Representations of finite dimensional algebras and singularity theory

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1 Introduction

This survey deals with classes of algebras, finite or infinite dimensional over a base field k, arising in the representation theory of finite dimensional algebras:

1. finite dimensional tame hereditary [14, 51] resp. canonical algebras [51, 21, 52, 40] more generally algebras with a separating tubular family [39, 41],
2. preprojective algebras of the path algebra of an extended Dynkin quiver [23], more generally preprojective algebras of a (tame) hereditary algebra [16, 8, 53],
3. surface singularities [12, 22],
4. algebras of automorphic forms [44, 47, 34],
5. two-dimensional factorial algebras [45, 32], see also [46, 56, 42].

We show that a common focus for these classes of rings is provided by the concept of an *exceptional curve*, which generalizes the notion of a weighted projective line from [21], and which for algebraically closed base fields agrees with the latter notion. Roughly speaking, an exceptional curve is a noncommutative curve which is smooth, projective and allows an exceptional sequence of coherent sheaves. Such curves are quite rare since among the nonsingular projective (commutative) curves over an algebraically closed field only the projective line is exceptional. We need to pass to not algebraically closed base fields or to a noncommutative geometric setting in order to get a richer supply of exceptional curves. Note that our choice of terminology is influenced from representation theory, the reader should therefore not confuse exceptional curves as they occur in this survey with those which traditionally have the same name in algebraic geometry, see [28].

The structure of the paper is as follows. In Section 2 we define the notion of an exceptional curve by specifying — as it is common for noncommutative algebraic geometry [54] — the properties of its abelian category of coherent sheaves, thought to arise from a finite number of module categories (of finitely genereated modules) over typically noncommutative noetherian rings by a gluing process. Noncommutativity allows that more than one simple sheaf may be concentrated in a single point; in case this will not happen we call the curve *homogeneous*, again following usage of the term in representation theory. The geometry of a homogeneous exceptional curve is shown in Section 3.3 to be controlled completely by the representation theory of a finite dimensional tame bimodule algebra, an important special case of a finite dimensional tame hereditary algebra (cf. [14, 51]).

In Sections 4 and 5 we show how to pass, in particular, from the homogeneous to the general case by a process called *insertion of weights*. As we are going to show, this process equips a nonsingular curve with a (quasi)-parabolic structure (cf. [58]), but in contrast to the usual treatment of a parabolic structure, which restricts to vector bundles, we extend this concept to all coherent sheaves. Section 7 forms the heart of this report and shows the wide range of applications of the concept of an exceptional curve.

Let k be a field. A k-algebra in this survey will always be associative with a unit element. Modules are usually right modules. We denote by $\mathrm{mod}(A)$ the category of finitely presented and by Mod(A) the category of all right A-modules. If A is moreover graded by an abelian group H, then $\mathrm{mod}^H(A)$ resp. $\mathrm{Mod}^H(A)$ denote the corresponding categories of H-graded modules with degree zero morphisms.

By a k-category $\mathcal{C}$ we mean an additive category where the morphism spaces are k-spaces and composition is k-bilinear. We call $\mathcal{C}$ *small* if the isomorphism classes of objects from $\mathcal{C}$ form a set. The small categories arising in this article will further be assumed to be *locally finite* over k, meaning that all morphism spaces — and in case of an abelian category also all extension spaces — do have finite k-dimension. Our main examples in that respect are the category mod(A) for a finite dimensional k-algebra A, the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves over a non-singular projective variety or occasionally a full subcategory of one of those. An abelian k-category $\mathcal{C}$ is called *connected* if it is not possible to represent $\mathcal{C}$ as a (co)product of non-zero k-categories.

If $\mathcal{C}$ has finite global dimension, the Grothendieck group $\mathrm{K}_0(\mathcal{C})$, formed with respect to short exact sequences, is equipped with a homological bilinear form, called *Euler form*, given on classes of objects by the expression

$$\langle [X],[Y]\rangle = \sum_{i=0}^{\infty}(-1)^i \dim_k \mathrm{Ext}^i(X,Y),$$

where bracket notation $[X]$ refers to the class in the Grothendieck group. An automorphism Φ of $\mathrm{K}_0(\mathcal{C})$, satisfying $\langle y,x\rangle = -\langle x,\Phi y\rangle$ for all x, y is called *Coxeter transformation* for $\mathcal{C}$ or $\mathrm{K}_0(\mathcal{C})$. Such an automorphism always exists if the Euler form is non-degenerate and in this case is further uniquely determined.

For the basic notions of the representation theory of finite dimensional algebras we refer to [5, 20] and [51]. We mention explicitly that the notion of an *almost-split sequence* $0 \to A \to B \to C \to 0$ makes sense in any small abelian k-category $\mathcal{C}$. It just means that the sequence has indecomposable end terms and does not split, and further that for any indecomposable object X in $\mathcal{C}$ each non-isomorphism $f : X \to C$ lifts to B, equivalently that each non-isomorphism $f : A \to Y$ into an indecomposable object Y extends to B. If almost-split sequences exist, the assignments $C \mapsto A$ and $A \mapsto C$ are called *Auslander-Reiten translation*s and denoted τ resp. τ^-. For a module category mod(Λ) over a finite dimensional algebra, almost-split sequences always exist, moreover the *Auslander-Reiten formula* states

$$\mathrm{DExt}^1(X,Y) = \overline{\mathrm{Hom}}(Y,\tau X) = \underline{\mathrm{Hom}}(\tau^- Y, X)$$

where $\underline{\mathrm{Hom}}$ (resp. $\overline{\mathrm{Hom}}$) refers to morphisms in the stable category modulo projectives (resp. injectives). Note that the formula resembles Serre duality for curves.

A morphism $f : X \to Y$ is called *irreducible* in $\mathcal{C}$ if for any factorization $f = \beta \circ \alpha$ in $\mathcal{C}$ the morphism α is a split monomorphism or β is a split epimorphism. Assuming that almost-split sequences exist for $\mathcal{C}$, the *Auslander-Reiten quiver* $\Gamma_{\mathcal{C}}$ of $\mathcal{C}$ has the isomorphism classes of indecomposable objects as vertices, while the arrows are determined by the existence of irreducible maps. The term *component*, in this context, refers to a connected component of $\Gamma_{\mathcal{C}}$.

For a hereditary abelian category $\mathcal{H}$, *hereditary* means that $\mathrm{Ext}^2_{\mathcal{H}}(-,-) = 0$, its (bounded) *derived category* $\mathrm{D}^b(\mathcal{H})$ can be obtained as follows: For each integer n we take a copy $\mathcal{H}[n]$ of $\mathcal{H}$ with objects denoted $X[n]$, $X \in \mathcal{H}$, then form the union of all $\mathcal{H}[n]$, which is a k-linear category with morphisms given by $\mathrm{Hom}(X[n],Y[m]) = \mathrm{Ext}^{m-n}_{\mathcal{H}}(X,Y)$, and finally close under finite direct sums. Note that we will use the bracket notation $\mathcal{A}[n]$ also for full subcategories. Most derived categories occurring in this paper are equivalent to derived categories of a hereditary category; for general information on derived categories we refer to [24, 25].

