The Aldous diffusion on continuum trees

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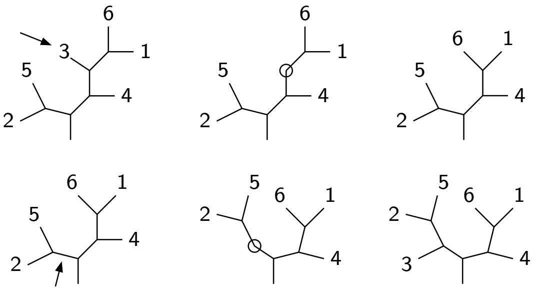
Part 1

The Aldous diffusion conjecture

Aldous down-up chain



Markov chain on rooted leaf-labeled binary trees. Each transition has two parts.

- Down-move: delete unif random leaf, contract away parent branch point.
- Up-move: select unif random edge, insert branch point, grow out new leaf-edge.

Results

Proposition (Aldous '01)

This is stationary with unif distrib on leaf-labeled binary trees.

Theorem (Schweinsberg '01)

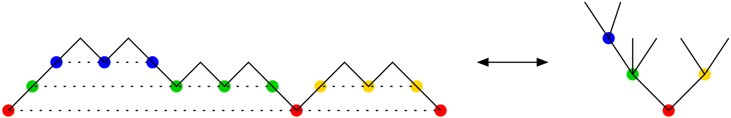
Relaxation time of Aldous chain on n-leaf trees is $\Theta(n^2)$.

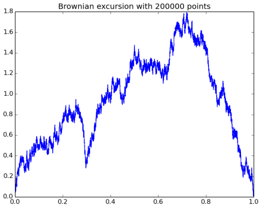
Conjecture (Aldous '99)

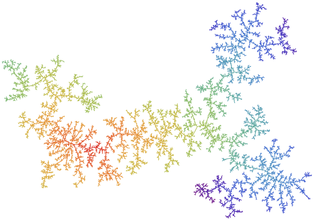
This Markov chain has a continuum analogue: a continuum random tree-valued diffusion, stationary w/ law of Brownian CRT.

What is a Brownian CRT? Aldous, Le Gall, ...

- Tree as a metric space with edge length $1/\sqrt{n}$. $n \to \infty$.
- Harris path representation (Harris '52):







(CRT Figure due to I. Kortchemski)

History and context

- Theoretical motivation: to construct a fundamental object – "Brownian motion on $\mathbb{R}$-tree space".
- Applied motivation: Aldous diffusion and projected processes are useful for inference on phylogenetic trees and genetic modeling. E.g., Ethier-Kurtz-Petrov diffusion.
- See: Evans-Winter '06, Evans-Pitman-Winter '06, Crane '14.
- Very recent related work: Löhr-Mytnik-Winter '18. Analysis without metric.

Our result

- We have a pathwise construction of the continuum-tree-valued analogue to the Aldous chain, stationary under BCRT (among other features).
- Forman-P.-Rizzolo-Winkel. "Aldous diffusion I: A projective system of continuum k-tree evolutions." ArXiv:1089.07756 [math.PR].
- For the remainder of this talk, we discuss this construction.

Key challenge: perfectly ephemeral leaves

- Time scaling is by n^2, where n is number of leaves.
- Takes $O(n \log(n))$ moves to replace every leaf. In $O(n^2)$ moves, w/ high probability, every leaf is replaced.
- Challenge: moves defined in terms of leaves, but in limit leaves die instantly. Makes it difficult to describe limiting object.
- Strategy: re-orient; focus on branch points.

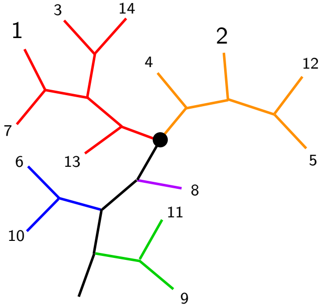
Part 2

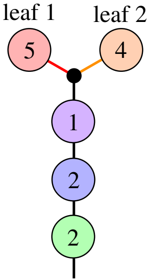
Projections and Intertwinings

Intutition

- Brownian CRT can be constructed as a projective limit of consistent finite trees.
- Idea goes back to original construction of Aldous.
- One can try a similar strategy in dynamics.

