\{Y\}(t) = y(t)$. Using this notation,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s-1} + \frac{-1}{s-2}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) = 3e^t - e^{2t},$$

where we have used linearity of $\mathcal{L}\{-\}$ and (from Appendix C) the formula $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$.

In the above example, we implicitly assumed that $Y(s)$ determines $y(t)$. In fact, this is the case, as the next theorem shows.

THEOREM 4. *Suppose that f and g are continuous. Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. If for some $c > 0$, $F(s) = G(s)$ for all $s > c$, then $f(t) = g(t)$ for all $t > 0$.*

REMARK 11.2. This theorem is *not* obvious, and in fact the proof is difficult in beyond the scope of this course.

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \text{ (the \textit{exponential shift formula})}. \tag{11.10}$$

PROOF.

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$

$\square$

EXAMPLE 11.5. Since $\mathcal{L}\{\cos(bt)\} = s/(s^2+b^2)$,

$$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2}.$$

Similarly,

$$\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}.$$

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\{tf(t)\}) = -F'(s). \tag{11.11}$$

PROOF. It's easiest to work backwards as follows:

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}\left(e^{-st}f(t)\right)dt = -\int_0^\infty e^{-st}tf(t)dt = -\mathcal{L}\{tf(t)\}.$$

$\square$

EXAMPLE 11.6. Since $\mathcal{L}\{1\} = \dfrac{1}{s}$, $\mathcal{L}\{t\} = -\left(\dfrac{1}{s}\right)' = \dfrac{1}{s^2}$.

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\{t\cdot t\} - \left(\dfrac{1}{s^2}\right)' = \dfrac{2}{s^3}$.

More generally, suppose $\mathcal{L}\left\{t^{n-1}\right\} = \dfrac{(n-1)!}{s^n}$. Then, $\mathcal{L}\left\{t^n\right\} = \mathcal{L}\left\{t \cdot t^{n-1}\right\} - \left(\dfrac{(n-1)!}{s^n}\right)' = \dfrac{(n)!}{s^{n+1}}$. Therefore, by mathematical induction,

$$\mathcal{L}\left\{t^n\right\} = \frac{(n)!}{s^{n+1}}\,.$$

for all positive integers.

EXAMPLE 11.7. Since, $\mathcal{L}\left\{t^3\right\} = \dfrac{6}{s^4}$,

$$\mathcal{L}\left\{t^3 e^{5t}\right\} = \frac{6}{(s-5)^4}$$

EXAMPLE 11.8. Since, $\mathcal{L}\left\{\sin(at)\right\} = \dfrac{a}{s^2+a^2}$,

$$\mathcal{L}\left\{t\sin(at)\right\} = -\left(\frac{a}{s^2+a^2}\right)' = \frac{(2as}{(s^2+a^2)^2}$$

Similarly, since $\mathcal{L}\left\{\cos(at)\right\} = \dfrac{s}{s^2+a^2}$,

$$\mathcal{L}\left\{t\cos(at)\right\} = -\left(\frac{s}{s^2+a^2}\right)' = \frac{(s^2-a^2}{(s^2+a^2)^2}$$

Suppose $\mathcal{L}\left\{f(t)\right\} = F(s)$, then

$$\mathcal{L}\left\{f(at)\right\} = \frac{1}{a}F(s/a)\,.$$

PROOF. Compute as follows, using the "u-substitution" $u = at$, $du = adt$:

$$\mathcal{L}\left\{f(at)\right\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-s(u/a)} f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)du = \frac{1}{a}F(s/a)\,.$$

□

EXAMPLE 11.9. Because $\mathcal{L}\left\{\cos(t)\right\} = \dfrac{s}{s^2+1}$, it follows that

$$\mathcal{L}\left\{\cos(at)\right\} = \frac{1}{a}\frac{(s/a)}{(s/a)^2+1} = \frac{s}{s^2+a^2}\,.$$

11.3. Computing the Inverse Laplace Transform

In Section 11.2, we found a general formula for the Laplace transform of the solution of an initial value problem. To find the solution, itself, we have to compute the inverse Laplace transform.

Computing the inverse Laplace transform often involves the *partial fraction expansion*[16] of the Laplace transform.

EXAMPLE 11.10. Find the inverse Laplace transform of $F(s) = \dfrac{3s}{s^2-s-6}$.

SOLUTION Compute the partial fractions expansion of $F(s)$ as follows:

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2)+B(s-3)}{(s-3)(s+2)}$$

[16]This is a good time to read Appendix B, which presents a quick review of partial fractions.

Comparing numerators gives $3s = A(s+2) + B(s-3)$. Set $s = 3$ to conclude that $9 = A(5)$ or $A = 5/9$. Set $s = -2$ to conclude that $-6 = B(-5)$ or $B = 6/5$. Hence

$$\frac{3s}{s^2 - s - 6} = \frac{5/9}{s-3} + \frac{6/5}{s+2}\,.$$

We can now use the table of Laplace transforms to compute as follows:

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2 - s - 6}\right\} = \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{5}{9}e^{3t} + \frac{6}{5}e^{-2t}\,.$$

Example 11.11. Find the inverse Laplace transform of $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2+4)}$.

Solution First compute the partial fractions expansion of $F(s)$:

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2+4} = \frac{A(s^2+4) + s(Bs+C)}{s(s^2+4)} = \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)}$$

Comparing coefficients of powers of s in the numerator, we find that

$$A = 3\,, \quad C = -4\,, \text{ and } B = 9 - 3 = 5\,.$$

Therefore,

$$F(s) = \frac{3}{s} + \frac{5s - 4}{s^2+4} = 3\left(\frac{1}{s}\right) + 5\left(\frac{s}{s^2+4}\right) - \frac{4}{2}\left(\frac{2}{s^2+4}\right)\,.$$

From the table of Laplace transforms, it now follows that

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{4}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 3 + 5\cos(2t) - 2\sin(2t)\,. \end{aligned}$$

Example 11.12. Find the inverse Laplace transform of $F(s) = \dfrac{2s-3}{s^2 + 2s + 10}$.

Solution The denominator has complex roots, so complete the square and rewrite $F(s)$ as follows:

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2 + 9} = \frac{2(s+1) - 5}{(s+1)^2 + 9}$$

Therefore,

$$F(s) = 2\left(\frac{(s+1)}{(s+1)^2+9}\right) - \frac{5}{3}\left(\frac{3}{(s+1)^2+9}\right)$$

From the table of Laplace transforms, it now follows that the inverse Laplace transform of $F(s)$ is

$$f(t) = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

EXAMPLE 11.13. Find the inverse Laplace transform of $Y(s) = \dfrac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$.

SOLUTION First compute the partial fractions expansion of $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As + B}{s^2+1} + \frac{Cs + D}{s^2+4} \\ &= \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+4)} = \frac{(A+C)s^3 + (B+D)s^2 + (4A+D)s + (4B+D)}{(s^2+1)(s^2+4)} \end{aligned}$$

Comparing the coefficients of powers of s in numerators results in the system of four equations in four unknowns

$$A + C = 2\,, \quad B + D = 1\,, \quad 4A + C = 8\,, \quad 4B + D = 6\,,$$

which we can solve to obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Therefore,

$$Y(s) = 2\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right).$$

Consequently, the inverse Laplace transform of $Y(s)$ is

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)\,.$$

11.4. Initial Value Problems with Continuous Forcing Function

Below are some examples illustrating the use of Laplace transforms for solving initial value problems. All of these examples could (sometimes more easily) be done using the method of undetermined coefficients. The purpose of these examples is mainly to illustrate the method. In later sections, more interesting examples are presented where the forcing function is not continuous and the method of undetermined coefficients does not apply.

EXAMPLE 11.14. Solve the initial value problem $y'' + 4y = \cos(3t)$, $y(0) = 0$, $y'(0) = 0$.

SOLUTION Applying $\mathcal{L}\{-\}$ gives

$$(s^2+4)Y(s) = \frac{s}{s^2+9}, \text{ or } Y(s) = \frac{s}{(s^2+4)(s^2+9)} = \frac{s/5}{s^2+4} - \frac{s/5}{s^2+9}.$$

From the table of Laplace transforms,

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \frac{1}{5}(\cos(2t) - \cos(3t))\,.$$

EXAMPLE 11.15. Solve the initial value problem $y'' + 4y = \cos(2t)$, $y(0) = 0$, $y'(0) = 0$.

SOLUTION Applying $\mathcal{L}\{-\}$ gives

$$(s^2+4)Y(s) = \frac{s}{s^2+4}, \text{ or } Y(s) = \frac{s}{(s^2+4)^2}.$$

Using the table of Laplace transforms:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2\cdot 2\cdot s}{(s^2+2^2)^2}\right\} = \frac{1}{4}t\sin(2t).$$

EXAMPLE 11.16. Laplace transforms can also be used to solve linear constant coefficient first order initial value problems. For instance, consider the initial value problem

$$y' + 2y = \cos(t)\,, \quad y(0) = 1\,.$$

Computing the Laplace transform of both sides gives

$$sY(s) - 1 + 2Y(s) = \frac{s}{s^2+1}\,,$$

which can be solved for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{s}{(s+2)(s^2+1)} + \frac{1}{s+2} = \frac{(2/5)s + (1/5)}{s^2+1} - \frac{2/5}{s+2} + \frac{1}{s+2} \\ &= \frac{(2/5)s}{s^2+1} + \frac{1/5}{s^2+1} + \frac{3/5}{s+2}\,. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} &= \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) + \frac{3}{5}e^{-2t} \end{aligned}$$

EXAMPLE 11.17. Solve the initial value problem

$$y'' - 3y' + 2y = 2e^{-3t}\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

SOLUTION Applying $\mathcal{L}\left\{-\right\}$ and setting $Y(s) = \mathcal{L}\left\{y\right\}$ yields the equations

$$(s^2Y(s) - s) - 3(sY(s) - 1) + 2Y(s) = \frac{2}{s+3}\,,$$

which simplifies to

$$(s^2 - 3s + 2)Y(s) - s + 3 = \frac{2}{s+3}\,.$$

Solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{2}{(s+3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2} \\ &= \frac{s^2-7}{(s-1)(s-2)(s+3)} = \frac{3/2}{s-1} + \frac{-3/5}{s-2} + \frac{1/10}{s+3}. \end{aligned}$$

Consequently,

$$y(t) = \frac{3}{2}e^{t} - \frac{3}{5}e^{2t} + \frac{1}{10}e^{-3t}\,.$$

EXAMPLE 11.18. Solve the initial value problem $y'' + 2y' + 2y = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$.

SOLUTION Proceeding as in the previous example, apply $\mathcal{L}\left\{-\right\}$, solve for $Y(s)$, compute the partial fractions expansion for $Y(s)$, and finally, compute the inverse Laplace transform.

Here's the (somewhat messy!) computation omitting some algebra:

$$\begin{aligned} (s^2Y(s) - s) + 2(sY(s) - 1) + 2Y(s) &= \frac{s}{s^2+4} \\ (s^2 + 2s + 2)Y(s) &= \frac{s}{s^2+4} + s + 2\,. \end{aligned}$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{s}{(s^2+4)(s^2+2s+2)} + \frac{s+2}{s^2+2s+2} \\ &= \frac{As+B}{s^2+4} + \frac{C(s+1)+D}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1}. \\ &= \frac{-\frac{1}{10}s+\frac{4}{10}}{s^2+4} + \frac{\frac{1}{10}(s+1)-\frac{3}{10}}{(s+1)^2+1} + \frac{(s+1)+1}{(s+1)^2+1} \\ &= -\frac{1}{10}\,\frac{s}{s^2+4} + \frac{2}{10}\,\frac{2}{s^2+4} + \frac{11}{10}\,\frac{s+1}{(s+1)^2+1} + \frac{7}{10}\,\frac{1}{(s+1)^2+1}. \end{aligned}$$

Using the table of Laplace transforms gives

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = -\frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + \frac{11}{10}e^{-t}\cos(t) + \frac{7}{10}e^{-t}\sin(t).$$

11.5. The Laplace Transform of Piecewise Continuous Functions

The Laplace transform is a useful tool when the forcing function $f(t)$ is piecewise continuous. Piecewise continuous forcing functions, such as those pictured in Figure 11.1, routinely occur in engineering applications, particularly in engineering applications involving signal processing.