2 Exceptional curves

This section is addressed to people not specialized in algebraic geometry, and intends to cover in a quick hence fragmentary way those aspects which are important to link these concepts with the range of representation theoretic topics mentioned in the introduction. For a proper treatment of the concepts involved we refer the reader for instance to [54, 2].

The main purpose of this section is to introduce the notion of an *exceptional curve* $\mathbb{X}$ and to put this notion into proper context. Roughly speaking we will call a possibly noncommutative curve $\mathbb{X}$ exceptional if it is noetherian, smooth, projective, and admits an exceptional sequence of coherent sheaves. We start to explain these notions, and the concept of an exceptional curve, in more detail.

2.1 Noncommutative spaces

We define a (possibly) *noncommutative space* $\mathbb{X}$ through an abelian category $\mathcal{C}$ which we are going to interpret as a category of *coherent sheaves* on $\mathbb{X}$. We are only interested in the situation where the category $\mathcal{C}$ is *noetherian*, meaning that each object in $\mathcal{C}$ satisfies the ascending chain condition for subobjects. Slightly deviating from standard terminology, we will in this case say that $\mathbb{X}$ is a *noetherian space*.

Calling $\mathcal{C}$ a category of coherent sheaves is, for the moment, mainly a façon de parler. It is however possible, following the dictionary provided by ([18], chap. VI, see also [54]) for the case of a commutative noetherian (pre)scheme $\mathbb{X}$, to recover the underlying space and its Zariski topology from the category $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ of coherent sheaves on $\mathbb{X}$. For the case of a curve, the case we are mainly interested in, we will specify its underlying set of points in the proper context of Section 2.5. The categories of coherent sheaves $\mathcal{C}$ arising in this article will (with a single exception discussed in Section 7.8) arise by a gluing process from module categories $\mathrm{mod}(A)$ over noetherian rings.

A *basic gluing* ([18], chap. VI) amounts to form the *pull-back* $\mathcal{C}$ of two abelian categories $\mathcal{C}_1$ and $\mathcal{C}_2$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow \rho_1 \\ \mathcal{C}_2 & \xrightarrow{\rho_2} & \mathcal{C}_{12} \end{array}$$

along two exact functors $\rho_1 : \mathcal{C}_1 \to \mathcal{C}_{12}$, $\rho_2 : \mathcal{C}_2 \to \mathcal{C}_{12}$ to a third abelian category $\mathcal{C}_{12}$, where additionally each ρ_i is assumed to induce an equivalence $\mathcal{C}/\ker(\rho_i) \cong \mathcal{C}_{12}$, so that $\mathcal{C}_{12}$ becomes a quotient category of both $\mathcal{C}_1$ and $\mathcal{C}_2$. The pull-back $\mathcal{C}$ is again an abelian category:

The objects of $\mathcal{C}$ are triples (C_1, C_2, α), where C_i is an object of $\mathcal{C}_i$ and α is an isomorphism from $\rho_1 C_1$ to $\rho_2 C_2$. A morphism from (C_1, C_2, α) to (C_1', C_2', α') is a pair (u_1, u_2) of morphisms $u_1 : C_1 \to C_1'$, $u_2 : C_2 \to C_2'$ such that $(\rho_2 u_2) \circ \alpha = \alpha' \circ (\rho_1 u_1)$. Composition of morphisms is componentwise.

In intuitive interpretation, a basic gluing corresponds to an open covering $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ with $\mathcal{C} = \mathrm{coh}(\mathbb{X})$, $\mathcal{C}_1 = \mathrm{coh}(\mathbb{X}_1)$, $\mathcal{C}_2 = \mathrm{coh}(\mathbb{X}_2)$ and $\mathcal{C}_{12} = \mathrm{coh}(\mathbb{X}_1 \cap \mathbb{X}_2)$. Starting with the module categories $\mathrm{mod}(A)$ for noetherian rings A, all categories obtained from those by a finite number of basic gluings are considered to qualify as categories of coherent sheaves (cf. [18], chap. VI for the commutative setting). Even not having specified what $\mathbb{X}$ is, we will write $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ for such a category in order to stay close to geometric intuition.

We illustrate the construction by an example: the k-algebra homomorphisms $r_1 : k[X_1] \to k[X, X^{-1}]$, $r_2 : k[X_2] \to k[X, X^{-1}]$ with $r_1(X_1) = X$, $r_2(X_2) = X^{-1}$ induce by extension of scalars exact functors $\rho_i : \mathrm{mod}(k[X_i]) \to \mathrm{mod}(k[X, X^{-1}])$. The category $\mathcal{C}$ obtained by gluing $\mathrm{mod}(k[X_1])$ and $\mathrm{mod}(k[X_2])$ along ρ_1, ρ_2 is equivalent to the category of coherent sheaves on the projective line $\mathbb{P}_1(k)$, and the gluing data correspond to a an open covering of $\mathbb{P}_1(k)$ by two affine lines.

A space $\mathbb{X}$ as above is called *commutative* if $\mathrm{coh}(\mathbb{X})$ can be obtained by a gluing process starting with categories $\mathrm{mod}(A_i)$ over rings which are commutative, otherwise $\mathbb{X}$ is called *noncommutative*. If $\mathcal{C}$ has two non-isomorphic simple objects S_1, S_2 with $\mathrm{Ext}^1(S_1, S_2) \neq 0$, then $\mathbb{X}$ is forced to be noncommutative and, in fact, that such situations happen is to a large extent typical for a noncommutative setting.

From the technical point of view it is often more convenient to glue full module categories $\mathrm{Mod}(A_i)$, resulting in what then is called a category of *quasi-coherent sheaves*. Also for a proper treatment of homological questions we will need the category $\vec{\mathcal{C}}$ of quasi-coherent sheaves corresponding to $\mathcal{C}$. In the noetherian situation, which we assume, this category can be recovered as the category of all contravariant left exact additive functors from $\mathcal{C}$ to abelian groups [18]. The category $\vec{\mathcal{C}}$ is a Grothendieck category which is locally noetherian, in particular is locally finitely presented. Therefore $\mathcal{C}$ can be identified with the full subcategory of finitely presented objects of $\vec{\mathcal{C}}$, and each object of $\vec{\mathcal{C}}$ is a filtered direct limit of objects from $\mathcal{C}$. As a Grothendieck category $\vec{\mathcal{C}}$ has sufficiently many injective objects, whereas $\mathcal{C}$ may not have any non-zero projectives or injectives.