Spinal projection (discrete regime)





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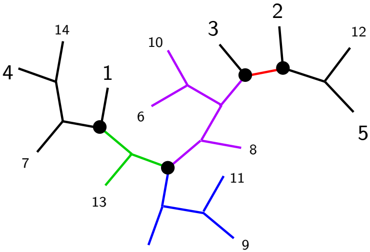

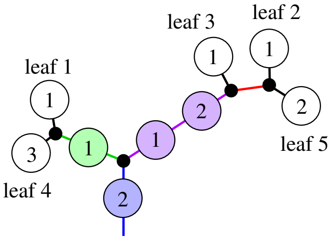

Taking the limit

Idea: Fix j and consider what happens when $n \to \infty$ in the projected trees.

- Take proportions of leaf masses in each component.

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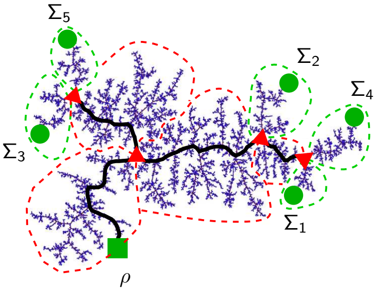
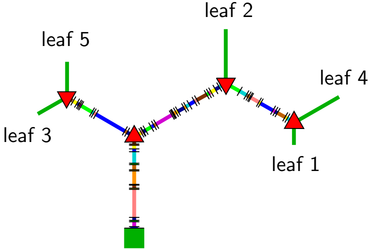
Then let $j \to \infty$.

Spinal projection (continuum regime)

Continuum 5-tree w/ interval partitions.





Interval partition (IP) β of $[0, M]$: a collection of disjoint, open intervals that cover $[0, M]$ up to Leb-null set.

Interval partitions

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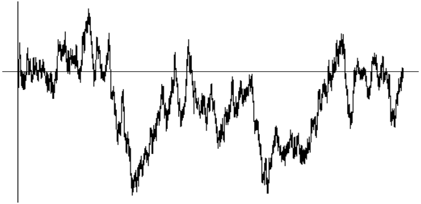
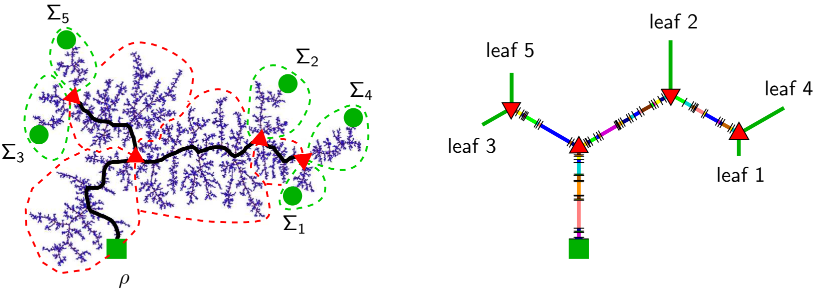
Example: Excursion intervals of standard Brownian bridge.



Call this a Poisson-Dirichlet$\left(\frac{1}{2}, \frac{1}{2}\right)$ interval partition, $\texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$.

Spinal projection of BCRT; Pitman-Winkel '15



- $\texttt{Dirichlet}\left(\frac{1}{2},\ldots,\frac{1}{2}\right)$ mass split among the 5 external and 4 internal components.
- Split the mass in each internal edge into an indep. $\texttt{PDIP}\left(\frac{1}{2},\frac{1}{2}\right)$.

We can recover path lengths from this picture, as diversities of interval partitions,

$$\mathrm{Div}(\beta) = \lim_{h\to 0} \sqrt{h}\#\{U \in \beta : \mathrm{Leb}(U) > h\}.$$

Projected diffusion on interval partitions

- One can recover the tree metric from diversity of interval partitions.
- The Aldous diffusion projected to interval partitions is also Markov.
- Select j leaves. Construct process of interval partitions from the projected masses.
- If we can describe it, and repeat consistency over j, that gives a projective limit as $j \to \infty$.
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- What is the dynamics on each interval partition?