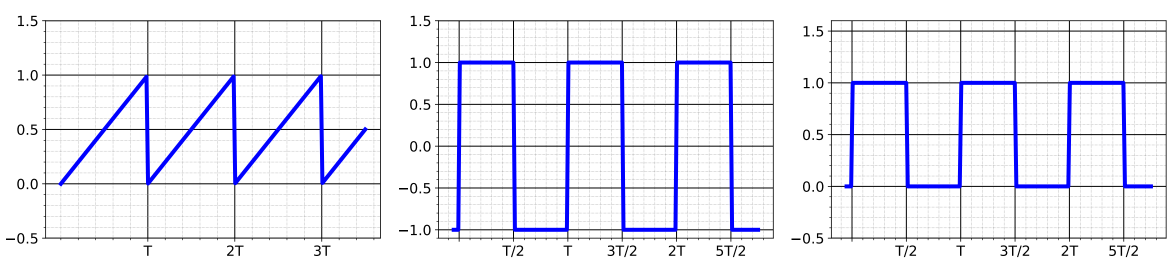


FIGURE 11.1. From left to right: a sawtooth wave, a square wave, and a pulse wave.

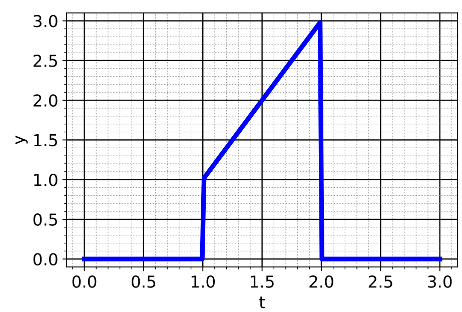
The *Heaviside step function*, denoted by $u_a(t)$ is the basic building block for constructing piecewise continuous function. It is defined as follows:

$$u_a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a\,. \end{cases}$$

The difference $u_a(t) - u_b(t)$, $b > a$, of two Heaviside step functions forms a pulse.

EXAMPLE 11.19. Let $f(t)$ be the function defined by $f(t) = \begin{cases} 0 & \text{if } t < 1, \\ 2t-1 & \text{if } 1 \leq t < 2, \\ 0 & \text{if } t \geq 2, \end{cases}$ then $f(t) = (2t-1)(u_1(t) - u_2(t))$.

FIGURE 11.2. The Heaviside step function $u_a(t)$ and its difference $u_a(t) - u_b(t)$ are the basic building blocks for constructing piecewise continuous functions.



EXAMPLE 11.20. The Heaviside step function is particularly useful in representing waves commonly found in engineering applications, such as *sawtooth waves*, *square waves*, and *pulse waves.*, illustrated in Figure 11.1. A sawtooth wave of period T and amplitude 1 can be represented as follows

$$f_{saw}(t) = \frac{t}{T} - \sum_{k=1}^{\infty} u_{kT}(t)\,; \tag{11.12a}$$

while a square wave of period T and amplitude 1 can be represented by

$$f_{sqr}(t) = u_0(t) + 2\sum_{k=1}^{\infty}(-1)^k u_{kT/2}(t)\,; \tag{11.12b}$$

and a pulse wave of period T and amplitude 1 can we represented by

$$f_{pulse}(t) = u_0(t) + \sum_{k=1}^{\infty}(-1)^k u_{kT/2}(t)\,. \tag{11.12c}$$

PROPOSITION 5. *The Laplace transform of $u_a(t)$ is e^{-as}/s. If $f(t)$ is a function with Laplace transform $F(s)$, then*

$$\mathcal{L}\left\{u_a(t)f(t-a)\right\} = e^{-as}F(s). \tag{11.13}$$

PROOF. The integral defining the Laplace transform is

$$\mathcal{L}\left\{u_a(t)f(t-a)\right\} = \int_0^\infty e^{-st}u_a(t)f(t-a)\,dt = \int_a^\infty e^{-st}f(t-a)\,dt.$$

Now make a change of variables: let $w = t - a$. When $t = a$, $w = 0$, and when $t = \infty$, $w = \infty$, so the integral becomes

$$\int_0^\infty e^{-s(w+a)}f(w)\,dw = \int_0^\infty e^{-sw}e^{-sa}f(w)\,dw = e^{-sa}\int_0^\infty e^{-sw}f(w)\,dw = e^{-sa}\mathcal{L}\left\{f\right\}.$$

The formula for the Laplace transform of $u_a(t)$ is a special case: set $f(t) = 1$ and recall that $1/s$ is the Laplace transform of 1, $\square$

EXAMPLE 11.21. If $f(t) = u_1(t)(t-1)$, then $\mathcal{L}\{f\} = e^{-s}/s^2$.

EXAMPLE 11.22. Suppose $f(t) = (u_1(t) - u_2(t))(2t-1)$. To make use of Proposition 5, rewrite $f(t)$ as follows

$$f(t) = u_1(t)(2t-1) - u_2(t))(2t-1) = u_1(t)(2(t-1)+1) - u_2(t)(2(t-2)+3)\,.$$

Proposition 5 then yields the formula $\mathcal{L}\{f\} = e^{-s}\left(\dfrac{2}{s^2} + \dfrac{1}{s}\right) - e^{-2s}\left(\dfrac{2}{s^2} + \dfrac{3}{s}\right)$.

EXAMPLE 11.23. The Laplace transforms of the sawtooth, square, and pulse waves are, respectively,

$$F_{saw}(s) = \frac{1}{Ts^2} - \left(\sum_{k=1}^{\infty} e^{-kTs}\right)\frac{1}{s}\,,$$

$$F_{sqr}(s) = \frac{1}{s} + \left(\sum_{k=1}^{\infty} (-1)e^{-k(T/2)s}\right)\frac{2}{s}\,,$$

and

$$F_{saw}(s) = \frac{1}{s} + \left(\sum_{k=1}^{\infty} (-1)^k e^{-k(T/2)s}\right)\frac{1}{s}\,.$$

REMARK 11.3. The following variant of the formula (11.13) is occasionally useful:

$$\mathcal{L}\{u_a(t)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\} \tag{11.14}$$

To show this, let $g(t) = f(t+a)$. Then $f(t) = g(t-a)$. Applying (11.13) to $g(t)$ shows

$$\mathcal{L}\{u_a(t)f(t)\} = \mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}\;.$$

For instance,

$$\begin{aligned}\mathcal{L}\{u_3(t)(2t-1)\} &= e^{-2s}\mathcal{L}\{2(t+3)-1\} = e^{-2s}\mathcal{L}\{2t+5\}\\ &= e^{-2s}\left(2\mathcal{L}\{t\} + 5\mathcal{L}\{1\}\right) = e^{-2s}\left(\frac{2}{s^2} + \frac{5}{s}\right)\,.\end{aligned}$$

11.6. Initial Value Problems with Piecewise Continuous Forcing Functions

Consider the differential equation

$$ay'' + by' + cy = f(t)\,,$$

where $f(t)$ is piecewise-continuous. What does it mean for $y(t)$ to be a solution of an equation like this? If $f(t)$ is discontinuous at some point $t = t_0$, will $y''(t)$ even be defined there? If not, how can the equation be satisfied? To avoid these issues, declare a function $y(t)$ to be a solution to an equation like this if

- $y(t)$ is continuous everywhere,
- $y'(t)$ is continuous everywhere, and
- $y(t)$ satisfies the differential equation at every point where the right-hand side $f(t)$ is continuous.

Thus $y''(t)$ need not be defined (and in practice usually won't be defined) at points of discontinuity of the right side; b it will, however, be defined at all other points.

EXAMPLE 11.24. Solve the initial value problem

$$y'' + 5y' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0\,,$$

where $f(t) = u_1(t) - u_{10}(t)$.

SOLUTION Applying the Laplace transform yields the equation

$$(s^2 + 5s + 4)Y(s) = \frac{e^{-s}}{s} - \frac{e^{-10s}}{s}\,.$$

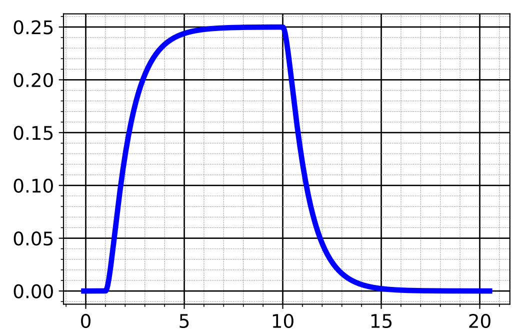
Hence,

$$Y(s) = \left(e^{-s} - e^{-10s}\right)\frac{1}{s(s+1)(s+4)} = \left(e^{-s} - e^{-10s}\right)\left(\frac{1/4}{s} + \frac{-1/3}{s+1} + \frac{1/12}{s+4}\right).$$

Let $p(t) = 1/4 - 1/3e^{-t} + 1/12e^{-4t}$, so that $p(t)$ is the inverse Laplace transform of the last term on the right. Then

$$\begin{aligned} y(t) &= u_1(t)p(t-1) - u_{10}(t)p(t-10) \\ &= u_1(t)\left(\frac{1}{4} - \frac{1}{3}e^{-t+1} + \frac{1}{12}e^{-4t+4}\right) + u_{10}(t)\left(\frac{1}{4} - \frac{1}{3}e^{-t+10} + \frac{1}{12}e^{-4t+40}\right). \end{aligned}$$

The solution is graphed below.



EXAMPLE 11.25. Solve the initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = 2,\ y'(0) = 0\,, \text{ where } f(t) = \begin{cases} 0 & \text{if } t < 1, \\ t-1 & \text{if } 1 \le t < 2, \\ 0 & \text{if } t \ge 2. \end{cases}$$

SOLUTION In this case,

$$f(t) = u_1(t)(t-1) - u_2(t)(t-1) = u_1(t)(t-1) - u_2(t)((t-2)+1).$$

Apply $\mathcal{L}\{-\}$ to the differential equation:

$$\begin{aligned} (s^2Y - 2s) - (3sY - 6) + 2Y &= (e^{-s} - e^{-2s})\frac{1}{s^2} - e^{-2s}\frac{1}{s} \\ (s^2 - 3s + 2)Y - 2s + 6 &= (e^{-s} - e^{-2s})\frac{1}{s^2} - e^{-2s}\frac{1}{s},. \end{aligned}$$

Solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{2s-6}{s^2-3s+2} + \left(e^{-s} - e^{-2s}\right)\frac{1}{s^2(s^2-3s+2)} - e^{-2s}\frac{1}{s(s^2-3s+2)} \\ &= \frac{4}{s-1} + \frac{-2}{s-2} + (e^{-s} - e^{-2s})\left(\frac{1/2}{s^2} + \frac{3/4}{s} + \frac{-1}{s-1} + \frac{1/4}{s-2}\right) - e^{-2s}\left(\frac{1/2}{s} + \frac{-1}{s-1} + \frac{1/2}{s-2}\right). \end{aligned}$$

