The Aldous diffusion on continuum trees

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Part 1

The Aldous diffusion conjecture
Aldous down-up chain

Markov chain on rooted leaf-labeled binary trees. Each transition has two parts.

- **Down-move**: delete unif random leaf, contract away parent branch point.
- **Up-move**: select unif random edge, insert branch point, grow out new leaf-edge.
Results

Proposition (Aldous ‘01)
This is stationary with unif distrib on leaf-labeled binary trees.

Theorem (Schweinsberg ‘01)
Relaxation time of Aldous chain on $n$-leaf trees is $\Theta(n^2)$.

Conjecture (Aldous ‘99)
This Markov chain has a continuum analogue: a continuum random tree-valued diffusion, stationary w/ law of Brownian CRT.
What is a Brownian CRT? Aldous, Le Gall, ...

- Tree as a metric space with edge length $1/\sqrt{n}$. $n \to \infty$.
- Harris path representation (Harris '52):

(CRT Figure due to I. Kortchemski)
History and context

- Theoretical motivation: to construct a fundamental object – “Brownian motion on $\mathbb{R}$-tree space”.

- Applied motivation: Aldous diffusion and projected processes are useful for inference on phylogenetic trees and genetic modeling. E.g., Ethier-Kurtz-Petrov diffusion.

- See: Evans-Winter ’06, Evans-Pitman-Winter ’06, Crane ’14.

Our result

- We have a pathwise construction of the continuum-tree-valued analogue to the Aldous chain, stationary under BCRT (among other features).


- For the remainder of this talk, we discuss this construction.
Key challenge: perfectly ephemeral leaves

- Time scaling is by $n^2$, where $n$ is number of leaves.
- Takes $O(n \log(n))$ moves to replace every leaf. In $O(n^2)$ moves, w/ high probability, every leaf is replaced.
- Challenge: moves defined in terms of leaves, but in limit leaves die instantly. Makes it difficult to describe limiting object.
- Strategy: re-orient; focus on branch points.
Part 2

Projections and Intertwinings
Intuition

- Brownian CRT can be constructed as a projective limit of consistent finite trees.
- Idea goes back to original construction of Aldous.
- One can try a similar strategy in dynamics.
Spinal projection (discrete regime)
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Taking the limit

Idea: Fix $j$ and consider what happens when $n \to \infty$ in the projected trees.

- Take proportions of leaf masses in each component.
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- What to do when a coordinate hits zero?
- FPRW: solves by resampling.
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- FPRW: There is a way to do this consistently over $j$ that allows taking projective limits. *Intertwining.*

Then let $j \to \infty$. 
Spinal projection (continuum regime)

Continuum 5-tree w/ interval partitions.

Interval partition (IP) $\beta$ of $[0, M]$: a collection of disjoint, open intervals that cover $[0, M]$ up to Leb-null set.
Interval partitions

Interval partition \((\text{IP}) \beta\) of \([0, M]\): a collection of disjoint, open intervals that cover \([0, M]\) up to Leb-null set.

Example: Excursion intervals of standard Brownian bridge.

Call this a Poisson-Dirichlet\((\frac{1}{2}, \frac{1}{2})\) interval partition, \(\text{PDIP} \left(\frac{1}{2}, \frac{1}{2}\right)\).
Spinal projection of BCRT; Pitman-Winkel ’15

- Dirichlet\((\frac{1}{2}, \ldots, \frac{1}{2})\) mass split among the 5 external and 4 internal components.
- Split the mass in each internal edge into an indep. PDIP\((\frac{1}{2}, \frac{1}{2})\).

We can recover path lengths from this picture, as diversities of interval partitions,

\[
\text{Div}(\beta) = \lim_{h \to 0} \sqrt{h} \# \{ U \in \beta : \text{Leb}(U) > h \}.
\]
Projected diffusion on interval partitions

- One can recover the tree metric from diversity of interval partitions.
- The Aldous diffusion projected to interval partitions is also Markov.
- Select $j$ leaves. Construct process of interval partitions from the projected masses.
- If we can describe it, and repeat consistency over $j$, that gives a projective limit as $j \to \infty$.
- The limit is the Aldous diffusion itself.
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- The limit is the Aldous diffusion itself.
- What is the dynamics on each interval partition?
Part 3

Dynamics on interval partitions
Projected chains and Chinese Restaurants

- Due to Dubins-Pitman
- $\text{CRP}(\alpha, \theta)$, $\alpha \in [0, 1)$, $\theta > -\alpha$. E.g., $\alpha = \frac{1}{2}, \theta = \frac{1}{2}$.
- Customer $n$ will join table $w/ m$ other customers $w/ \text{weight } m - \alpha$.
- Or, sit at empty table $w/ \text{weight } \theta + \alpha(\# \text{ of tables})$.