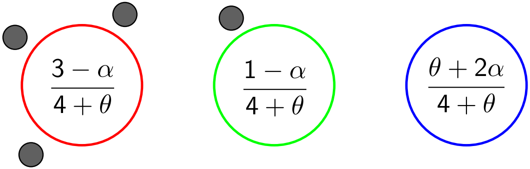
Part 3

Dynamics on interval partitions

Projected chains and Chinese Restaurants

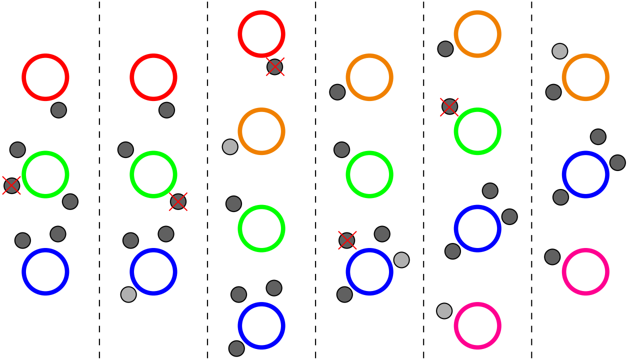
- Due to Dubins-Pitman
- $\texttt{CRP}(\alpha, \theta)$, $\alpha \in [0, 1)$, $\theta > -\alpha$. E.g., $\alpha = \frac{1}{2}, \theta = \frac{1}{2}$.
- Customer n will join table w/ m other customers w/ weight $m - \alpha$.
- Or, sit at empty table w/ weight $\theta + \alpha(\#$ of tables).

Probabilities of customer 5 joining each table



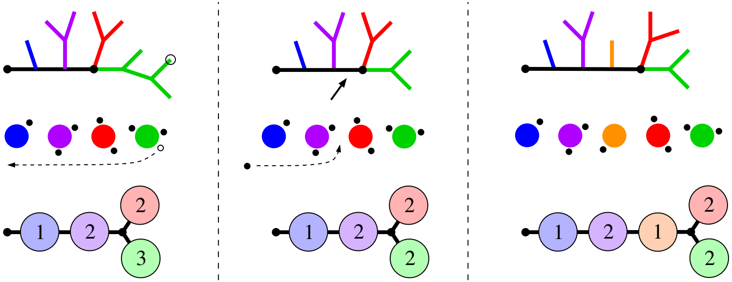
A Chinese restaurant with re-seating

- Markov chain on composition/ partitions of $[n]$.
- Transition rule: uniform random customer leaves, then re-enters according to $\texttt{CRP}(\alpha, \theta)$ seating rule.

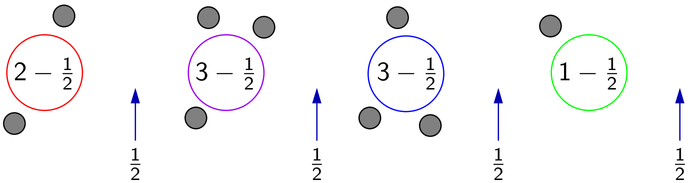


- See Petrov '09; Borodin-Olshanski '09

Aldous chain as re-seating



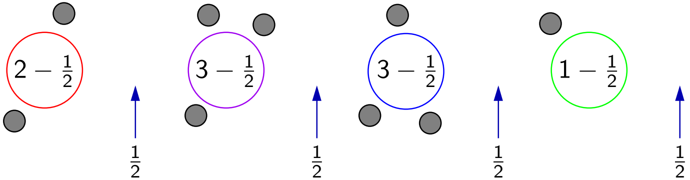
Poissonized down-up CRP



- each customer leaves after $\texttt{Exponential}(1)$ time,
- for a table w/ m customers, add customers with rate $m - \frac{1}{2}$,
- between any two tables, insert new tables w/ rate $\frac{1}{2}$.