If we let $p(t)$ denote the inverse Laplace transform of the sum of fractions in the left-hand parentheses, and $q(t)$ the the inverse Laplace transform of the terms in the right-hand set, then

$$p(t) = \frac{1}{2}t + \frac{3}{4} - e^t + \frac{1}{4}e^{2t}, \quad q(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t},$$

and the solution $y(t)$ can be written as follows:

$$y(t) = 4e^t - 2e^{2t} + u_1(t)p(t-1) - u_2(t)p(t-2) - u_2(t)q(t-2).$$

After lots of algebra this reduces to

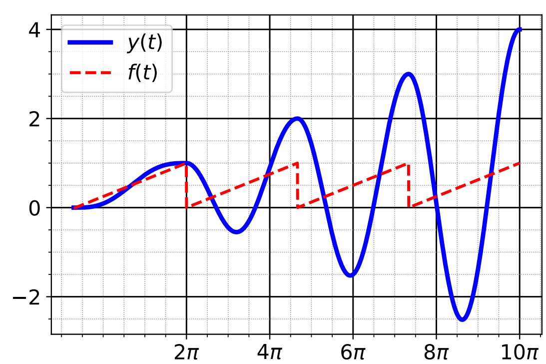
$$y(t) = \begin{cases} 4e^t - 2e^{2t} & \text{if } 0 \le t < 1, \\ 4e^t - 2e^{2t} + \frac{1}{2}(t-1) + \frac{3}{4} - e^{t-1} + \frac{1}{4}e^{2t-2} & \text{if } 1 \le t < 2, \\ (4 - e - 2e^{-2})e^t + \left(-2 + \frac{1}{4}e^{-2} - \frac{3}{4}e^{-4}\right)e^{2t} & \text{if } t \ge 2. \end{cases}$$

Example 11.26. Find the solution to the initial value problem $y'' + y = f(t)\,, \quad y(0) = 0, \quad y'(0) = 0,$ where $f(t)$ is the sawtooth wave with period $T = 2\pi$:

$$f(t) = \frac{t}{2\pi} - \sum_{k=1}^{\infty} u_{2k\pi}(t).$$

Solution Applying the Laplace transform to this initial value problem and solving for $Y(s)$, we find that

$$\begin{aligned} Y(s) &= \frac{1}{2\pi s^2(s^2+1)} - \frac{1}{s^2+1}\left(\sum_{k=1}^{\infty} e^{-2k\pi s}\right)\frac{1}{s} \\ &= \frac{1}{2\pi}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - \left(\sum_{k=1}^{\infty} e^{-2k\pi s}\right)\left(\frac{1}{s} - \frac{s}{(s^2+1)}\right). \end{aligned}$$



The inverse transform of $Y(s)$ is then

$$y(t) = \frac{1}{2\pi}(t - \sin(t)) - \sum_{k=1}^{\infty} u_{2k\pi}(t)h(t - 2k\pi)$$

where $h(t) = 1 - \cos(t)$, the inverse Laplace transform of $\dfrac{1}{s} - \dfrac{s}{s^2+1}$. The solution $y(t)$ together with the forcing function $f(t)$ are graphed above.

EXAMPLE 11.27. Suppose that $f(t)$ is defined by $f(t) = \begin{cases} 100\sin(40t) & \text{when } 0 \le t < 7, \\ 0 & \text{when } t \ge 7. \end{cases}$

Solve the initial value problem $y'' + 3y' + 2y = f(t), \quad y(0) = 0,\ y'(0) = 0.$

SOLUTION In this case, it's easier to work with complex-valued functions. Since $f(t) = (1-u_7(t))100\sin(40t) = 100(1-u_7(t))\mathrm{Re}\,(-ie^{i40t})$, the solution of the original initial value problem is the real part of the solution of the initial value problem

$$z'' + 3z' + 2z = g(t)\,, \quad z(0) = 0, z'(0) = 0\,,$$

where $g(t) = -100i(1-u_7(t))e^{i40t}$. Applying the Laplace transform gives

$$Z(s) = \frac{G(s)}{s^2+3s+2} = \frac{G(s)}{(s+2)(s+1)}\,.$$

The Laplace transform of $g(t)$ is

$$\begin{aligned} G(s) &= -100i\left(\mathcal{L}\left\{e^{i40t}\right\} - e^{-7s}\mathcal{L}\left\{e^{i40(t+7)}\right\}\right) = -100i\left(1-e^{280i}e^{-7s}\right)\mathcal{L}\left\{e^{i40t}\right\} \\ &= -\left(1-e^{280i}e^{-7s}\right)\frac{100i}{s-40i}\,. \end{aligned}$$

Therefore, (by a messy partial fractions computation, that can be skipped[17])

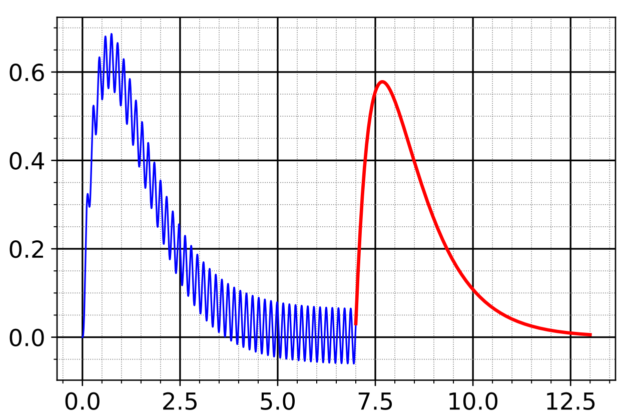
$$\begin{aligned} Z(s) &= \left(1-e^{280i}e^{-7s}\right)\frac{(-100i)}{(s-40i)(s+2)(s+1)} \\ &= \left(1-e^{280i}e^{-7s}\right)\left(\frac{-0.00467+0.0622i}{s-40i} - \frac{2.4938+0.1247i}{s+2} + \frac{2.498+0.06246i}{s+1}\right) \\ &= \left(1-e^{280i}e^{-7s}\right)\mathcal{L}\left\{h(t)\right\} \end{aligned}$$

where $\quad h(t) = (-0.00467+0.0622i)e^{i40t} - (2.4938+0.1247i)e^{-2t} + (2.498+0.06246i)e^{-t}$

Hence, $z(t) = h(t) - e^{280i}u_7(t)h(t-7)$

Finally,

$$y(t) = \mathrm{Re}\,(z(t)) = \begin{cases} -2.49e^{-2t} + 2.50e^{-t} - 0.0622\sin(40t) - 0.00467\cos(40t), & \text{if } 0 \le t < 7, \\ -2.25e^{-2(t-7)} + 2.28e^{-(t-7)}, & \text{if } t \ge 7. \end{cases}$$



[17] The computation without using complex-valued functions is worse!

11.7. The Dirac Delta Function/Impulse Response

Laplace transform techniques are useful in cases where the forcing function $f(t)$ represents "impulses" of short duration.

As a motivating example, consider an object of mass m (in kilograms) free to move in a straight line. Let $x(t)$ be the position (in meters) of the object at time t seconds and let $v(t)$ be its velocity. Suppose also that $x(0) = 0$ and $v(0) = 0$.

Suppose that (as shown in Figure 11.3) at time $t = a$ a positive force is exerted on the object for ε seconds and vanishes for $t > a + \varepsilon$, where $\varepsilon > 0$ is assumed to be a small number. For instance, the object could be a football or baseball suddenly struck by a foot or a bat.

Label this force $f_\varepsilon(t)$, and assume that $f_\varepsilon(t)$ satisfies the following condition:

$$\int_a^{a+\varepsilon} f_\varepsilon(t)\,dt = J\,,$$

where J is a fixed constant. This integral is called an *impulse* and has the dimensions of momentum (Newton-seconds or kilogram-meters/second).

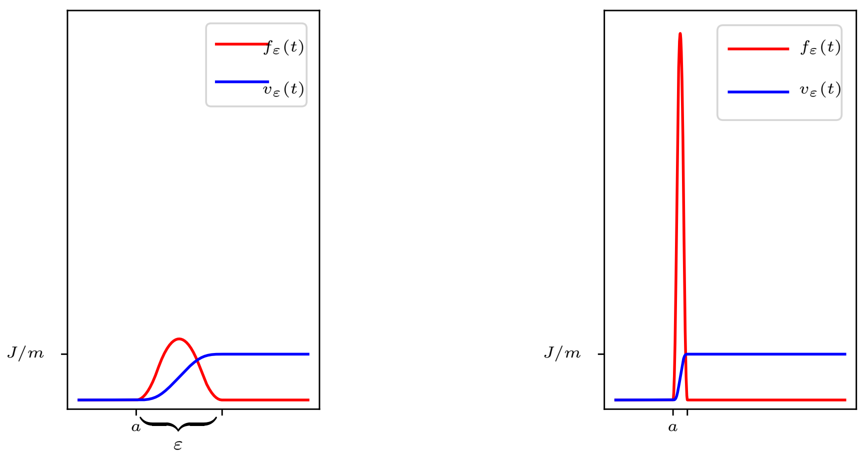


FIGURE 11.3. As ε approaches zero, $y_\varepsilon(t)$ approaches $(J/m)u_a(t)$, a multiple of the Heaviside step function.

In this situation, Newton's second law of motion assumes the simple form

$$m\frac{dv}{dt} = f(t)\,, \quad v(0) = 0\,,$$

which we can integrate to find

$$mv(t) = \int_0^t f(\tau)\,d\tau\,.$$

It is useful to discuss the process as t increases: $v(t) = 0$ until $t = a$, at which time $v(t)$ increases until time $t = a + \varepsilon$. After that time, $v(t) = J/m$ because no force is being exerted on the object after that time.

Imagine now what happens if the impulse J stays constant, but ε approaches zero. To keep J constant, the values of $f_\varepsilon(t)$ have to become large on the interval $a \le t < a+\varepsilon$. For very small values of ε, the graph of $v(t)$ will become almost indistinguishable from the graph of the step function $(J/m)u_a(t)$. The specific choice of $f_\varepsilon(t)$ is unimportant: we only need to insist that it vanishes outside the interval $a \le t < a+\varepsilon$ and that its integral remains equal to J.

To understand the behavior of the Laplace transform of $f_\varepsilon(t)$ as ε approaches 0, assume for simplicity, assume that $m = 1$, $J = 1$, and that $f_\varepsilon(t)$ has the special form:

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\left(u_a(t) - u_{a+\varepsilon}(t)\right) = \begin{cases} 0 & t < a, \\ 1/\varepsilon & a \le t < a+\varepsilon, \\ 0 & t > a+\varepsilon. \end{cases}$$

The Laplace transform of $f_\varepsilon(t)$ can then be computed as follows:

$$\mathcal{L}\left\{f_\varepsilon(t)\right\} = \frac{1}{\varepsilon}\left(\mathcal{L}\left\{u_a(t)\right\} - \mathcal{L}\left\{u_{a+\varepsilon}(t)\right\}\right) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right) = e^{-as}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

Using l'Hôpital's rule, the limit as ε approaches zero of the Laplace transforms is easily found:

$$\lim_{\varepsilon\to 0}\mathcal{L}\left\{f_\varepsilon(t)\right\} = e^{-as}\lim_{\varepsilon\to 0}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-as}\lim_{\varepsilon\to 0}\left(\frac{se^{-\varepsilon s}}{s}\right) = e^{-as}.$$

Roughly speaking, the *Dirac Delta Function* $\delta_a(t)$ is defined by

$$\delta_a(t) = \lim_{\varepsilon\to 0^+} f_\varepsilon(t).$$

Although this is not a well-defined function[18], it does have a well-defined Laplace transform:

$$\mathcal{L}\left\{\delta_a(t)\right\} = e^{-as}. \tag{11.15}$$

REMARK 11.4. When $a = 0$, the subscript is dropped and the notation $\delta(t)$ is used. The identity (11.15) then reduces to

$$\mathcal{L}\left\{\delta(t)\right\} = 1.$$

An alternate notation for $\delta_a(t)$ is $\delta(t-a)$. Then the delta function satisfies the identity

$$\mathcal{L}\left\{\delta(t-a)\right\} = e^{-as}\mathcal{L}\left\{\delta(t)\right\} = e^{-as}\,1 = e^{-as}$$

which is consistent with the general formula $\mathcal{L}\left\{u_a(t)f(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t)\right\}$.