**Probabilities of customer 5 joining each table**

- \( \frac{3 - \alpha}{4 + \theta} \)
- \( \frac{1 - \alpha}{4 + \theta} \)
- \( \frac{\theta + 2\alpha}{4 + \theta} \)
A Chinese restaurant with re-seating

- Markov chain on composition/ partitions of $[n]$.
- Transition rule: uniform random customer leaves, then re-enters according to $\text{CRP}(\alpha, \theta)$ seating rule.

- See Petrov ’09; Borodin-Olshanski ’09
Aldous chain as re-seating
Poissonized down-up CRP

- each customer leaves after Exponential(1) time,
- for a table with $m$ customers, add customers with rate $m - \frac{1}{2}$,
- between any two tables, insert new tables with rate $\frac{1}{2}$. 

\[
\begin{align*}
2 - \frac{1}{2} & \\
3 - \frac{1}{2} & \\
3 - \frac{1}{2} & \\
1 - \frac{1}{2} & 
\end{align*}
\]
Table populations

Tables evolve independently of each other. Population of each is a birth-and-death chain.

When it has population \( m \), increases w/ rate \( m - \frac{1}{2} \), decreases w/ rate \( m \). Birth-and-death chain.
Coding the Poissonized, ordered CRP
Convergence

In scaling limits:

- Law of **birth-and-death chain** of table populations in re-seating, starting from 1, converges to $\text{BESQ}(-1)$ excursion measure, Bessel square diffusion with drift $-1$.

- Draw lines connecting deaths and births of tables. Converges to **spectrally positive Stable**($\frac{3}{2}$).
Spindles on scaffolding

- Decorate jumps of Stable $(3/2)$ by ind. BESQ($-1$) excursions.
- Scaffolding - Lévy process.
- Spindles - independent excursions hanging on jumps of scaffolding.
The Skewer map

For $y \in \mathbb{R}$, to get the level $y$ skewer:

- Draw a line across picture at level $y$.
- From left to right, collect cross-sections of spindles.
- Slide together, as if on a skewer, to remove gaps.
- A stochastic process on interval partitions.
The Skewer process

- As line moves up from level 0, interval partition evolves continuously.
- Diversity = number of existing tables = local time of Stable(3/2) = tree metric on the spine.
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Outline of the construction of the Aldous diffusion by FPRW; chatty version

- Poissonize: leaves die and born independently.
- Project on $j$ leaves to get $(j - 1)$ independent skewer processes and $j$ leaf masses.
- Each skewer process is a diffusion. Each mass is BESQ.
- Show consistency over $j$ by intertwining.
- DePoissonize by scaling and time-change.
- Take projective limit. Obtain process stationary with Brownian CRT.
- Prove limit is Markov and continuous is GH topology.
Building the limit

Evolving interval partitions generate a tree-valued process.
Part 4

Application: Ethier-Kurtz-Petrov diffusions
Consider interval partition (IP) of $[0, 1]$ (mass one).
Consider decreasing order stats of interval mass.

Kingman simplex:

\[ \nabla_\infty = \left\{ x_1 \geq x_2 \geq \ldots, \sum_{i \in \mathbb{N}} x_i = 1 \right\} . \]

PDIP gives Poisson-Dirichlet distributions on $\nabla_\infty$.

PDIP $(1/2, 1/2) \rightarrow \text{PD} (1/2, 1/2)$.

PDIP $(\alpha, \theta) \rightarrow \text{PD} (\alpha, \theta)$, $0 \leq \alpha < 1$, $\theta > -\alpha$. 
Diffusions on the Kingman simplex

- Diffusions on $\nabla_\infty$ reversible with respect to $PD(\alpha, \theta)$?
- Ethier-Kurtz ’81, Petrov ’10 - generator for EKP $(\alpha, \theta)$:
  \[
  \sum_{i \geq 1} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j \geq 1} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i \geq 1} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.
  \]
- Also see Bertoin ’08, Borodin and Olshanski ’09, Feng-Sun ’10, Feng-Sun-Wang-Xu ’11, Ruggiero and coauthors ’09, ’13,’14.
- Mostly analytical or Dirichlet form techniques.
- Understanding on path behavior missing.
Diffusions without ranking?

- **Theorem (FPRW)**

  The de-Poissonized skewer process of interval partitions, when ranked gives $EKP\left(\frac{1}{2}, \frac{1}{2}\right)$ diffusion on the Kingman simplex.

  - Provides pathwise description.
  - Can be generalized to all $(\alpha, \theta)$ (future work).
  - Advantage of not ranking: provides better understanding of evolution of small blocks.
  - Allows us to settle some conjectures by previous authors.
  - E.g. continuity of diversity process.
Vielen Dank!