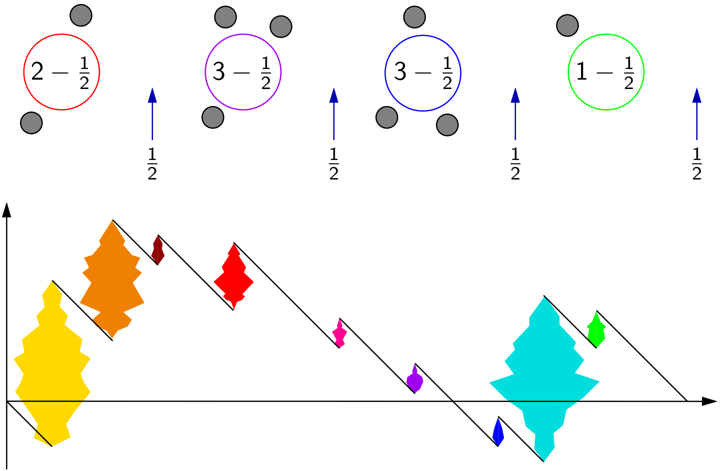
Table populations

Tables evolve independently of each other. Population of each is a birth-and-death chain.



When it has population m, increases w/ rate $m - \frac{1}{2}$, decreases w/ rate m. Birth-and-death chain.

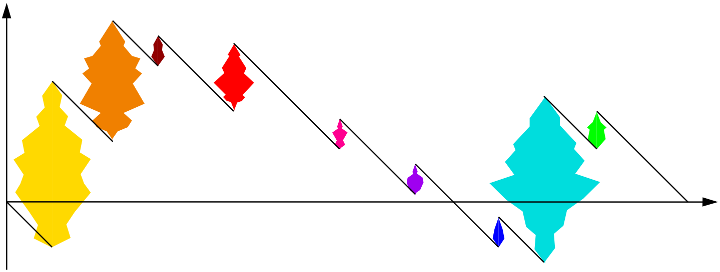
Coding the Poissonized, ordered CRP



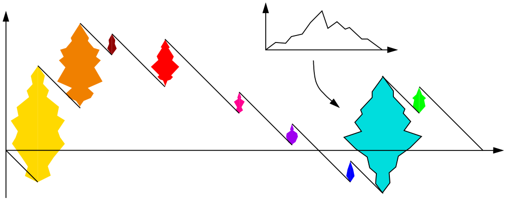
Convergence

In scaling limits:

- Law of birth-and-death chain of table populations in re-seating, starting from 1, converges to $\mathtt{BESQ}(-1)$ excursion measure, Bessel square diffusion with drift -1.
- Draw lines connecting deaths and births of tables. Converges to spectrally positive $\mathtt{Stable}\left(\frac{3}{2}\right)$.



Spindles on scaffolding

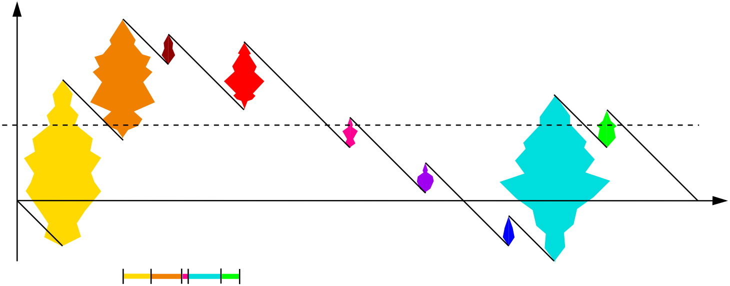


- Decorate jumps of $\texttt{Stable}\,(3/2)$ by ind. $\texttt{BESQ}\,(-1)$ excursions.
- Scaffolding - Lévy process.
- Spindles - independent excursions hanging on jumps of scaffolding.