EXAMPLE 11.28. Laplace transforms give a way to model the dynamics of a force that acts instantaneously on an object of mass m:

$$m\frac{dv}{dt} = \lim_{\varepsilon\to 0} f_\varepsilon(t) = J\delta_a(t), \quad v(0) = 0.$$

Taking Laplace transforms gives

$$msV(s) = Je^{-as} \implies V(s) = (J/m)\frac{e^{-as}}{s}.$$

Therefore,

$$v(t) = (J/m)\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = (J/m)u_a(t).$$

Rather than solving for $v(t)$, one can apply Newton's second law of motion:

$$mx''(t) = J\delta_a(t), \quad x(0) = 0, \quad x'(0) = 0.$$

[18] It is something called a "generalized function" or a "distribution," not an actual function.

Taking Laplace transforms gives $X(s) = (J/m)e^{-as}/s^2$. Consequently,

$$x(t) = (J/m)\mathcal{L}^{-1}\left\{e^{-as}/s^2\right\} = (J/m)(t-a)u_a(t)\,,$$

as expected.

EXAMPLE 11.29. Solve the initial value problem

$$y'' + 2y' + 2y = \delta_1(t), \quad y(0) = 0,\ y'(0) = 0.$$

SOLUTION Apply the Laplace transform and solve for $Y(s)$:

$$(s^2 + 2s + 2)Y(s) = e^{-s} \implies Y(s) = e^{-s}\frac{1}{s^2 + 2s + 2}.$$

Complete the square and write $Y(s)$ in the form

$$Y(s) = e^{-s}\frac{1}{s^2 + 2s + 2} = e^{-s}\frac{1}{(s+1)^2 + 1}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = u_1(t)h(t-1)\,,$$

where

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin(t).$$

So

$$y(t) = u_1(t)e^{-(t-1)}\sin(t-1) = \begin{cases} 0 & \text{if } t < 1, \\ e^{-(t-1)}\sin(t-1) & \text{if } t \geq 1. \end{cases}$$

Note that this function is continuous everywhere, but it is not differentiable at $t = 1$. This is not surprising, because $t = 1$ is when the delta function is applied—this example models what happens in a damped mass-spring system when you hit the mass with a hammer.

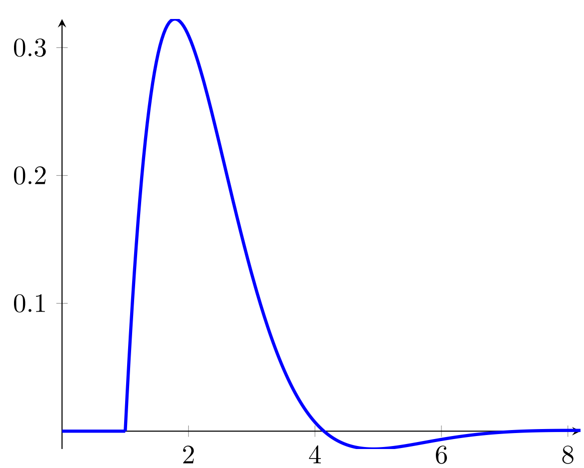


FIGURE 11.4. The graph of $y(t) = u_1(t)e^{-(t-1)}\sin(t-1)$.

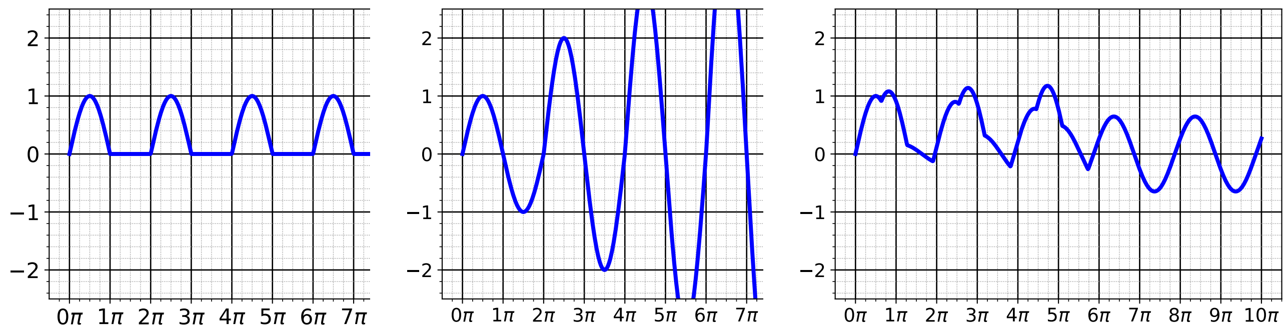


FIGURE 11.5. From left to right the solution of the initial value problem (11.16) for $T=\pi$, $T=2\pi$, and $T=2.0$. Resonance occurs when $T=2\pi$, the natural frequency of the oscillator.

11.8. Modeling Examples

EXAMPLE 11.30. Consider a mass-spring system, with mass $m=1$ kilogram and spring constant $k=1$ Newton/meter. Suppose, in addition, the mass is repeatedly struck with a unit impulse every T seconds. The following initial value problem models this situation:

$$y''+y=f(t)\,,\quad y(0)=y'(0)=0\,,\tag{11.16}$$

where $f(t)=\displaystyle\sum_{j=0}^{\infty}\delta_{jT}(t)$, and $T>0$. The natural frequency of this harmonic oscillator is 2π. Therefore, we expect to observe some sort of resonance when $T=2\pi$ (see Figure 11.5).

Taking Laplace transforms gives

$$Y(s)=\sum_{j=0}^{\infty}\frac{e^{-jTs}}{s^2+1}\,.$$

Taking inverse Laplace transforms yields the solution

$$y(t)=\sum_{j=0}^{\infty}u_{jT}(t)\sin(t-jT)\,.$$

EXAMPLE 11.31. (A MIXING PROBLEM) Suppose a large tank contains algae that grows exponentially with a doubling time of 24 hours. The tank initially contains 100 kilograms of algae. Every 12 hours, h kilograms are instantaneously removed. How large can h be so that this process can be repeated indefinitely?

SOLUTION. Let t denote time in hours and let $y(t)$ denote the total mass of algae in the tank at time t. Then $y(t)$ is a solution of the initial value problem

$$y'=ky-\sum_{j=1}^{\infty}h\,\delta_{12j}(t)\,,\quad y(0)=100\,,\tag{11.17}$$

where k and h are to be determined. Observe that we modeled instantaneously removing h kilograms of algae at time $t=12j$ by the impulse $-h\,\delta_{12j}(t)$.

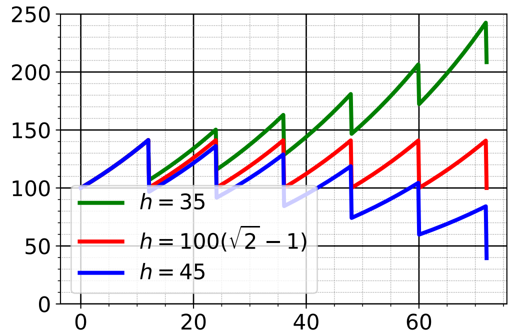


FIGURE 11.6. The solution of the initial value problem (11.17) for values of h below, at, and above the critical value $h = 100(\sqrt{2} - 1) \approx 41.4$ kilograms.

Taking the Laplace transform of the initial value problem gives

$$(s-k)Y(s) - 100 = -\sum_{j=1}^{\infty} h e^{-12js} \implies Y(s) = \frac{100}{s-k} - h\sum_{j=1}^{\infty} \frac{e^{-12js}}{s-k}\,.$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= 100e^{kt} - h\sum_{j=1}^{\infty} \mathcal{L}^{-1}\left\{e^{-12jt}/(s-k)\right\} \\ &= 100e^{kt} - h\sum_{j=1}^{\infty} u_{12j}(t)e^{k(t-12j)} = \left(100 - h\sum_{j=1}^{\infty} u_{12j}(t)e^{-12jk}\right) e^{kt}\,. \end{aligned} \tag{11.18}$$

If the term in parentheses ever became negative, then the tank would be empty, so the condition on h is that the term in parentheses be positive for all t, no matter how large. Since $u_{12j}(t) = 1$ for t large, this amounts to the condition

$$100 - h\sum_{j=1}^{\infty} e^{-12jk} > 0 \text{ or } h < \frac{100}{\sum_{j=1}^{\infty} e^{-12jk}}\,.$$

Using the sum formula for the geometric series $\sum_{j=1}^{\infty} r^j = \frac{r}{1-r}$, with $r = e^{-12k}$, this can be rewritten as

$$h < \frac{100(1 - e^{-12k})}{e^{-12k}} = 100(e^{12k} - 1)$$

Since the doubling time is 24 hours, $e^{24k} = 2$, so $k = \ln(2)/24$. Hence $e^{12k} = e^{\ln(2)/2} = \sqrt{2}$. We conclude that

$$h < 100(\sqrt{2} - 1) = 100(0.4121) \approx 41.4 \text{ kilograms.}$$

11.9. Convolutions

Equation (3.10) of Section 3.2.2, gave the formula

$$y(t) = e^{-kt}\int_0^t e^{ku} f(u)\,du + y_0\,e^{-kt}$$

for the solution of the first order initial value problem

$$y' + ky = f(t)\,, \quad y(0) = y_0,.$$

If $y(0) = 0$, then the formula simplifies to

$$y(t) = e^{-kt}\int_0^t e^{ku} f(u)\, du = \int_0^t f(u) e^{-k(t-u)}\, du\,.$$

Recall that the function e^{-kt} is the solution of the initial value problem

$$y' + ky = 0\,, \quad y(0) = 1\,.$$

There is a similar formula for the solution of the initial value problem

$$ay'' + by' + cy = f(t)\,, \quad y(0) = y'(0) = 0\ :$$

$$y(t) = \int_0^t f(u) g(t-u)\, du\,, \tag{11.19}$$

where $g(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1/a\,.$$

The right-hand side of Equation (11.19) is called the *convolution* of the functions $f(t)$ and $g(t)$.

More generally, if $f(t)$ and $g(t)$ are any two functions defined for $t \geq 0$, then their convolution is defined to be

$$(f * g)(t) = \int_0^t f(u) g(t-u)\, du\,. \tag{11.20}$$

The asterisk $*$ does not mean ordinary multiplication: it is a new operation, *convolution*, defined by the integral on the right side.)

As it applies to differential equations, the most important property of convolution is given by the following theorem:

THEOREM 6 (The Convolution Theorem). *If* $\mathcal{L}\{f\} = F(s)$ *and* $\mathcal{L}\{g\} = G(s)$, *then*

$$\mathcal{L}\{f * g\} = F(s)G(s).$$

PROOF. (*Skip this proof if you haven't taken Math 126.*) Compute as follows, using the definition of the Laplace transform, followed by the formula for convolution:

$$\begin{aligned}\mathcal{L}\{f * g\} = \int_0^\infty e^{-st}(f * g)\, dt &= \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)du\right) dt \\ &= \int_0^\infty \left(\int_0^t f(u)g(t-u)e^{-st}du\right) dt = \iint_R f(u)g(t-u)e^{-st}dudt\,.\end{aligned}$$

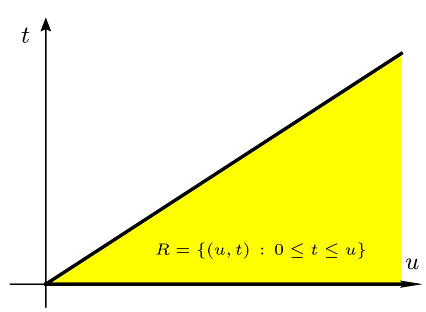


FIGURE 11.7.