Notice that the above scheme allows for *different scales of non-commutativity* of a space $\mathbb{X}$: the rings involved in the gluing process can be requested to be commutative, or to be finitely generated as modules over their center, or to satisfy a polynomial identity, or ... In fact, our examples, except the one in Section 7.8, will all belong to the second class, hence stay quite close to commutativity.

2.2 Nonsingular spaces

Let $\mathbb{X}$ be a noncommutative noetherian space and $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ its noetherian category of coherent sheaves. We define $\mathrm{Ext}^n(C, -)$ as the n-th right derived functor of $\mathrm{Hom}(C, -)$ by means of injective resolutions in $\vec{\mathcal{C}}$. Within $\mathcal{C}$ the resulting values of $\mathrm{Ext}^n(X, Y)$ can alternatively be calculated by means of n-fold Yoneda extensions.

Definition 2.1 A noetherian space $\mathbb{X}$ is called smooth of dimension d if $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ has finite global dimension d, i.e. $\mathrm{Ext}^d_{\mathcal{C}}(-,-) \neq 0$ for $\mathcal{C}$ but $\mathrm{Ext}^{d+1}_{\mathcal{C}}(-,-) = 0$.

For a *smooth curve* ($d = 1$) the category $\mathcal{C} = \mathrm{coh}(\mathbb{X})$ is thus hereditary and noetherian.

For a commutative space $\mathbb{X}$ with a noetherian category of coherent sheaves the above concept of smooth spaces and their dimension yields the usual geometric concept. For the noncommutative spaces, close to commuativity we are going to discuss, the concept also reflects the geometrical meaning quite well. In general situations, further apart from commutativity, a more restrictive concept of smoothness is needed, cf. [2].

2.3 Projective spaces

Projectivity (relative to a base field k) is an important finiteness condition on a space $\mathbb{X}$ or its category $\mathcal{C}$ of coherent sheaves. It implies, in particular, that $\mathcal{C}$ has morphism and extension spaces which are finite dimensional over k. The definition however is technical, and less obvious than the requirements discussed up to now. The definition to follow is basically modelled after Serre's treatment of projective spaces in [57]. We restrict to the case where $\mathbb{X}$ is smooth.

Let k be a field and let H be a finitely generated abelian group of rank one which may have torsion. We further assume that H is equipped with a (partial) order, compatible with the group structure. We reserve the term *H-graded k-algebra* for the following setting:

R is a k-algebra equipped with a decomposition $R = \bigoplus_{h\in H} R_l$ such that each R_h is a finite dimensional k-space, further $R_h \cdot R_l \subseteq R_{l+h}$ holds for all h, l and finally $R_h \neq 0$ implies $h > 0$ in the given ordering of H. We do not request that $R_0 = k$ or that R is generated — as a k-algebra — by elements of degree one.

Definition 2.2 A smooth space $\mathbb{X}$ is called *projective over k* if there exists a finitely generated noetherian k-algebra $R = \bigoplus_{h\in H} R_h$, graded by an ordered abelian group H of rank one, such that $\mathrm{coh}(\mathbb{X})$ is isomorphic to the quotient category $\mathrm{mod}^H(R)/\mathrm{mod}^H_0(R)$.

Here, $\mathrm{mod}^H(R)$ denotes the category of finitely generated H-graded right R-modules. Therefore each $M \in \mathrm{mod}^H(R)$ is equipped with a decomposition $M = \bigoplus_{h\in H} M_h$ into k-subspaces such that $M_h \cdot R_l \subseteq M_{h+l}$ holds for all h, l in H. Our request on R forces the M_h to be finite dimensional over k. The subindex zero in the expression $\mathrm{mod}^H_0(R)$ refers to the full subcategory of modules of finite length (here the same as finite k-dimension). This subcategory is closed under subobjects, quotients and extensions, hence is a *Serre subcategory* of $\mathrm{mod}^H(R)$. The resulting *quotient category* is formed in the sense of Serre-Grothendieck-Gabriel, see [18] or [49]. It is not difficult to verify the next assertion.

Proposition 2.3 *Assume that R is a finite module over its center C and $C = k[x_1, \dots, x_n]$, where the x_i are assumed to be homogeneous of degree > 0, then the quotient category $\mathrm{mod}^H(R)/\mathrm{mod}_0^H(R)$ arises from a finite number of module categories $\mathrm{mod}(R_i)$ by gluing, where each R_i is obtained from R by central localization.*

2.4 Exceptional spaces

An object E in a small abelian k-category is called *exceptional* if the endomorphism algebra of E is a skew field and moreover E has no self-extensions, i.e. $\mathrm{Ext}^n(E, E) = 0$ for all $n > 0$. This notion extends to objects of the bounded derived category, where the Ext-condition needs to be replaced by the request $\mathrm{Hom}(E, E[n]) = 0$ for all $n \neq 0$.

Definition 2.4 A smooth projective space $\mathbb{X}$ is called *exceptional* if the bounded derived category of $\mathrm{coh}(\mathbb{X})$ admits a *complete exceptional sequence* $E_1, \dots, E_n$. We thus request that

1. Each E_i is exceptional;
2. $\mathrm{Ext}^l(E_k, E_i) = 0$ for $i < k$ and for all l;
3. The objects $E_1, \dots, E_n$ generate the bounded derived category of $\mathrm{coh}(\mathbb{X})$ as a triangulated category.

A complete exceptional sequence can be viewed as a system of building blocks for the derived category or for $\mathrm{coh}(\mathbf{X})$ if the E_i lie in $\mathrm{coh}(\mathbf{X})$. In particular, the classes $[E_1], \dots, [E_n]$ form a $\mathbb{Z}$-basis of the Grothendieck group $\mathrm{K}_0(\mathrm{coh}(\mathbb{X}))$. Exceptional spaces accordingly are quite rare especially in the range of commutative spaces. For instance, the only commutative exceptional curve over an algebraically closed field is the projective line $\mathbb{P}_1(k)$. By contrast, and this is in the focus of this survey, there is a rich supply of noncommutative exceptional curves.

Further examples of commutative exceptional spaces are provided by the projective n-space [9], the quadrics [30] and Grassmannians [29]. For additional information in this direction we refer to [55], further to [7] for examples of noncommutative exceptional spaces of dimension ≥ 2.

Closely related to the notion of an exceptional sequence is the notion of a *tilting object* $\bigoplus_{i=1}^n E_i$, where conditions 1. and 2. of Definition 2.4 are replaced by the request $\mathrm{Ext}^l(E_i, E_j) = 0$ for all i, j and all $l \neq 0$. For the situations, studied in this paper, the indecomposable summands E_i of a tilting object can be rearranged to form an exceptional sequence (cf. [27], Lemma 4.1).