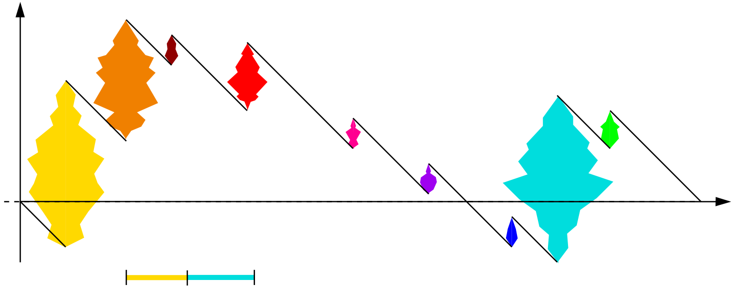
The Skewer map

For $y \in \mathbb{R}$, to get the level y skewer:

- Draw a line across picture at level y.
- From left to right, collect cross-sections of spindles.
- Slide together, as if on a skewer, to remove gaps.
- A stochastic process on interval partitions.



The Skewer process



- As line moves up from level 0, interval partition evolves continuously.
- Diversity=number of existing tables=local time of Stable(3/2)=tree metric on the spine.

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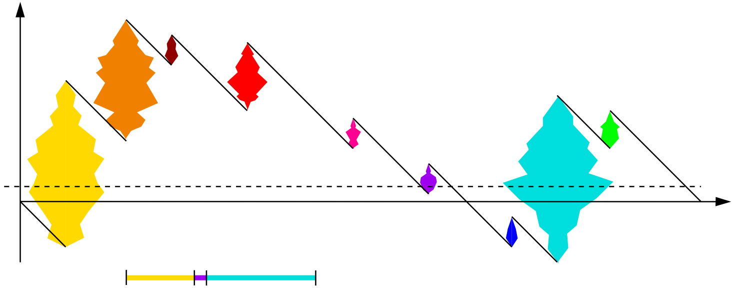

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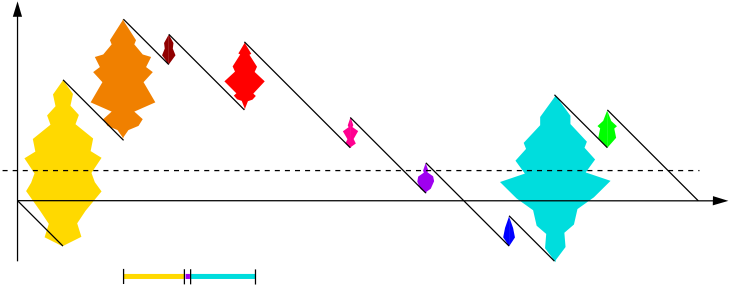

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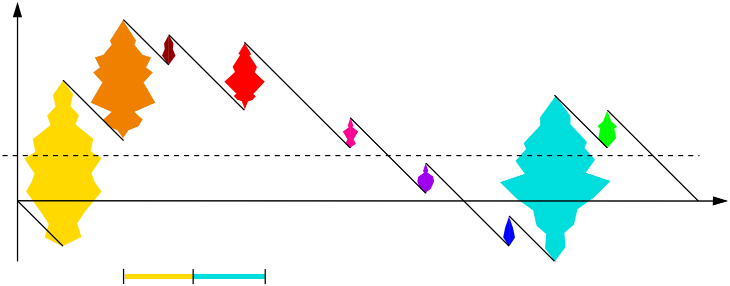

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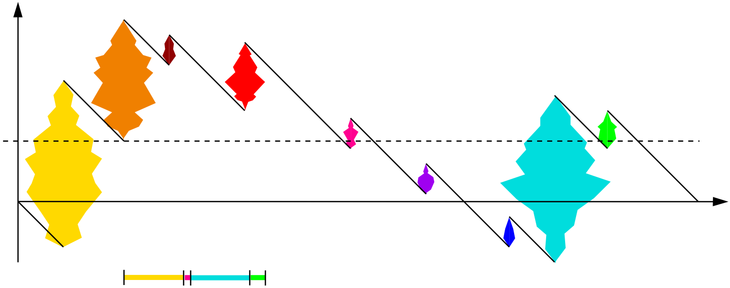