This is a double integral over the infinite region R in Figure 11.7. Now change variables, letting $v = t-u$, so $t = u+v$ and $dv = dt$:

$$\begin{aligned}\mathcal{L}\{f*g\} &= \int_0^\infty \int_0^\infty f(u)g(v)e^{-s(u+v)}\,dv\,du \\ &= \left(\int_0^\infty e^{-su}f(u)\,du\right)\left(\int_0^\infty e^{-sv}g(v)\,dv\right) = \mathcal{L}\{f\}\,\mathcal{L}\{g\}\,.\end{aligned}$$

□

A number of properties of convolution follow immediately from The Convolution Theorem:

COROLLARY 7. *Let $f(t)$, $g(t)$, and $h(t)$ be continuous functions. Then the following identities hold:*

$$f*g = g*f \tag{11.21a}$$

$$(f*g)*h = f*(g*h) \tag{11.21b}$$

PROOF. By Theorem 4, to prove each identity, we need only show that the left-hand side and the right-hand side have the same Laplace transform:

$$\begin{aligned}&\text{(i)}\ \ \mathcal{L}\{f*g\} = F(s)G(s) = G(s)F(s) = \mathcal{L}\{g*f\}\\ &\text{(ii)}\ \ \mathcal{L}\{(f*g)*h\} = \mathcal{L}\{f*g\}\,\mathcal{L}\{h\} = F(s)G(s)H(s)\\ &\qquad \mathcal{L}\{f*(g*h)\} = \mathcal{L}\{f\}\,\mathcal{L}\{g*h\} = F(s)G(s)H(s)\,.\end{aligned}$$

□

Equation (11.19) follows immediately from the Convolution Theorem. For, consider the initial value problem

$$ay'' + by' + cy = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

Apply the Laplace operator to get

$$(as^2 + bs + c)Y(s) = F(s)\,.$$

Therefore,

$$Y(s) = F(s)G(s)\,, \quad \text{where } G(s) = \frac{1}{as^2+bs+c}.$$

Let $g(t) = \mathcal{L}^{-1}\{G(s)\}$. It then follows from The Convolution Theorem that

$$y(t) = (f*g)(t)\,.$$

That $g(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1/a$$

follows by taking Laplace transforms of the initial value problem

$$a(s^2Y(s) - y'(0) - sy(0)) + b(sY(s) - y(0)) + cY(s) = 0\,.$$

Solving for $Y(s)$ and recalling that $y(0) = 0$ and $y'(0) = 1/a$, shows that $Y(s) = G(s)$. Hence, the solution is $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

REMARK 11.5. The function $g(t)$ is perhaps best viewed as the solution of the initial value problem

$$ay'' + by' + cy = \delta(t)\,, y(0) = y'(0) = 0\,.$$

For taking the Laplace transform of this initial value problem also yields $G(s)$:

$$(as^2 + bs + c)Y(s) = 1 \text{ or } Y(s) = \frac{1}{as^2+bs+c} = G(s)\,.$$

EXAMPLE 11.32. Consider $y'' + 3y' + 2y = \sin(t)$, $y(0) = 0$, $y'(0) = 0$. Then

$$G(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2},$$

So

$$g(t) = \mathcal{L}^{-1}\left\{G\right\}(t) = e^{-t} - e^{-2t}\,,$$

and the solution is, therefore, given by the convolution

$$y(t) = (e^{-t} - e^{-2t}) * \sin(t).$$

REMARK 11.6. The solution to the initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0,\ y'(0) = 0. \tag{11.22a}$$

is called the *(unit) impulse response function*; it is often denoted by $g(t)$. Taking the Laplace transform of (11.22a) shows that

$$G(s) = \mathcal{L}\left\{g(t)\right\} = \frac{1}{as^2 + bs + c}\,. \tag{11.22b}$$

$G(s)$ is called the *transfer function.*

There is a rather nice formula for $g(t)$ in terms of the roots $r_1, r_2 = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$ of the characteristic polynomial $as^2 + bs + c$. Let $\rho = \dfrac{b}{2a}$.

$$g(t) = \begin{cases} \dfrac{e^{-\rho t}}{a}\dfrac{\sinh(wt)}{w}, & \text{if } b^2 - 4ac > 0,\ w = \frac{\sqrt{b^2-4ac}}{2a}, \\[2ex] \dfrac{e^{-\rho t}}{a} t, & \text{if } b^2 - 4ac = 0, \\[2ex] \dfrac{e^{-\rho t}}{a}\dfrac{\sin(\omega t)}{\omega}, & \text{if } b^2 - 4ac < 0,\ \omega = \frac{\sqrt{4ac-b^2}}{2a}. \end{cases} \tag{11.22c}$$

PROOF. (i) If $b^2 - 4ac > 0$, let $w = \dfrac{\sqrt{b^2 - 4ac}}{2a}$. Then

$$G(s) = \frac{1}{as^2 + bs + c} = \frac{1}{a(s + \rho + w)(s + \rho - w)} = \frac{1}{2aw}\left(\frac{1}{s + \rho - w} - \frac{1}{s + \rho + w}\right).$$

Hence,

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = \frac{1}{2aw}\left(e^{-(\rho - w)\,t} - e^{-(\rho + w)\,t}\right) = \frac{e^{-\rho t}}{aw}\sinh(w\,t)$$

(ii) If $b^2 - 4ac = 0$, then $G(s) = \dfrac{1}{a(s+\rho)^2}$. Therefore

$$g(t) = \frac{1}{a}\mathcal{L}^{-1}\left\{\frac{1}{s - (-\rho)}\right\} = \frac{1}{a} t e^{-\rho t}\,.$$

(iii) If $b^2 - 4ac < 0$, let $\omega = \dfrac{\sqrt{4ac - b^2}}{2a}$. Then

$$G(s) = \frac{1}{as^2 + bs + c} = \frac{1}{a(s + (\rho + \omega i)(s + \rho - \omega i)} = \frac{1}{2a\omega i}\left(\frac{1}{s + \rho - \omega i} - \frac{1}{s + \rho + \omega i}\right).$$

Hence, by Equation (11.6),

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = \frac{1}{2a\omega i}\left(e^{-(\rho - \omega i)\,t} - e^{-(\rho + \omega i)\,t}\right) = \frac{e^{-\rho t}}{a\omega}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) = \frac{e^{-\rho t}}{a\omega}\sin(\omega t)\,.$$

$\square$

The *state-free solution* is the solution to the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0,\ y'(0) = 0. \tag{11.23}$$

Taking Laplace transforms gives

$$Y(s) = G(s)F(s)\,. \tag{11.24}$$

By The Convolution Theorem, the state-free solution is the function $(f * g)(t)$. The *input-free solution* is the solution to

$$ay'' + by' + cy = 0, \quad y(0) = y_0,\ y'(0) = y_0'. \tag{11.25}$$

PROPOSITION 8. *The solution of the initial value problem*

$$ay'' + by' + cy = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = y_0'$$

is the sum of the state-free and input-free solutions:

$$y(t) = (f * g)(t) + ay_0\, g'(t) + (ay_0' + by_0)\, g(t)\,. \tag{11.26}$$

PROOF. Taking the Laplace transform gives of the initial value problem gives the formula

$$Y(s) = F(s)G(s) + (ay_0\, s + (ay_0' + by_0)G(s)\,.$$

for the Laplace transform of the solution. On the other hand, taking the Laplace transform of (11.26) using the convolution theorem gives the same thing. Consequently, the two functions agree and (11.26) is the solution of the initial value problem. ☐

EXAMPLE 11.33. Consider $y'' + 4y = f(t)$, $y(0) = 2$, $y'(0) = 3$. Then $g(t) = \mathcal{L}^{-1}\left\{1/(s^2+4)\right\} = 1/2\sin(2t)$. The state-free solution is

$$\frac{1}{2}\sin(2t) * f(t) = \frac{1}{2}\int_0^t \sin(2u) f(t-u)\, du = \frac{1}{2}\int_0^t \sin(2(t-u)) f(u)\, du.$$

The input-free solution is

$$ay_0\, g'(t) + (ay_0' + by_0)g(t) = 2\cos(2t) + \frac{3}{2}\sin(2t).$$

Therefore,

$$y(t) = 2\cos(2t) + \frac{3}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-u)) f(u)\, du.$$

Exercises 1.

(1) Using Laplace Transforms, find the solution of each of the following initial value problems,
  (a) $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 0$.
  (b) $y'' - 3y' + 2y = t$, $y(0) = 0$, $y'(0) = 0$.
  (c) $y'' - 3y' + 2y = e^{2t}$, $y(0) = 0$, $y'(0) = 0$.
(2) Evaluate each of the following Laplace transforms or inverse Laplace transforms
  (a) $\mathcal{L}\left\{2t + u_2 \cos(t-2)\right\}$
  (b) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-2)(s+1)}\right\}$
  (c) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2+2s+5}\right\}$
  (d) $\mathcal{L}\left\{t\sin(2t)\right\}$
  (e) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{4s^2+4s+5}\right\}$
  (f) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s(s^2+4)}\right\}$
  (g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4s+5)}\right\}$
(3) Solve the initial value problem $y'' - y = u_4(t) - u_5(t)$, $y(0) = 1$, $y'(0) = 0$.
(4) Consider the following initial value problem:
$$y'' + 2y' + 5y = u_2(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$
  (a) Let $Y(s)$ denote the Laplace transform for the solution. Find $Y(s)$.
  (b) Find the solution $y(t)$ by computing the inverse Laplace transform of $Y(s)$.
  (c) Give the numerical value of $y(3)$. (*Use a calculator for this part.*)
(5) Compute $y(7)$, where $y(t)$ is the solution of the the initial value problem
$$y'' - y = u_4(t) - u_5(t) + u_6(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$
(6) Compute $y(10\pi)$, where $y(t)$ is the solution of the the initial value problem
$$y'' + y = \delta(t-\pi)) - \delta(t-2\pi) + \delta(t-3\pi)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,.$$
(7) Consider the initial value problem
$$y'' + y = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$
where
$$f(t) = 1 + 2\sum_{k=1}^{\infty}(-1)^k u_{k\pi}(t)$$
  (a) Draw the graph of $f(t)$ for $0 \le t \le 6\pi$.
  (b) Find a formula for $F(s)$, the Laplace transform of $f(t)$.
  (c) Find $Y(s)$, the Laplace transform of the solution $y(t)$. Express your answer in the form
$$Y(s) = H(s)\sum_{k=0}^{\infty} e^{-kTs}\,.$$
  (d) Find $h(t)$, the inverse Laplace transform of $H(s)$ (from part (c)) and use this to find a formula for the solution $y(t)$.
  (e) Graph $y(t)$ for $0 \le t < 8\pi$. (Although $y(t)$ is expressed as an infinite series, most terms in the series vanish for $t < 8\pi$.)
(8) Consider the initial value problem
$$y'' + 0.2y' + y = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 0\,,$$
where $f(t)$ is the same as in the previous problem. Repeat steps (b)–(e) of that problem.

(9) When making prescriptions for drugs that will be taken over a prolonged period of time it is necessary to take into account the fact that the concentration of a drug in the bloodstream grows after each subsequent dose. In this problem you derive a formula in standard use by physicians.

Let c_0 be the concentration of a drug immediately after the first dose (this is proportional to the size of the dose and the weight of the patient and is information known for all commonly used drugs). After t units of time the concentration will be given by the formula $c = c_0 e^{-rt}$ where r is a constant that depends on the drug (this is just the law of exponential decay and again the value of r is known for all commonly used drugs).