2.5 Exceptional curves

Let k be an arbitrary field. In accordance with the preceding discussion we now fix the notion of an *exceptional curve* $\mathbb{X}$ over k by the following requests on its associated category $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ of coherent sheaves:

1. $\mathcal{H}$ is a connected small abelian k-category with morphism spaces that are finite dimensional over k.
2. $\mathcal{H}$ is hereditary and noetherian, and there exists an equivalence $\tau : \mathcal{H} \to \mathcal{H}$ such that Serre duality $\mathrm{DExt}^1(X, Y) = \mathrm{Hom}(Y, \tau X)$ holds.
3. $\mathcal{H}$ admits a complete exceptional sequence.

It is possible to skip the request on Serre duality by replacing condition 3. through the related condition "$\mathcal{H}$ has no non-zero projectives and admits a tilting object" (compare [36]).

Let $\mathcal{H}_0$ denote the full subcategory of $\mathcal{H}$ consisting of all objects of finite length, and let $\mathcal{H}_+$ consist of all objects from $\mathcal{H}$ with zero socle. We term the members from $\mathcal{H}_0$ *torsion sheaves* and those of $\mathcal{H}_+$ *vector bundles.*

Combining the techniques of [36], dealing with the case of an algebraically closed base field, and of [41], which treats the related question of a finite dimensional algebra with a separating tubular family over an arbitrary field, we get the following information on $\mathcal{H}$.

1. Each object in $\mathcal{H}$ decomposes into a finite number of indecomposable objects with local endomorphism rings. Moreover each indecomposable object is either a bundle or a torsion sheaf.

2. The category $\mathcal{H}_0$ of torsion sheaves decomposes into a coproduct $\coprod_{x\in\mathbb{X}}\mathcal{U}_x$ of connected uniserial categories $\mathcal{U}_x$, each having a finite number $p(x)$ of simple objects (compare [19]). The latter means that each object of $\mathcal{U}_x$ has a unique composition series in $\mathcal{U}_x$, hence is determined up to isomorphism by its simple $\mathcal{U}_x$-socle and its length in the category $\mathcal{U}_x$.

   Moreover, finite generation of $\mathrm{K}_0(\mathcal{H})$ implies that $p(x) = 1$ for almost all $x \in \mathbb{X}$.

3. We interpret the index set $\mathbb{X}$ as the *point set of the curve* $\mathbb{X}$ associated with $\mathcal{H}$. The members of $\mathcal{U}_x$ are said to be *concentrated in* x. By definition, the Zariski topology has $\mathbb{X}$ and its finite subsets as the closed sets.

4. Defining w_x as the sum of all classes of simple objects from $\mathcal{U}_x$ in the Grothendieck group $\mathrm{K}_0(\mathcal{H})$, all classes $w_x = [S_x]$ are proportional, hence all are in the same rank one direct factor $\mathbb{Z}.w$ of the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ ([36] Lemma 6).

5. Defining the normalized *rank function* on $\mathrm{K}_0(\mathcal{H})$ as $\frac{1}{\kappa}\langle -, w\rangle$, where $\mathbb{Z}.\kappa = \langle \mathrm{K}_0(\mathcal{H}), w\rangle$, there exists a *line bundle* L, i.e. an indecomposable object from $\mathcal{H}$ of rank one ([36] lemma 6).

6. Each vector bundle E has a filtration $E_0 \subset E_1 \subset \cdots \subset E_r = E$ whose factors E_i/E_{i-1} are line bundles and r equals the rank of E.

7. Because of *Serre duality* $\mathrm{D}\mathrm{Ext}^1(X, Y) = \mathrm{Hom}(Y, \tau X)$ the category $\mathcal{H}$ has almost-split sequences where the equivalence τ serves as the Auslander-Reiten translation.

8. $\mathcal{H}$ admits a tilting bundle T whose endomorphism algebra Λ is a canonical algebra in the sense of [52].

Conversely, [41] describes how to associate to each canonical algebra Λ a category $\mathcal{H}(\Lambda)$ of coherent sheaves on an exceptional curve $\mathbb{X}$ which parametrizes the "central" separating tubular family for Λ.

The Grothendieck group $\mathrm{K}_0(\mathcal{H})$ equipped with the Euler form is determined, via a process described in [35], by an invariant of $\mathbb{X}$, called the symbol $\sigma_{\mathbb{X}} = \left(\begin{array}{c|c} p_1,\ldots,p_t & \\ d_1,\ldots,d_t & \epsilon \\ f_1,\ldots,f_t & \end{array}\right)$ of $\mathbb{X}$. The symbol collects the following data:

1. The *weights* $p_i = p(x_i)$ of the finite collection $x_1, \ldots, x_t$ of exceptional points x of $\mathbb{X}$ with $p(x) > 1$.

2. $\mathcal{H}$ has a line bundle L, called *special*, such that for each $i = 1, \ldots, t$, there is up to isomorphism exactly one simple sheaf S_i concentrated at x_i with the property $\mathrm{Hom}(L, S_i) \neq 0$. Moreover, if S is the direct sum of a τ-orbit of simple sheaves, the middle term of the couniversal extension with L

$$0 \to L^d \to E \to S \to 0, \quad d = \dim_{\mathrm{End}(L)}\mathrm{Ext}^1(S, L)$$

has the form $E = \bar{L}^r$, where $\bar{L}$ is an indecomposable bundle, whose isomorphism class does not depend on the choice of S. We put $\epsilon = \left[\dim_{\mathrm{End}(L)}\mathrm{Hom}(L, \bar{L})\right]^{1/2}$ where $\epsilon > 0$. It follows that ϵ equals 1 or 2, see [35].

3. We put $f_i = \dim_{\mathrm{End}(L)}\mathrm{Hom}(L, S_i)$, $e_i = \dim_{\mathrm{End}(S_i)}\mathrm{Hom}(L, S_i)$ and $d_i = e_i\, f_i$.

From the symbol data one derives a Riemann-Roch formula [35] which determines the *genus* of $\mathbb{X}$ as

$$g_{\mathbb{X}} = 1 + \frac{\epsilon\, p}{2}\left[\sum_{i=1}^{t} d_i\left(1 - \frac{1}{p_i}\right) - \frac{2}{\epsilon}\right].$$

Here p denotes the least common multiple of the weights $p_1, \ldots, p_t$. The genus decides on the representation type:

1. If $g_{\mathbb{X}} < 1$, then $\mathcal{H}$ has a tilting bundle T whose endomorphism algebra is a tame hereditary algebra Σ. See Sections 3.1 and 7.1.
2. If $g_{\mathbb{X}} = 1$, then $\mathcal{H}$ has a tilting bundle T whose endomorphism algebra is a *tubular* canonical algebra Λ. If k is algebraically closed, the indecomposable objects in $\mathcal{H}$ (resp. $\mathrm{mod}(\Lambda)$) are completely classified in [38] (resp. [51]). Moreover, both cases are related since the bounded derived categories of $\mathcal{H}$ and $\mathrm{mod}(\Lambda)$ are equivalent. See also Section 7.6.
3. If $g_{\mathbb{X}} > 1$, then the classification problem for $\mathcal{H}$ is *wild*. We refer to [40] and Section 7.2 for further information.

