Chapter 7, section 7.

1. (7.2) Let $G$ be a finite group and let $H$ be a subgroup of order $p^m$ for a prime $p$ and integer $m \geq 1$. Show that $H$ is contained in a $p$-Sylow subgroup of $G$.

**Solution.** Let $K$ be a $p$-Sylow subgroup of $G$. The number of left $K$ cosets is not divisible by $p$ since it equals $\circ(G)/\circ(K)$ where $\circ(K)$ is the highest power of $p$ that divides $\circ(G)$. Let $H$ transform the set of left $K$ cosets by left composition. The orbits have sizes that divide the order of $H$, hence, the sizes are either 1 or a power of $p$. If each is a power of $p$, then $p$ would divide the number of left $K$-cosets. Therefore, some orbit consists of a single left coset $aK$. Hence, $HaK = aK$, i.e., $a^{-1}HaK = K$, which means that $a^{-1}Ha \subseteq K$, i.e., $H \subseteq aKa^{-1}$. Thus, $H$ is contained in the $p$-Sylow subgroup $aKa^{-1}$.

2. (7.5) Find the 2-Sylow subgroups in
(a). $D_{10}$.

**Solution.** A 2-Sylow subgroup is any subgroup with 4 elements. They are all conjugate, so we need only find one of them. The 180-degree rotation $\sigma$ is in the center of $D_{10}$. Let $\tau$ be any reflection in $D_{10}$. Then $\{I, \sigma\}\{I, \tau\}$ is a subgroup with 4 elements. If we conjugate that 2-Sylow subgroup with an element of $D_{10}$ we exchange $\tau$ for another reflection. Thus, as $\tau$ ranges over the set of reflections, $\{I, \sigma\}\{I, \tau\}$ ranges over the 2-Sylow subgroups.
(b). $S_4$.

**Solution.** The 2-Sylow subgroups are the subgroups with 8 elements. $D_4$ is a 2-Sylow subgroup. By Sylow theory, the number of 2-Sylow subgroups is congruent to 1 mod 2, and divides $24/8=3$. Thus, the number of 2-Sylow subgroups is 1 or 3. It is not 1 since conjugating $D_4$ by the permutation (12) gives a different 2-Sylow subgroup, as you can check. Thus, there are 3 2-Sylow subgroups in $S_4$.
(c). $S_6$.

**Solution.** The 2-Sylow subgroups have order $2^4 = 16$. To get one 2-Sylow subgroup, take $D_4\{I, (56)\}$, where $D_4$ permutes the numbers 1,2,3,4. Or we can pair any the three 2-Sylow subgroup in $S_4$ with $\{I, (56)\}$. To get the other 2-Sylow subgroups, instead of having $D_4$ permute 1,2,3,4, suppose that it permutes some other 4-element subset of $\{1,2,3,4,5,6\}$. There are 15 (6-choose-4) ways to choose 4-elements of 1,2,3,4,5,6, and each choice leads to three 2-Sylow subgroups. That gives $15(3) = 45$ 2-Sylow subgroups of $S_6$. 
3. (7.6) Exhibit a subgroup of $S_7$ that has order 21.

**Solution.** In a group of order 21, a subgroup of order 7 is normal since its index 3 is the smallest prime dividing the order of the group. All subgroups of order 7 in $S_7$ are conjugate since they are the 7-Sylow subgroups. That means that we can begin the construction of a group of order 21 with any subgroup of order 7, say with $H = <(1234567)>$. We need to find a subgroup of order 3 that normalizes $H$; that suggests that we compute the normalizer of $H$. $\sigma \in S_7$ normalizes $H = <(1234567)>$ if and only if

$$\sigma(1234567)\sigma^{-1} = (1234567)^m,$$

for some exponent $m$. If we look at $m = 2$, we find that $\sigma^3$ will commute with $(1234567)$. If $m = 2$, then

$$\sigma(1234567)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)\sigma(6)\sigma(7)) = (1234567)^2 = (1357246).$$

Matching up $(\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)\sigma(6)\sigma(7))$ with $(1357246)$, and taking $\sigma(1) = 1$, we see that $\sigma(2) = 3, \sigma(3) = 5, \sigma(4) = 7, \sigma(5) = 2, \sigma(6) = 4,$ and $\sigma(7) = 6$. That is, $\sigma = (235)(476)$, an element of order 3. Thus, since $(235)(476)$ normalizes $<(1234567)>$,

$$<(235)(476)> <(1234567)>$$

is a subgroup, and it has order $(3)(7)=21.$
4. (7.10) Show that the only simple groups of order less than 60 has prime order.

**Solution.** Let $G$ be a group of order is a nonprime number less than 60. We consider the nonprime numbers less than 60, listing those with the larger prime divisor first. The largest prime divisor of a nonprime number less than 60 is 29.

For 29, $\circ(G) = 29(2) = 58$. Since the index of the 29-Sylow subgroup is the smallest prime divisor of $\circ(G)$, the 29-Sylow subgroup is normal.

For 23, $\circ(G) = 23(2) = 46$. As for 29, the 23-Sylow subgroup is normal.

For 19, $\circ(G) = 19(2)$ or $\circ(G) = 19(3)$, and in both cases, the 19-Sylow subgroup is normal.

For 17, $\circ(G) = 17(2)$ or $\circ(G) = 17(3)$, and in both cases, the 17-Sylow subgroup is normal.