Now suppose that the same dose is taken every T units of time (e.g. every 4 hours). Let $y(t)$ denote the concentration of the drug in the bloodstream t hours after the first dose. Then $y(t)$ is a solution of the following initial value problem

$$y' + ry = f(t) = \sum_{k=0}^{\infty} c_0 \delta(t - kT)\,, y(0) = 0$$

(a) Compute $Y(s)$, the Laplace Transform of $y(t)$. (Note: it is an infinite series.)

(b) Now compute the inverse Laplace Transform to obtain a formula for $y(t)$ as another infinite series.

(c) Use part (b) to find a formula for $c_k = y(kT)$, the concentration of the drug right after a dose is administered at time $t = kT$.

(d) Initially, c_k will grow pretty rapidly, but it will eventually level off and approach

$$c_\infty = \lim_{k\to\infty} c_k\,.$$

Use the formula for the sum of a geometric series to find a formula for c_∞.

(e) Find the value of r if the half-life of the drug in the bloodstream is 3 hours.

(f) Use the result of the previous parts of the problem to obtain a graph of the ratio c_∞/c_0 as a function of T for a drug with a half-life of 3 hours. What is the time between doses if the stable concentration is twice the initial concentration?

Appendices

A. Basic Formulas from Algebra, Trigonometry, and Calculus

Algebra:

Completing the square: $X^2 + bX + c = (X + \frac{b}{2})^2 \; - \frac{b^2}{4} + c.$

Quadratic formula: roots of $aX^2 + bX + c$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exponents: $a^b \cdot a^c = a^{b+c}$; $\dfrac{a^b}{a^c} = a^{b-c}$; $(a^b)^c = a^{bc}$; $a^{1/b} = \sqrt[b]{a}$

Logarithms: $\ln(1) = 0$; $\ln(e) = 1$; $\ln(ab) = \ln(a) + \ln(b)$; $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$

Geometry:

Circle: circumference $= 2\pi r$; area $= \pi r^2$;

Sphere: vol $= \frac{4}{3}\pi r^3$; surface area $= 4\pi r^2$

Cylinder: vol $= \pi r^2 h$; lateral area $= 2\pi r h$; surface area $= 2\pi r h + 2\pi r^2$.

Cone: vol $= \frac{1}{3}\pi r^2 h$; lateral area $= \pi r\sqrt{r^2 + h^2}$; surface area $= \pi r\sqrt{r^2 + h^2} +$ πr^2

Analytic geometry

Point-slope formula for straight line: $y = y_0 + m(x - x_0)$

Equation for circle centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

Equation for ellipse centered at (h, k): $\dfrac{(x - a)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

Trigonometry

$\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$; $\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}}$; $\tan = \dfrac{\text{opposite}}{\text{adjacent}}$;

$\sec = \dfrac{1}{\cos}$; $\csc = \dfrac{1}{\sin}$; $\cot = \dfrac{1}{\tan}$; $\tan = \dfrac{\sin}{\cos}$; $\cot = \dfrac{\cos}{\sin}$;

$\sin(x) = \cos\left(\dfrac{\pi}{2} - x\right)$; $\cos(x) = \sin\left(\dfrac{\pi}{2} - x\right)$

$\sin(x + \pi) = -\sin(x)$; $\cos(x + \pi) = -\cos(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$; $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

;

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$; $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$;

$\sin(x) + \sin(y) = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{y-x}{2}\right)$ $\quad$ $\sin(x) - \sin(y) = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

$\cos(x) + \cos(y) = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{y-x}{2}\right)$ $\quad$ $\cos(x) - \cos(y) = 2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{y-x}{2}\right)$

$\sin^2(x) + \cos^2(x) = 1$; $\tan^2(x) + 1 = \sec^2(x)$; $1 + \cot^2(x) = \csc^2(x)$.

$\sin^2 x = \dfrac{1 - \cos(2x)}{2}$; $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

Values at common angles:

$\theta =$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(\theta) =$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos(\theta) =$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan(\theta) =$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—

The phase-shift formula:
$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \cos(\omega t + \phi) = A \cos(\omega(t - t_0))$,
where $t_0 = -\dfrac{\phi}{\omega}$; $A = \sqrt{C_1^2 + C_2^2}$,
$\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2 + C_2^2}}$. $\sin(\phi) = -\dfrac{C_2}{\sqrt{C_1^2 + C_2^2}}$, $\tan(\phi) = -\dfrac{C_2}{C_1}$

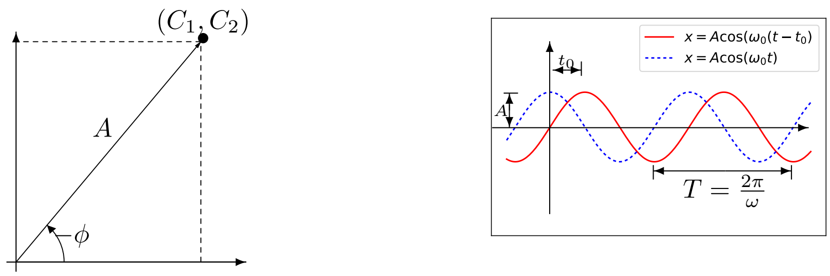
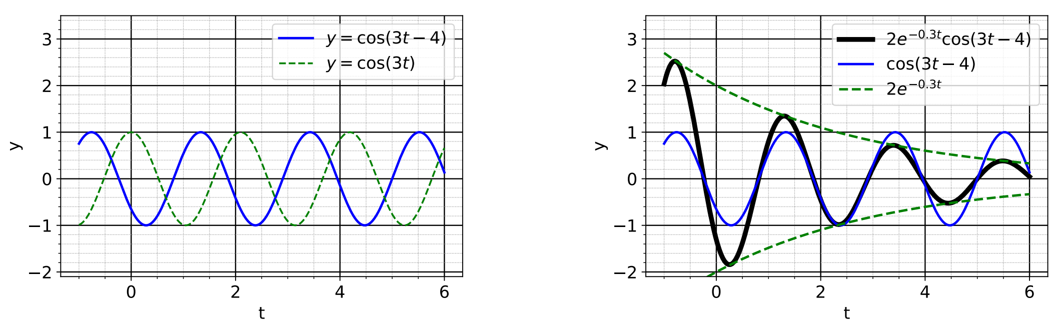


FIGURE A.1. The figure above illustrates how to graph the function $y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$. It is a "cosine curve" of amplitude $A = \sqrt{C_1^2 + C_2^2}$, period $T = \frac{2\pi}{\omega}$, shifted by $t_0 = -\frac{\phi}{\omega}$ units. The angle ϕ is called the phase angle or phase shift. Notice, however, that the actual time-shift is the quantity $t_0 = -\phi/\omega$ rather than ϕ. The curve is shifted to the right. Therefore, $\phi < 0$ and $t_0 > 0$. When ϕ is positive, the curve is shifted to the left.

EXAMPLE A.1. *Sketch the graph of the function* $y(t) = 2\, e^{-0.3t} \cos(3\, t - 4)$.



Step 1: Graph the function $f(t) = \cos(3t)$, a cosine function with period $2\pi/3 \approx 2$.
Step 2: Graph the function $f(t - 4/3) = \cos(3\, t - 4) = \cos(3(t - 4/3))$. This is the graph of $\cos(3t)$ shifted to the right by $4/3$ units.
Step 3: Next graph the functions $g(t) = 2e^{-0.3\, t}$ and $-g(t) = -2e^{-0,3\, t}$.
Step 4: Finally graph $y(t) = 2e^{-0.3t} \cos(3\, t - 4)$, which is the product $g(t) \cdot f(t - 4/3)$ —a shifted cosine function $\cos(3t - 4)$ with varying amplitude $g(t) = 2e^{-0.3\, t}$. In the right-hand figure below, the graph of $y(t)$ touches the graphs of $g(t)$ and $-g(t)$ (the dotted curves) when $\cos(3t - 4) = \pm 1$, which is where $3t - 4$ is an integer multiple of π:

$$3t - 4 = n\pi \qquad \Longleftrightarrow \qquad t = \frac{4}{3} + n\frac{\pi}{3}\,.$$

Calculus

Basic differentiation formulas:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}\,, \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\,.$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right), \text{ for } v \neq 0.$$

Chain rule:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(u)\,du = h\left(g(x)\right) g'(x) - h\left(f(x)\right) f'(x)$$

Derivatives of specific functions:

$$\frac{dx^n}{dx} = nx^{n-1};\ \frac{de^x}{dx} = e^x;\ \frac{dln|x|}{dx} = \frac{1}{x};$$
$$\frac{d\sin(x)}{dx} = \cos(x);\ \frac{d\cos(x)}{dx} = -\sin(x);\ \frac{d\tan(x)}{dx} = \sec^2(x);$$
$$\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}};\ \frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}.$$

Basic integration formulas:

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx;\ \int au\,dx = a\int u\,dx;$$

Substitution:

$$\int f\left(u(x)\right) u'(x)\,dx = F\left(u(x)\right), \text{ where } \int f(u)du = F(u);$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du;$$

Standard integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1);\ \int \frac{dx}{x} = \ln|x| + C;\ \int e^x\,dx = e^x + C;$$
$$\int \sin(x)\,dx = -\cos(x) + C;\ \int \cos(x)\,dx = \sin(x) + C;\ \int \tan(x)\,dx = -\ln|\cos(x)| + C;$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C;\ \int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C;$$
$$\int \frac{dx}{1+x^2} = \arctan(x) + C;\ \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C;$$
$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) + C$$

B. Review of Partial Fractions

When computing integrals and inverse Laplace transforms, *rational functions*, i.e. ratios of polynomials, arise:

$$R(s) = \frac{P(s)}{Q(s)} = \frac{p_n s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0}{q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0}$$

It is useful to express $R(s)$ as a sum of simple fractions, this is called the *partial fractions expansion* of $R(s)$. Here's how to do that:

Step 0: If $P(s)$ and $Q(s)$ have common factors, cancel them. If degree $P(s) \geq$ degree $Q(s)$, perform a long division.

Example:

$$\frac{(s-2)(s^5+1)}{(s-2)(s^4+2s^2+1)} = \frac{s^5+1}{s^4+2s^2+1} = s - \frac{2s^3+s-1}{s^4+2s^2+1}\,.$$

Step 1: If $Q(s)$ hasn't already been factored, factor it.

Example:

$$-\frac{2s^3+s-1}{s^4+2s^2+1} = -\frac{2s^3+s-1}{(s^2+1)^2}\,.$$

Step 2: For each factor in the denominator of the form $(s+a)^p$, with a real, include terms of the form

$$\frac{A_1}{(s+a)} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_p}{(s+a)^p}\,,$$

and for each term in $Q(s)$ of the form $(s^2+bs+c)^q$, where[19] $c > b^2/4$, include terms of the form

$$\frac{A_1(s+\frac{b}{2})+B_1}{(s^2+bs+c)} + \frac{A_2(s+\frac{b}{2})+B_2}{(s^2+bs+c)^2} + \ldots \frac{A_p(s+\frac{b}{2})+B_p}{(s^2+bs+c)^q}$$

in the partial fractions expansion.

Examples:

$$\begin{aligned}
-\frac{2s^3+s-1}{(s^2+1)^2} &= \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2}\,,\\
\frac{2s^3-s^2+2s}{(s-1)^2(s^2+s+1)} &= \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C(s+\frac{1}{2})+D}{(s^2+s+1)}\,,\\
\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D(s+1)+E}{(s^2+2s+2)}\,.\\
\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)^2} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D(s+1)+E}{(s^2+2s+2)} + \frac{F(s+1)+G}{(s^2+2s+2)^2}\,.
\end{aligned}$$

Step 3: Determine the unknown constants. There are two methods:

- The first method proceeds by collecting the terms in the partial fractions expansion and equating the numerator of the result with the numerator of the original fraction. This yields a system of equations that can be solved to determine the constants.