3 The homogeneous case

The sheaf theory of an exceptional curves which is homogeneous is strongly related to, and in fact completely controlled by the representation theory of a tame hereditary algebra of bimodule type whose category of regular modules has the same property of homogenuity. We thus start to discuss the representation theory of tame hereditary (bimodule) algebras.

3.1 Tame hereditary algebras

Let $\Lambda = \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ be the finite dimensional hereditary k-algebra of a *tame bimodule* ${}_F M_G$, i.e. we request that F, G are finite dimensional skew field extensions of k, and that ${}_F M_G$ is a bimodule such that k acts centrally on F, G, M and $\dim_F M \cdot \dim_G M = 4$ [14]. The pair $(\dim_F M, \dim M_G)$ is called the dimension type of M; tameness therefore is characterized by the dimension types $(2,2)$, $(1,4)$ or $(4,1)$.

If the base field k is algebraically closed, the only choice for M is the bimodule $M = {}_k k \oplus k_k$, and Λ is then called *Kronecker algebra.* The classification of indecomposable modules over this algebra amounts to the classification of pairs of linear maps, a problem solved by Kronecker in 1890. For the base field of real numbers the dimension types (1,4) and (4,1) can also be realized, for instance by the bimodules of real quaternions ${}_{\mathbb{H}}\mathbb{H}_{\mathbb{R}}$ and ${}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$. The classification of indecomposable modules in these two cases basically amounts to classify real subspaces of a quaternion vector space [15].

The tame bimodule algebras are special cases of connected finite dimensional tame hereditary k-algebras. In this paper we use *tame* in the sense "tame and representation-infinite", where tameness refers to the possibility to classify all indecomposable modules in any given dimension by a finite number of one-parameter families. For a finite dimensional hereditary algebra Λ it is equivalent that the quadratic form $q(x) = \langle x, x\rangle$ associated with the Euler form is positive semi-definite. An indecomposable Λ-module is called *preprojective* (resp. *preinjective*) if for some $n \geq 0$ it has the form $\tau^{-n}P$ (resp. $\tau^n Q$) for an indecomposable projective module P (resp. an indecomposable injective module Q). The remaining indecomposable modules are called *regular.*

Theorem 3.1 (Dlab-Ringel [14]) *Let Λ be a tame bimodule algebra, more generally a connected tame hereditary algebra. The indecomposable Λ-modules fall into the three classes of preprojective, regular, resp. preinjective modules $\mathcal{P}$, $\mathcal{R}$, resp. $\mathcal{I}$, where $\mathcal{R}$ decomposes into a coproduct $\coprod_{x\in\mathbb{X}}\mathcal{U}_x$ of connected uniserial categories. Moreover,* $\mathrm{mod}(\Lambda) = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}$. □

Here and later we use the notation $\vee$ to denote the closure of the union of $\mathcal{P}$, $\mathcal{R}$, $\mathcal{I}$ with respect to finite direct sums, but also to indicate that in the sequence $\mathcal{P}$, $\mathcal{R}$, $\mathcal{I}$ non-zero morphisms only exist from left to right. Hence there are no non-zero morphisms from objects from $\mathcal{R}$ to $\mathcal{P}$, from $\mathcal{I}$ to $\mathcal{P}$ or from $\mathcal{I}$ to $\mathcal{R}$.

Intuitively speaking, $\mathcal{P}$, $\mathcal{I}$ are discrete families, while $\mathcal{R}$ is a continuous family, thought to be parametrized by the index set of the decomposition $\coprod_{x\in\mathbb{X}}\mathcal{U}_x$. A major aim, in this context, is to understand the geometric structure of the parametrizing space $\mathbb{X}$. This amounts to specify a natural geometric structure on the space of regular Auslander-Reiten components for Λ.

3.2 The preprojective algebra

Let Λ be a connected hereditary algebra. We are interested in the preprojective algebras because — at least in the tame case — they yield projectivity of the parameter space $\mathbb{X}$ for the "continuous family" of regular components, conforming to the conventions of Section 2.3. If Λ is tame, the elements x with $\langle x, -\rangle = 0$ form a rank one direct factor $\mathbb{Z}.w$, the *radical* of $\mathrm{K}_0(\Lambda)$. Suitably normalized by a rational factor, the linear form $\langle -, w\rangle$ yields a surjective mapping $\mathrm{rk} : K_0(\Lambda) \to \mathbb{Z}$ called *rank,* which is strictly positive on indecomposable projectives. Moreover, there exists at least one indecomposable projective module of rank one.

The preprojective algebras are a special case of the general construction of the positively $\mathbb{Z}$-graded *orbit algebra* $\mathbb{A}(F; A) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}(A, F^n A)$ associated with a pair consisting of a k-linear endofunctor $F : \mathcal{A} \to \mathcal{A}$ of a small k-category $\mathcal{A}$ and an object A from $\mathcal{A}$ [8, 34]. The product of two homogeneous elements $u : A \to F^n A$ of degree n and $v : A \to F^m A$ of degree m in the orbit algebra is given by the composition $vu = [A \xrightarrow{u} F^n A \xrightarrow{F^m v} F^{n+m} A]$.

Definition 3.2 Let Λ be a connected hereditary algebra and let P denote either the Λ-module Λ or alternatively a projective Λ-module of rank 1 (the latter case requests that Λ is tame). Then the positively $\mathbb{Z}$-graded algebra associated with the Auslander-Reiten translation τ^-

$$\Pi(\Lambda, P) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(P, \tau^{-n} P)$$

is called a *preprojective algebra* for Λ. We write $\Pi(\Lambda)$ for $\Pi(\Lambda, \Lambda)$.

The preprojective algebra $\Pi(\Lambda)$ was introduced by Gelfand and Panomarev [23] for quivers using a combinatorial definition. Dlab and Ringel [16] extended the notion to the hereditary algebras given as tensor algebras of a species. Our definition as a graded algebra associated with the Auslander-Reiten translation τ^- is taken from [8] and the relationship between the various definitions is studied in [53]. Invoking [11] the following result follows from [8].