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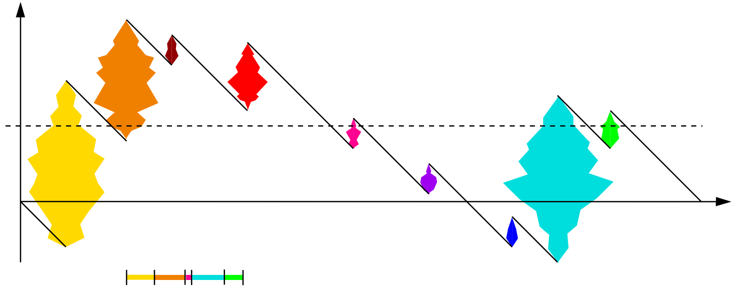

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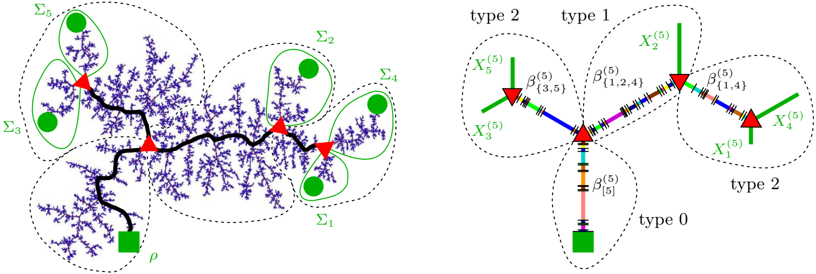
Outline of the construction of the Aldous diffusion by FPRW; chatty version

- Poissonize: leaves die and born independently.
- Project on j leaves to get $(j-1)$ independent skewer processes and j leaf masses.
- Each skewer process is a diffusion. Each mass is $\mathtt{BESQ}$.
- Show consistency over j by intertwining.
- DePoissonize by scaling and time-change.
- Take projective limit. Obtain process stationary with Brownian CRT.
- Prove limit is Markov and continuous is GH topology.

Building the limit

Evolving interval partitions generate a tree-valued process.



Part 4

Application: Ethier-Kurtz-Petrov diffusions

Ranked interval lengths and Poisson-Dirichlets

- Consider interval partition (IP) of $[0, 1]$ (mass one).
- Consider decreasing order stats of interval mass.
- Kingman simplex:

$$\nabla_\infty = \left\{ x_1 \geq x_2 \geq \dots, \sum_{i \in \mathbb{N}} x_i = 1 \right\}.$$

- $\texttt{PDIP}$ gives Poisson-Dirichlet distributions on ∇_∞.
- $\texttt{PDIP}\,(1/2, 1/2) \to \texttt{PD}\,(1/2, 1/2)$.
- $\texttt{PDIP}\,(\alpha, \theta) \to \texttt{PD}\,(\alpha, \theta)$, $0 \leq \alpha < 1$, $\theta > -\alpha$.

Diffusions on the Kingman simplex

- Diffusions on ∇_∞ reversible with respect to $\mathtt{PD}(\alpha, \theta)$?
- Ethier-Kurtz '81, Petrov '10 - generator for $\mathtt{EKP}(\alpha, \theta)$:

$$\sum_{i \geq 1} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j \geq 1} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i \geq 1} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.$$

- Also see Bertoin '08, Borodin and Olshanski '09, Feng-Sun '10, Feng-Sun-Wang-Xu '11, Ruggiero and coauthors '09, '13,'14.
- Mostly analytical or Dirichlet form techniques.
- Understanding on path behavior missing.

Diffusions without ranking?

- Theorem (FPRW)

  The de-Poissonized skewer process of interval partitions, when ranked gives $\texttt{EKP}\,(1/2, 1/2)$ *diffusion on the Kingman simplex.*

  - Provides pathwise description.
  - Can be generalized to all (α, θ) (future work).
  - Advantage of not ranking: provides better understanding of evolution of *small* blocks.
  - Allows us to settle some conjectures by previous authors.
  - E.g. continuity of diversity process.

Vielen Dank!