[19] These terms correspond to complex roots of $Q(s)$.

Example:

$$\frac{-2s^3-s+1}{(s^2+1)^2} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2} = \frac{As^3+Bs^2+(A+C)s+(B+D)}{(s^2+1)^2}.$$

So $A=-2 \quad B=0\,, \quad A+C=-1\,, \quad B+D=1.$

Hence, $\quad \dfrac{-2s^3-s+1}{(s^2+1)^2} = \dfrac{-2s}{(s^2+1)} + \dfrac{s+1}{(s^2+1)^2}.$

- The second (and often simpler) method is the "cover-up" method, which is best understood by example.

Example. $\dfrac{6s^3-3s^2+16s-3}{(s-1)(s+3)(s^2-2s+5)} = \dfrac{A}{s-1} + \dfrac{B}{s+3} + \dfrac{C(s-1)+D}{(s-1)^2+4}$

To determine A, multiply both sides by $(s-1)$ to get

$$\frac{6s^3-3s^2+16s-3}{(s+3)(s^2-2s+5)} = A + (s-1)\left\{\frac{B}{s+3} + \frac{C(s-1)+D}{(s-1)^2+4}\right\}$$

Set $s=1$ to get

$$A = \frac{6(1)^3-3(1)^2+16(1)-3}{15-(1)+(1)^2+(1)^3} = \frac{16}{16} = 1$$

Similarly, to find B, multiply by $(s+3)$ and set $s=-3$ to get

$$B = \frac{6(-3)^3-3(-3)^2+16(-3)-3}{(-1+(-3))(5-2(-3)+(-3)^2} = \frac{-240}{-80} = 3$$

To find both C and D, multiply by $(s-1)^2+4$ and set $s=1+2i$ (one of the roots). The terms involving A and B vanish, so:

$$C(2i)+D = \frac{6(1+2i)^3-3(1+2i)^2+16(1+2i)-3}{(1+2i)-1)((1+2i)+3)} = \frac{-44+ii}{-4+8i} = 3+4i$$

Hence, $C=2$ and $D=3$.

Example. $\dfrac{2s^3-s^2+4s-4}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}.$

To determine B, multiply by s^2 and set $s=0$:

$$\frac{2s^3-s^2+4s-4}{(s^2+4)} = sA+B+s^2\frac{Cs+D}{s^2+4} \implies B = \frac{-4}{4} = -1$$

To determine C and D, multiply by s^2+4 and set $s=2i$:

$$\frac{2s^3-s^2+4s-4}{s^2} = (s^2+4)\left(\frac{A}{s}+\frac{B}{s^2}\right) + Cs+D$$
$$\implies 2iC+D = 2i \implies C=1 \text{ and } D=0\,.$$

We now have $\dfrac{A}{s} - \dfrac{1}{s^2} + \dfrac{s}{s^2+4} = \dfrac{2s^3-s^2+4s-4}{s^2(s^2+4)}$ and we can determine A by setting s equal to another value and solving for A. For instance, setting $s=1$ gives $A-1+\dfrac{1}{5} = \dfrac{1}{5} \implies A=1.$

C. Table of Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\dfrac{1}{s}$	e^{bt}	$\dfrac{1}{s-b}$
$t^n,\ n = 1, 2, 3\ldots$	$\dfrac{n!}{s^{n+1}}$	$t^n e^{bt}$	$\dfrac{n!}{(s-b)^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{bt}\sin(at)$	$\dfrac{a}{(s-b)^2+a^2}$	$e^{bt}\cos(at)$	$\dfrac{(s-b)}{(s-b)^2+a^2}$
$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
$e^{bt}\sinh(at)$	$\dfrac{a}{(s-b)^2-a^2}$	$e^{bt}\cosh(at)$	$\dfrac{(s-b)}{(s-b)^2-a^2}$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$	$\delta(t-c)$	e^{-cs}

General Formulas			
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	$f(at)$	$\dfrac{1}{a}F(s/a)$
$e^{bt}f(t)$	$F(s-b)$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$\displaystyle\int_0^t f(u)\,du$	$\dfrac{F(s)}{s}$	$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s) - f(0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f * g(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	$f(t+T) = f(t)$	$\dfrac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

D. Answers to selected problems

SOLUTION (1.1):
(a) mass $= \dfrac{640}{32} = 20$slugs. (b) $20\dfrac{dv}{dt} = 20 - 2v$, $v(0) = 0$ft/sec.

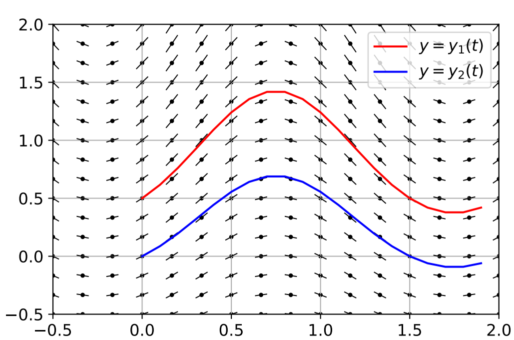
SOLUTION (1.2): $10^{-5}\dfrac{dI}{dt} + 100I = 0$ or $\dfrac{dI}{dt} + (10^7)I = 0$.

SOLUTION (1.3):
(a) $r^2 + 4r - 21 = (r+7)(r-3) = 0$ so $r = 3$ and $r = -7$. So two solutions are e^{3t} and e^{-7t}.
(b) If Ae^{rt} is a solution then $(Ae^{rt})'' + 4(Ae^{rt}) = (r^2+4)Ae^{rt} = 24e^{2t}$. So $r = 2$ and $(r^2+4)A = 24$. This implies that $A = \dfrac{24}{2^2+4)} = 3$. So $y = 3e^{2t}$ is a solution.

SOLUTION (1.4):
$(Ce^{rt})' + 3(Ce^{rt}) = C(r+3)e^{rt} = 0$. So $r = -3$ and $y(t) = Ce^{-3t}$. But then $y(1) = Ce^{-3} = 2$, so $C = 2e^3$. Therefore, $y(t) = 2e^3e^{-3t} = 2e^{-3(t-1)}$ is the solution of the initial value problem.

SOLUTION (1.5):
First find r_1 and r_2:
$(r^2 + 3r + 2 = (r+2)(r+1) = 0$, so $r_1 = -2$, $r_2 = -1$; hence $y(t) = C_1r^{-2t} + C_2e^{-t}$. The initial conditions force $C_1 + C_2 = 2$ and $-2C_1 - C_2 = 1$. Adding the two equations gives $-C_1 = 3$, so $C_1 = -3$ and $-3 + C_2 = 2$. Thus $C_2 = 5$, and the solution is $y(t) = -3e^{-2t} + 5e^{-t}$.

SOLUTION (2.1):



SOLUTION (2.2):
$y(1) \approx 1.706$.

SOLUTION (3.1):
(a) $y(t) = \displaystyle\int (1-t)(2-t)dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + C$
$y(1) = \dfrac{5}{6} + C = 1 \implies C = \dfrac{1}{6}$, so
$y(t) = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 2t + \dfrac{1}{6}$
(b) $y(t) = 2 + Ce^{t^2/2-t}$
(c) $y(t) = \dfrac{2e^{2t}}{2e^{2t}-1}$.
(d) $y(t) = \dfrac{y_0be^{abt}}{b + y_0(e^{abt}-1}$.
(f) $y(x) = \tan(\sin(x) + C)$.

SOLUTION (3.2):
(a) $y(x) = -\dfrac{5}{2}e^{-3x} + Ce^{-x}$
(b) $y(t) = \dfrac{t^4 + 2t^2 + 5}{4(t^2+1)}$
(c) $w(t) = (t + C)e^{t^2}$
(d) $z(x) = \dfrac{1}{15}\sin(6x) - \dfrac{2}{15}\cos(6x) + Ce^{-3x}$
(e) $y(t) = -e^{2t} + Ce^{3t}$
(f) $y(t) = \dfrac{1}{5}e^{2t} + Ce^{-3t}$
(g) $y(t) = \dfrac{1}{3}t^2 - \dfrac{2}{9}t + \dfrac{2}{27} + Ce^{-3t}$.
(h) (i)$y(t) = 6 + Ce^{-t/2}$, (ii) $6(1 - e^{-t/2})$, 6, $6 + 4e^{-t/2}$.

SOLUTION (4.1):
(a) $y' = 0.02y$, (b) $y(t) = Ce^{0.02t}$, (c) In one year: 1,020200, in 20 years: 1,491,820, doubles every 34.66 years.

SOLUTION (4.2):
Let $P(t)$ be the population of the Earth (in billions) t years after 1650 CE. Under the assumptions of the problem, P satisfies the initial value problem

$$P' = kP\,, \quad P(0) = 0.250\,.$$

So $P(t) = 0.250e^{kt}$. Let T be the time when the limit is reached. Then

$$0.250e^{kT} = 25 \implies e^{kT} = \frac{25}{0.25} = 100 \implies T = \frac{\ln(100)}{k}\,.$$

To continue, we need to know the value of k. But $P(300) = 0.250e^{k300} = 2.5$, so taking logarithms gives $300k = \ln\left(\dfrac{2.5}{0.250}\right) = \ln(10)$, so $k = \frac{\ln(10)}{300}$.
Therefore,

$$T = \frac{\ln(100)}{\ln(10)/300} = 300\frac{2\ln(10)}{\ln(10)} = 600 \text{ years,}$$

or in the year 2250 CE.

SOLUTION (4.3): (c) The maximum they will lend is \$131,026.

Solution (4.4): After 7.26 hours.
Solution (4.5): (a) $y' = -\frac{1}{100}y$,
(b) $y(t) = 200e^{-t/100}$,
(c) 68.315 minutes,
(d) 460.5 minutes.
Solution (4.6): (a) $0.04(1 - \exp(-8.3 \times 10^{-5}t))$,
(b) after 36.05 minutes.
Solution (4.7): (a) Let $v(t)$ denote the speed (so a positive number) in miles per hour, with t the number of seconds after its speed was 500 mph. Then $v' = -kv$, $v(0) = 500$, where k is a constant. $v(t) = \frac{500}{1+500kt}$, $k = \frac{1}{10000}$, so $v(t) = \frac{500}{1+t/20}$
(c) 125 miles per hour
(d) We have to convert miles per hour to miles per second, then integrate. So distance traveled $= \displaystyle\int_0^{60} \frac{1}{3600}\frac{500}{1+t/20}\,dt \approx 3.85$ miles.
Solution (4.8):
(a) Let L denote the number of licenses issued, and assume that each license is used to catch 5 fish per year. Then the differential equation is now

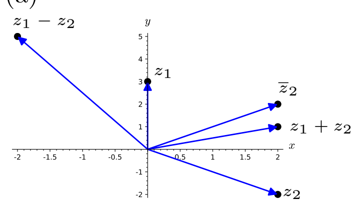
$$\frac{dP}{dt} = 2\left(1 - \frac{P}{1000}\right) - 5L$$

The maximum value of the right-hand side is attained when $P = 500$, when it assumes the value $500 - 5L$. If L is greater than 100, then $\frac{dP}{dr}$ will be negative no matter the value of P. So 100 is the absolute maximum number of licenses that can be issued.
(b) If the fish population is 500 when these licenses are issued then if nothing happens to decrease the population below 500, the fish population will stay at 500.
(c) If the fish population ever drops below 500 then the population will drop to zero. So in reality, the number of licenses issued should be less than 100.
Solution (4.9):
(a) $v(t) = 10(1 - e^{-t/10}$. (b) 10 miles per hour.

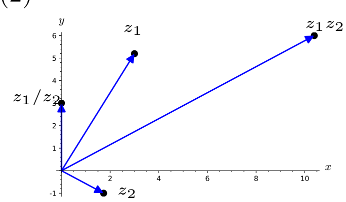
Solution (5.1): $T(r) = 50\dfrac{\ln(2r)}{\ln(2)}$.