Theorem 3.3 *Let Λ be a connected* tame hereditary *algebra, and P be either Λ or an indecomposable projective Λ-module of rank one. Then the following assertions hold:*

1. *The preprojective algebra $\Pi = \Pi(\Lambda, P)$ is two-sided noetherian and has Krull dimension two, and for $P = \Lambda$ also global dimension two. Moreover, $\Pi(\Lambda)$ is module-finite over its center which is an affine k-algebra.*
2. *The quotient category $\mathrm{mod}^{\mathbb{Z}}(\Pi)/\mathrm{mod}_0^{\mathbb{Z}}(\Pi)$ is, in each of the two cases, equivalent to the category $\mathcal{H}(\Lambda) = \mathcal{I}[-1] \vee \mathcal{P} \vee \mathcal{R}$, where $\mathcal{P}$, $\mathcal{R}$, $\mathcal{I}$ are the categories of preprojective, regular and preinjective Λ-modules, respectively.*
3. *$\mathcal{H}(\Lambda)$ is a connected abelian k-category which is hereditary noetherian with Serre duality $\mathrm{DExt}^1(X, Y) = \mathrm{Hom}(Y, \tau X)$. $\mathcal{H}(\Lambda)$ has a tilting object T with $\Lambda \cong \mathrm{End}(T)$.* □

As the notation indicates, $\mathcal{H}(\Lambda)$ is formed in the derived category of $\mathrm{mod}(\Lambda)$. The theorem implies that $\mathcal{H}$ has the interpretation of a category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on an exceptional curve $\mathbb{X}$. If Λ is the Kronecker algebra over k, then $\mathbb{X}$ equals the projective line $\mathbb{P}_1(k)$, so is a commutative curve.

We illustrate the two kinds of preprojective algebras for the case where Λ is the Kronecker algebra. The preprojective component has the shape

$$\begin{array}{ccccccccccccc} & & P_1 & & & & P_3 & & & & P_5 & & \cdots \\ & \nearrow\!\nearrow & & \searrow\!\searrow & & \nearrow\!\nearrow & & \searrow\!\searrow & & \nearrow\!\nearrow & & \searrow\!\searrow & & \nearrow\!\nearrow \\ P_0 & & & & P_2 & & & & P_4 & & & & P_6 & \end{array}$$

where $\tau^- P_n = P_{n+2}$ and, denoting the arrows from P_n to P_{n+1} consistently by X and Y, the category of indecomposable preprojective Λ-modules is generated by X and Y with relations $XY = YX$, whenever this composition makes sense. Hence $\mathrm{Hom}(P_0, P_n)$ equals R_n the $(n+1)$-dimensional space of homogeneous polynomials in X, Y of total degree n, and therefore we get isomorphisms

$$\Pi(\Lambda) \cong \bigoplus_{n=0}^{\infty} \begin{pmatrix} R_{2n} & R_{2n-1} \\ R_{2n+1} & R_{2n} \end{pmatrix} \text{ and } \Pi(\Lambda, P) \cong \bigoplus_{n=0}^{\infty} R_{2n} = k[X^2, XY, Y^2]$$

as $\mathbb{Z}$-graded algebras.

Remark 3.4 We assume that k is algebraically closed. Then the isomorphism class of $\Pi(\Lambda, P)$ does not depend on the choice of the projective rank one module P. This follows from the fact that the associated curve in this case is a weighted projective line, and therefore the automorphism group of $\mathcal{H}$ acts transitively on the isomorphism classes of line bundles [21].

Moreover, the small preprojective algebra $\Pi' = \Pi(\Lambda, P)$, where P is of rank one, in this case is a *commutative* algebra which is affine over k (see subsection 7.1) and is always of infinite global dimension, in contrast to the fact that the full preprojective algebra $\Pi = \Pi(\Lambda, \Lambda)$ always has global dimension two. The equivalences

$$\mathrm{coh}(\mathbb{X}) \cong \frac{\mathrm{mod}^{\mathbb{Z}}(\Pi)}{\mathrm{mod}_0^{\mathbb{Z}}(\Pi)} \cong \frac{\mathrm{mod}^{\mathbb{Z}}(\Pi')}{\mathrm{mod}_0^{\mathbb{Z}}(\Pi')}$$

can be interpreted in geometric terms: alternatively $\mathbb{X}$ can be obtained from a *singular commutative* surface $\mathbb{F}'$ with $\mathrm{coh}(\mathbb{F}') = \mathrm{mod}(\Pi')$ or from a *nonsingular noncommutative* surface $\mathbb{F}$ with $\mathrm{coh}(\mathbb{F}) = \mathrm{mod}(\Pi)$ as the quotient of a k^*-action.

3.3 Homogeneous exceptional curves

Let $\mathbb{X}$ be any noncommutative curve which is exceptional (in particular projective over k and smooth). Accordingly $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ is a connected small abelian category with Serre duality which is noetherian, hereditary and has an exceptional sequence. Let $\mathcal{H}_0$ denote the full subcategory of $\mathcal{H}$ consisting of the objects of finite length. Following the terminology from representation theory we say that $\mathbb{X}$ is *homogeneous* if, in the decomposition of $\mathcal{H}_0 = \coprod_{x \in \mathbb{X}} \mathcal{U}_x$ into connected uniserial subcategories, each $\mathcal{U}_x$ has only one simple object. It is equivalent to say that $\mathrm{Ext}^1(S_1, S_2) = 0$ for each pair of nonisomorphic simple objects. The following result characterizes the exceptional homogeneous curves.

Theorem 3.5 *Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ for an exceptional curve $\mathbb{X}$ and assume that $\mathrm{Ext}^1(S_1, S_2) = 0$ for each pair of nonisomorphic simple objects. Then there exists a tame bimodule algebra Λ such that $\mathcal{H} \cong \mathcal{H}(\Lambda)$ in the notations of Theorem 3.3.*

Proof. By assumption $\mathcal{H}_0$ decomposes as a coproduct $\coprod_{x \in \mathbb{X}} \mathcal{U}_x$ of connected uniserial categories, where each $\mathcal{U}_x$ has exactly one simple object S_x. The proof involves several steps, where we use the notations of Section 2.5.

1. Since $\mathbb{X}$ is homogeneous, the class of each simple is a multiple of w, hence $\mathrm{K}_0(\mathcal{H}) = \mathbb{Z}.[L] \oplus \mathbb{Z}.w$.

2. $\mathbb{X}$ has a rational point x, i.e. $[S_x] = w$. Moreover, the couniversal extension $0 \to L^\epsilon \to \bar{L} \to S_x \to 0$, $\epsilon = \dim_{\mathrm{End}(L)}\mathrm{Ext}^1(S_x, L)$, of S_x by L has an indecomposable middle term $\bar{L}$, the companion bundle of L.

3. $T = L \oplus \bar{L}$ is a tilting object in $\mathcal{H}$, $F = \mathrm{End}(L)$, $G = \mathrm{End}(\bar{L})$ are division rings and the (F, G)-bimodule $M = \mathrm{Hom}(L, \bar{L})$ satisfies $\dim_F M \cdot \dim M_G = 4$:

   With respect to the $\mathbb{Z}$-basis $p_1 = [L]$, $p_2 = [\bar{L}]$ of $\mathrm{K}_0(\mathcal{H})$, the Euler form is given by the matrix $C = \begin{pmatrix} \langle p_1, p_1\rangle & \langle p_1, p_2\rangle \\ 0 & \langle p_2, p_2\rangle \end{pmatrix}$. Correspondingly, the Coxeter transformation is given by the matrix $\Phi = -C^{-1}C^{tr}$ which has w as a fixed point, hence admits 1 as an eigenvalue. This in turn implies that $4\langle p_1, p_1\rangle\langle p_2, p_2\rangle = \langle p_1, p_2\rangle^2$, and therefore $\dim_{F_1} M \cdot \dim M_{F_2} = 4$.