For 13, $\circ(G) = 13(2)$, $\circ(G) = 13(3)$, or $\circ(G) = 13(4)$. As above, for the first two cases, the 13-Sylow subgroup is normal. When $\circ(G) = 13(4)$, the number of 13-Sylow subgroups divides 4 and is congruent to 1 mod 13. Hence, there is only one 13-Sylow subgroup, and so, it is normal.

For 11, $\circ(G) = 11(2)$, $\circ(G) = 11(3)$, $\circ(G) = 11(4)$, or $\circ(G) = 11(5)$. The argument for 29 works for all the cases except for $\circ(G) = 11(4)$, which uses the argument for $\circ(G) = 13(4)$. In all cases, the 11-Sylow subgroup is normal.

For 7, $\circ(G) = 7(n)$, where $n = 2, ..., 7, 8$. If $n$ is prime, the 7-Sylow subgroup is normal by the minimal prime argument. If $n \leq 6$ is not prime, then since the number of 7-Sylow subgroups is congruent to 1 mod 7 and divides $n \leq 6$, which imply that there is just 1 $p$-Sylow subgroup, i.e., the p-Sylow subgroup is normal.

Looking over the different cases above, we see a general argument: if $\circ(G)$ is a product $pm$, where $m < p$, then the p-Sylow subgroup is normal.

If $\circ(G) = 7^2$, then a subgroup of order 7 is normal since its index, which is 7, is the smallest prime divisor of $\circ(G)$.

More generally, if $\circ(G) = p^n$ for some $n \geq 2$, $G$ is not simple: by Cauchy’s theorem, $G$ has a subgroup $H$ of order $p^{n-1}$, which is normal in $G$ since its index $p$ in $G$ is the smallest prime divisor of $G$. 


If \( \circ(G) = 7(8) = 56 \), then there is either 1 or 8 Sylow subgroups. If there is 1, it is normal. If there are 8, there are 6(8) = 48 elements of order 7, leaving only 56-48=8 elements to form a subgroup of order 8. Hence, those elements form a normal subgroup since the elements not of order 8 are transformed among themselves by conjugation, which preserves order. Hence, the 2-Sylow subgroup will be normal.

For 5, in the groups of order 5n, where \( n = 2, 3, 4, 8, 9 \), there is only 1 5-Sylow subgroup by the computation of Sylow theory, and so, it is normal.

If \( \circ(G) = 5(5) = 5^2 \), G is not simple by the general power of a prime argument above.

If \( \circ(G) = 5(6) = 30 \), G has 1 or 6 5-Sylow subgroups. If G has 6 5-Sylow subgroups, then it has 24 elements of order 5, leaving 6 elements that are permuted among themselves by conjugation. By Cauchy, there is a subgroup of order 3, which is normal since otherwise, there would be at least 4 3-Sylow subgroups by Sylow theory, which would give at least 8 elements of order 3, while only 6 elements are available. Pairing that normal subgroup with a subgroup of order 2, we find that the corresponding subgroup has order 6, and consists of the 6 elements not of order 5. Hence, that 6 element subgroup is normal since it is stable under conjugation.

If \( \circ(G) = 5(10) = 5^2(2) \), the 5-Sylow subgroup is normal since its index is 2.

For 3, \( \circ(G) = 3^n2^m \), with \( n \geq 1 \).

If \( m = 0 \) and \( n \geq 2 \), the group is not simple since the order is a power of a prime.

If \( m = 1 \), the 3-Sylow subgroup is normal, since its index is 2.

If \( m = 2 \) and \( n = 1 \), there are either 1 or 4 3-Sylow subgroups. If 1, the 3-Sylow is normal.

If 4, there are 8 elements of order 3, and the remaining 4 elements must form a subgroup of order 4, which is then unique.

If \( m = 2 \) and \( n = 2 \) (G has order \( 3^22^2 = 36 \)), then there are either 1 or 4 3-Sylow subgroups. If there are 4 3-Sylow subgroups, then the stabilizer subgroup \( H \) of a 3-Sylow subgroup under conjugation has 36/4 = 9 elements. To find a normal subgroup, take the homomorphism \( \phi : G \rightarrow Perm(G/H) \). Since G has 36 elements and \( Perm(G/H) \) has 4! = 24 elements, the kernel of \( \phi \) has more than one element, and since the kernel is contained in the subgroup \( H \), it is not the whole group. Thus, the kernel is a normal subgroup of G that shows that G is not simple.

If \( m = 3 \) and \( n = 1 \) (G has order \( 3(2^3) = 24 \)), then there are either 1 or 4 3-Sylow subgroups. If G has 4 3-Sylow subgroups, then the stabilizer \( H \) of a 3-Sylow subgroup under conjugation
has $24/4 = 6$ elements. Take the homomorphism $\phi : G \to \text{Perm}(G/H)$ from one 24-element group to another 24-element group. If the mapping is bijective, then $G$ is $S_4$, which is not simple since it contains the normal subgroup $A_4$. If it is not bijective, then the kernel of the homomorphism shows that $G$ is not simple.

If $m = 4$ and $n = 1$ ($G$ has order $3(2^4) = 48$, then there are either 1, 4, or 16 3-Sylow subgroups. If there are 16, then $G$ has $16(2) = 32$-elements of order 3, leaving only 16 elements to form a 2-Sylow subgroup. Hence, there is room for only one 2-Sylow subgroup, which is then normal.

If there are 4 3-Sylow subgroups, then the stabilizer subgroup $H$ has 12 elements and $G/H$ has 4 elements. Just as for groups of order 36, that leads to a normal subgroup of $G$.

For 2, $\sigma(G) = 2^n$ for $n \geq 2$. Those groups are not simple since their order is a power of a prime.