Solution (6.1): (1) $(x+iy)\left(\frac{x-iy}{x^2+y^2}\right) = 1$.
(2)(a) $-2i$, (b)$-\dfrac{2}{5}$, (c) $\dfrac{1}{2}i$, (d) -4.
(3) (a) This is a computation:
$z + \overline{z} = (x+iy) + (x-iy)$
$= (x+x) + (y-y)i = 2x$, which, by definition is $2\mathrm{Re}\,(z)$. (b) We need only check that the real and imaginary parts of the left and right hand side agree.
left-hand side $= \overline{z_1 z_2}$
$= \overline{(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i}$
$= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i$
right-hand side $= \overline{z_1}\,\overline{z_2}$
$= (x_1 - y_1 i)(x_2 - y_2 i)$
$= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1 i$.
(b) By definition $|z| = \sqrt{z\overline{z}}$, so we can compute as follows: $|z_1z_2| = \sqrt{(z_1z_2)\overline{z_1z_2}} = \sqrt{z_1z_2\overline{z_1}\,\overline{z_2}}$
$= \sqrt{z_1\overline{z_1}z_2\overline{z_2}} = \sqrt{z_1\overline{z_1}}\sqrt{z_2\overline{z_2}} = |z_1||z_2|$.
(5) $z = \sqrt{\dfrac{\sqrt{2}+1}{2}} + \dfrac{1}{\sqrt{2\sqrt{2}+2}}i$ and
$z = -\sqrt{\dfrac{\sqrt{2}+1}{2}} - \dfrac{1}{\sqrt{2\sqrt{2}+2}}i$.

Solution (6.2):
(1) (a)

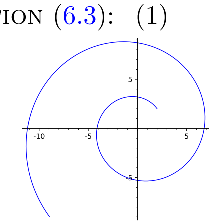


(b) $|z_1 + z_2| = \sqrt{29}$, $|z_1 - z_2| = \sqrt{5}$.
(c) $z_1 = 3\,e^{\pi i/2}$, $z_2 = 2\sqrt{2}e^{-\pi i/4}$.
(2)



(3) $z = r\cos(\theta) + ir\sin(\theta)$. So
$\overline{z} = r\cos(\theta) - ir\sin(\theta) = r\cos(-\theta) + ir\sin(-\theta)$
$= re^{-i\theta}$.
$z^{-1} = \dfrac{r\cos(\theta) - ir\sin(\theta)}{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = \frac{r\cos(\theta) - ir\sin(\theta)}{r^2}$
$= r^{-1}(\cos(-\theta) + i\sin(-\theta)) = r^{-1}e^{-i\theta}$.

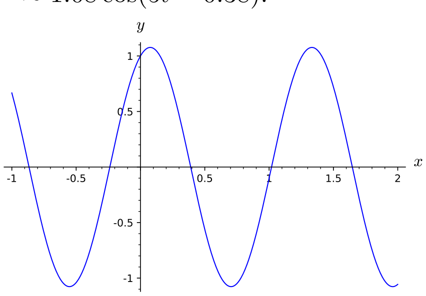
Solution (6.3): (1)



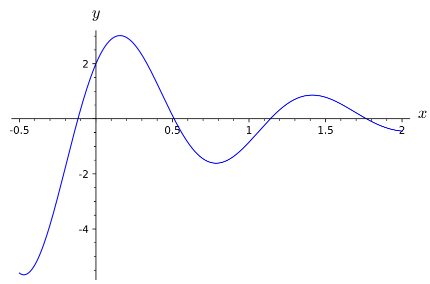
(2) $x(t) = \mathrm{Re}\,(3-5i)e^{(-2+4i)t}$
$= \sqrt{34}e^{-2t}\cos(4t - \tan^{-1}(5/3))$.

(3) $x(t) = \mathrm{Re}\,(2+3i)e^{(-1+4i)t}$.
Therefore, $x''(t) = \mathrm{Re}\,(2+3i)(-1+4i)^2 e^{(-1+4i)t}$
$= \mathrm{Re}\,(-14+5i)e^{(-1+4i)t}$
$= -(15\cos(4t) + 5\sin(4t))e^{-t}$
(4) $\int_0^\pi e^{t/\pi}\sin(t)\,dt = \mathrm{Im}\left(\int_0^\pi e^{(1/\pi+i)t}\,dt\right)$
$= \mathrm{Im}\left(\frac{1}{1/\pi+i}\; e^{(1/\pi+i)t}\Big|_0^\pi\right)$
$= \mathrm{Im}\left(\frac{1}{1/\pi+i}\left(e^{(1+\pi i)}-1\right)\right) = \mathrm{Im}\left(-\frac{\pi(e+1)}{1+\pi i}\right)$
$= \frac{\pi^2(e+1)}{\pi^2+1}$

Solution (7.1): (1) $y(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{2t}$
(2) $y(t) = \frac{1}{2}e^{3(t-1)} - \frac{1}{2}e^{t-1}$
(3) $y(t) = C_1 e^{-3t} + C_2 e^{-t}$
(4) $y(t) = C_1 e^{-\sqrt{3}t} + C_2 e^{\sqrt{3}t}$
Solution (7.2): (1) $y(t) = (t+4)e^{-3t}$
(2) $y(t) = (t-1)e^{2(t-1)}$
(3) $y(t) = (C_1 t + C_2)e^{-2t}$
(4) $y(t) = (C_1 t + C_2)e^{t}$
Solution (7.3):
The roots of the characteristic polynomial are $0 \pm 5i$, so the general solution is
$y(t) = \mathrm{Re}\,Ce^{5it}$.
The initial conditions are $y(0) = \mathrm{Re}\,(C) = 1$ and $y'(0) = \mathrm{Re}\,(5iC) = 2$. From the first initial condition, we find that $C = 1 + C_2 i$.
Hence, $\mathrm{Re}\,(5iC) = -5C_2 = 2$, so $C = 1 - \frac{2}{5}i$
In polar form $C = \frac{\sqrt{29}}{5}e^{-\tan^{-1}(2/5)i}$.
Therefore, $y(t) = \frac{\sqrt{29}}{5}\cos\left(t - \tan^{-1}(2/5)\right)$
$\approx 1.08\cos(5t - 0.38)$.



Solution (7.4): $y(t) = \mathrm{Re}\left((2-3i)e^{(-1+5i)t}\right)$.
$y(t) = \sqrt{13}\,e^{-t}\cos\left(5t - \tan^{-1}(3/2)\right)$.



Solution (7.5):
(1) $y(t) = \mathrm{Re}\left(\left(1 - \frac{2}{\sqrt{3}}i\right)e^{(-1+\sqrt{3}i)t}\right)$
or $y(t) = \left(\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right)e^{-t}$
(2) $y(t) = \mathrm{Re}\left(-\frac{i}{\sqrt{3}}e^{(2+\sqrt{3}i)(t-1)}\right)$
or $y(t) = \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}(t-1)\right)e^{2(t-1)}$
(3) $y(t) = \mathrm{Re}\left(Ce^{(-2+2i)t}\right)$
or $y(t) = (C_1\cos(2t) + C_2\sin(2t))\,e^{-2t}$
(4) $y(t) = \mathrm{Re}\left(Ce^{(-3+2i)t}\right)$
or $y(t) = (C_1\cos(2t) + C_2\sin(2t))\,e^{-3t}$

Solution (8.1):
(a) $A = 3$, (b) $\omega_0 = \pi$
(c) $\phi = \pi/2$.
Solution (8.2):
(a) $u(0) = 1.0$ meters, $u'(0) = 0$ meters/sec.
(b) $T = 0.5$ sec.
(c) $k = 80\pi^2 \approx 789.6$ Newtons/meter
Solution (8.3):
(a) $d_{eq} = \frac{10}{\pi} \approx 3.183$ meter.
(b) $100d'' + 100\pi d = 1000$.
(c) $100y'' + 100\pi y = 0$, $y(0) = 1$, $y'(0) = 0$.
(d) $y(t) = \cos(\sqrt{\pi}\,t)$.
Solution (8.4):
(a) Force $= 83126.5y$ dynes.
(b) $500y'' + 83126.5y = 0$. (c)
$y(t) = A\cos(12.894t + \phi)$.
(d) $T = 0.4873$ seconds.
Solution (8.5): The weight will oscillate with a period of $T = \frac{\sqrt{5}\pi}{2} \approx 3.5124$ seconds, a frequency of $\frac{2}{\sqrt{5}\pi}$ cycles per second, stretching to a maximum amount of 20 feet and returning to its unstretched position, with an amplitude of 10 feet.
Solution (8.6):
$x(t) = 2e^{-2.5t}(\cos(1.936t) + 1.033\sin(1.936t))$ or
$x(t) = 2.25e^{-2.5t}\cos(1.936t - 0.477)$.
Solution (8.7):

(a) $\gamma = 2$ N-sec/meter
(b) $y() = \frac{1}{2}te^{-t}$.
(c) $\frac{1}{2}e^{-1} \approx 0.1839$ meter.
Solution (8.8):
(a) $\omega_d \approx 1.96$, $\zeta \approx 0.2,\omega_0$
(b) $x'' + 0.8x' + 4x = 0$
(c) $x'(0) \approx 7.8$
Solution (8.9):
(a) 43 bounces (the period is approx 1.40 seconds)
(b) $4000\sqrt{5} \approx 8944$ N-sec/meter
(c) $\frac{1}{2\sqrt{5}e} \approx 0.082$ meters.
Solution (9.1): (a)
$y(t) = \frac{t^2}{3} + \frac{1}{9} + C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t)$
(b) $y(t) = \sin(t) - t\cos(t)$.
(c) $y(t) = e^t\left(-\frac{3}{13}\cos(t) + \frac{2}{13}\sin(t)\right)$
$+e^{-t/2}\left(C_1\cos(\sqrt{3}t/2) + C_2\sin(\sqrt{3}t/2)\right)$
(d) $y(t) = te^t + C_1e^t + C_2$
(e) $y(t) = \frac{t}{4}e^{2t} + \frac{5}{16}e^{-2t} + \frac{11}{16}e^{2t}$
(f) $y(t) = \frac{t^2}{2}e^{2t} + (C_1t + C_2)e^{2t}$
(g) $y(t) = 3\cos(t) + \frac{4}{3}\sin(t) - 3\cos(2t) - 2\sin(2t)$
(h) $y(t) = \sin(4t) - \cos(4t) + \cos(2t) - 2\sin(2t)$
(i) $y(t) = \frac{e^{-t}}{8}\cos(2t) + (C_1t + C_2)e^t$
Solution (10.1):
(a) $k = \frac{5}{16\pi^2}$ N-sec/meter
(b) $\omega = 5\pi\text{sec}^{-1}$
(c) $F_0 \approx 14.4$ N
Solution (10.2):
$\theta(t) = \frac{\cos(t) - \cos(\sqrt{g}t)}{g-1} \approx$
$0.011(\cos(t) - \cos(3.13t))$. Solution (10.3):
$y(t) = \frac{1}{5}(e^{-3t} + 3\sin(t) - \cos(t))$.
Solution (10.4):
$u(t) = \frac{\sqrt{377}}{87}\cos(3t + \tan^{-1}(16/11) - \pi)$
$\approx 0.223\cos(3t - 2.173)$.
Solution (10.5):
$x(t) = \frac{\gamma^2\omega^2\cos(\omega t) - m\gamma\omega(\omega_0^2 - \omega^2)\sin(\omega t)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$
or $x(t) = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2) + \gamma^2\omega^2}}\cos(\omega t + \phi)$,
where $\phi = \tan^{-1}\left(\frac{m(\omega_0^2 - \omega^2)}{\gamma\omega}\right)$.

Index