4. Since further $\mathrm{Hom}(\bar{L}, L) = 0$, the endomorphism algebra of T is isomorphic to a tame hereditary bimodule algebra Λ. Since Λ equals the endomorphism ring of a tilting object in $\mathcal{H}$ and in $\mathcal{H}(\Lambda)$, we get an equivalence of derived categories $\mathrm{D}^b(\mathcal{H}) \to \mathrm{D}^b(\mathcal{H}(\Lambda))$, preserving the rank, accordingly an equivalence $\mathcal{H} \cong \mathcal{H}(\Lambda)$, cf. [36] for details. □

We refer to [13] for the investigation of related commutative curves.

4 Modification of the weight type

Let $\mathbb{X}$ be a smooth noncommutative projective curve, in particular $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ is a hereditary noetherian category with Serre duality. We have already seen, that the genus and thus the complexity of the classification problem for the coherent sheaves on $\mathbb{X}$ largely depends on the weight function of $\mathbb{X}$ attaching to each point of $\mathbb{X}$ the number $p(x)$ of simple sheaves concentrated in x. It is therefore important to dispose of tools changing the weight type of a given $\mathbb{X}$. *Reduction of the weight type* is easily achieved through perpendicular calculus [22]: Let S be a simple sheaf which is concentrated in a point x of weight $p(x) > 1$, then $\mathrm{Ext}^1(S, S) = 0$, hence S is exceptional, and the full subcategory $S^\perp$ of $\mathcal{H}$ consisting of all objects of X satisfying $\mathrm{Hom}(S, X) = 0 = \mathrm{Ext}^1(S, X)$ is again a category of coherent sheaves over an exceptional curve $\mathbb{X}'$ having the same underlying point set as $\mathbb{X}$ but reduced weight. In more detail x has weight $p(x) - 1$ for $\mathbb{X}'$ while the weight of a point $y \neq x$ remains unchanged under the operation. The process can be iterated as long as there still exist exceptional sheaves in the same point x or in a different point, until finally a homogeneous exceptional curve is obtained, where the process of reduction of weights will stop.

For the converse construction which we call *insertion of weights* we need some preparation. First we encode the notion of a point $x \in \mathbb{X}$ in terms of a natural transformation of functors.

Recall that $\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$. Each point $x \in \mathbb{X}$ determines, by means of a mutation with respect to the simple objects from $\mathcal{U}_x$, a shift functor $\sigma_x : \mathcal{H} \to \mathcal{H}$, $E \mapsto E(x)$, together with a natural transformation $x : \mathrm{Id} \to \sigma_x$, also denoted by the symbol x (see [41, 38, 43]). For each bundle E, more generally for each sheaf E with $\mathrm{Hom}(\mathcal{U}_x, E) = 0$, these data are given by the $\mathcal{S}_x$-universal extension

$$0 \to E \stackrel{x_E}{\to} E(x) \to E_x \to 0,$$

where $\mathcal{S}_x$ is the semisimple category consisting of all finite direct sums of simple sheaves concentrated in x and E_x belongs to $\mathcal{S}_x$. If E is an indecomposable torsion object, σ_x acts as follows: If E is concentrated in a point y different from x then we have the $\mathcal{S}_x$-universal extension as above, hence $E(x) = E$ and $x_E = 1_E$. If y equals x, then $E(x) = \tau^- E$, and the kernel of x_E equals the simple socle of E.

4.1 The category of p-cycles

Let $\mathcal{H}$ be a category of coherent sheaves on a (noncommutative) exceptional curve as before. Fixing a point x of $\mathbb{X}$ and an integer $p \geq 1$ we are going to form the category $\bar{\mathcal{H}}$ of p-cycles in x which may be viewed as a category of coherent sheaves on a curve $\bar{\mathbb{X}}$, having the same underlying point set as $\mathbb{X}$, where the weights for points $y \neq x$ remains unchanged and the weight of x in $\bar{\mathbb{X}}$ equals p times the weight of x in $\mathbb{X}$. Intuitively speaking the effect of the following construction is to form a p-th root of the natural transformation $x_E : E \to E(x)$ corresponding to the point x, and thus relates algebraically to the construction of the Riemann surface of the p-th root function.

Definition 4.1 A *p-cycle E concentrated in x* is a diagram

$$\cdots \to E_n \xrightarrow{x_n} E_{n+1} \xrightarrow{x_{n+1}} E_{n+2} \xrightarrow{x_{n+2}} \cdots \longrightarrow E_{n+p} \xrightarrow{x_{n+p}} \cdots$$

which is *p-periodic* in the sense that $E_{n+p} = E_n(x)$, $x_{n+p} = x_n(x)$ and moreover $x_{n+p-1} \circ \cdots \circ x_n = x_{E_n}$ holds for each integer n.

A morphism $u : E \to F$ of p-cycles concentrated in the same point x is a sequence of morphisms $u_n : E_n \to F_n$ which is *p-periodic*, i.e. $u_{n+p} = u_n$ for each n, and such that each diagram

$$\begin{array}{ccc} E_n & \xrightarrow{x_n} & E_{n+1} \\ u_n \downarrow & & \downarrow u_{n+1} \\ F_n & \xrightarrow{x_n} & F_{n+1} \end{array}$$

commutes. We denote p-cycles in the form $E_0 \xrightarrow{x_0} E_1 \xrightarrow{x_1} \cdots \to E_{p-1} \xrightarrow{x_{p-1}} E_0(x)$ and the category of all p-cycles concentrated in x by $\overline{\mathcal{H}} = \mathcal{H}\begin{pmatrix} p \\ x \end{pmatrix}$.

Obviously $\overline{\mathcal{H}}$ is an abelian category, where exactness and formation of kernels and cokernels has a pointwise interpretation. Moreover we have a full exact embedding

$$j : \mathcal{H} \hookrightarrow \overline{\mathcal{H}}, \qquad E \mapsto \bar{E} = E = \cdots = E \xrightarrow{x_E} E(x).$$

We will therefore identify $\mathcal{H}$ with the resulting exact subcategory of $\overline{\mathcal{H}}$. We note that inclusion $j : \mathcal{H} \to \overline{\mathcal{H}}$ has a left adjoint ℓ and a right adjoint r which are both exact functors and are given by

$$\ell\left(E_0 \xrightarrow{x_0} E_1 \xrightarrow{x_1} \cdots \to E_{p-1} \xrightarrow{x_{p-1}} E_0(x)\right) = E_{p-1}$$

and

$$r\left(E_0 \xrightarrow{x_0} E_1 \xrightarrow{x_1} \cdots \to E_{p-1} \xrightarrow{x_{p-1}} E_0(x)\right) = E_0.$$

Lemma 4.2 *The category $\overline{\mathcal{H}}$ is connected, abelian and noetherian. The simple objects of $\overline{\mathcal{H}}$ occur in two types:*

1. *the simple objects of $\mathcal{H}$ which are concentrated in a point y different from x.*
2. *for each simple object S from $\mathcal{H}$, concentrated in x, the p simples*

$$\begin{array}{rc} S_1 : & 0 \to 0 \to \cdots \to 0 \to S \to 0 \\ S_2 : & 0 \to 0 \to \cdots \to S \to 0 \to 0 \\ \cdots & \cdots \\ S_{p-1} : & 0 \to S \to \cdots \to 0 \to 0 \to 0 \\ S_p : & S \to 0 \to \cdots \to 0 \to 0 \to S(x). \end{array}$$

Each S_i is exceptional and $\mathrm{End}(S_i) \cong \mathrm{End}(S) \cong \mathrm{Ext}^1(S_i, S_{i+1})$ *where the indices are taken modulo p.*

If $\mathcal{S} = \{S_1, \ldots, S_{p-1} | S$ simple in $\mathcal{H}$ and concentrated in $x\}$, then the extension closure $\langle \mathcal{S} \rangle$ of $\mathcal{S}$ is localizing in $\overline{\mathcal{H}}$. Moreover

a. *the quotient category $\overline{\mathcal{H}}/\langle \mathcal{S} \rangle \cong \mathcal{H}$ is isomorphic to $\mathcal{H}$, the isomorphism induced by $r : \overline{\mathcal{H}} \to \mathcal{H}$.*

b. *the right perpendicular category $\mathcal{S}^{\perp}$ formed in $\overline{\mathcal{H}}$ is equivalent to $\mathcal{H}$.*

Proof. Abelianness and noetherianness are obvious by pointwise consideration. It is straightforward from the construction and from the properties of $\mathcal{H}$ that the category $\mathcal{S}^{\perp}$ right perpendicular to $\mathcal{S}$ consists of exactly those p-cycles $E_0 \stackrel{x_0}{\to} E_1 \stackrel{x_1}{\to} \cdots \to E_{p-1} \stackrel{x_{p-1}}{\to} E_0(x)$ such that $x_0, \ldots, x_{p-2}$ are isomorphisms, hence — up to isomorphism — agree with the objects from $\mathcal{H}$. The remaining properties are straightforward to check. $\square$

We note that $\overline{\mathcal{H}}$ is again equipped with a natural shift automorphism $\overline{\sigma}_x : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$, sending a p-cycle $E_0 \stackrel{x_0}{\to} E_1 \stackrel{x_1}{\to} \cdots \to E_{p-1} \stackrel{x_{p-1}}{\to} E_0(x)$ to $E_1 \stackrel{x_1}{\to} E_2 \stackrel{x_2}{\to} \cdots \to E_p \stackrel{x_p}{\to} E_0(x) \stackrel{x_0}{\to} E_1(x)$. Note that $\overline{\sigma}_x(S_i) = S_{i+1}$, where the index i is taken modulo p. It is moreover straightforward to define a natural transformation $\mathrm{Id} \to \sigma$ in terms of the "cycle morphisms" $x_0, \ldots, x_{p-1}$.

Theorem 4.3 *The category $\overline{\mathcal{H}}$ shares with $\mathcal{H}$ the properties to be equivalent to a category of coherent sheaves on a smooth projective curve. Moreover, if $\mathcal{H}$ has a tilting object the same holds for $\overline{\mathcal{H}}$.*

Proof. We have already seen that $\overline{\mathcal{H}}$ is an abelian category which is noetherian; it is less obvious that $\overline{\mathcal{H}}$ is also hereditary:

Pass from $\mathcal{H}$ to the Grothendieck category $\vec{\mathcal{H}}$ of left exact functors from $\mathcal{H}^{\mathrm{op}}$ to abelian groups, such that $\mathcal{H}$ becomes the full subcategory of finitely presented objects of $\vec{\mathcal{H}}$. In a straightforward way extend σ_x and the corresponding notion of p-cycles in x from $\mathcal{H}$ to $\vec{\mathcal{H}}$. Invoking Zorn's lemma it is easily checked that a p-cycle E is an injective object if and only if all the E_i are injective objects in $\vec{\mathcal{H}}$ and moreover all the $x_i : E_i \to E$ are epimorphisms. This property is obviously preserved when passing to quotients. Hence the category of p-cycles in $\vec{\mathcal{H}}$, and therefore also $\overline{\mathcal{H}}$ is hereditary.

Note that inclusion $\mathcal{H} \hookrightarrow \overline{\mathcal{H}}$ has a left adjoint $\ell : \overline{\mathcal{H}} \to \mathcal{H}$, given with the previous notations by $\ell(E) = \overline{E}$. If E is projective in $\overline{\mathcal{H}}$ then $\ell(E)$ is projective in $\mathcal{H}$, so is zero, hence E belongs to $\langle \mathcal{S} \rangle$. This implies $E = 0$, and so $\overline{\mathcal{H}}$ has no non-zero projective objects.

Finally, the assumptions imply by a variation of the arguments from [36, 41] the existence of a tilting bundle T for $\mathcal{H}$. For each simple sheaf in $\mathcal{H}$ concentrated in x we form a filtration $0 = F_0(S) \subset F_1(S) \subset \cdots \subset F_p(S) = S$ such that $F_p(S)/F_{p-1}(S) \cong S_i$. Let $T(S) = F_1(S) \oplus \cdots \oplus F_{p-1}(S)$, then the direct sum of T and all $T(S)$, with S simple concentrated at x, is a tilting object in $\overline{\mathcal{H}}$. $\square$

Note that the formation of categories of p-cycles may be iterated and thus the process of insertion of weights may involve any finite set of points of the original curve. We illustrate what is going to happen by an example:

Assuming that k is algebraically closed, the preceding theorem together with the characterization of categories of weighted projective lines from [36] implies the following result. Let $\mathbb{X}$ be the usual projective line over k, and let $\lambda_1, \ldots, \lambda_t$ be a family of pairwise distinct points from $\mathbb{X}$ and $p_1, \ldots, p_t$ be a sequence of integers ≥ 1. Let $\mathbb{X}_0 = \mathbb{X}$ and let inductively denote $\mathbb{X}_i$ be the exceptional curve obtained from $\mathbb{X}_{i-1}$ by inserting weight p_i in λ_i, i.e. forming the category of p_i-cycles in $\mathrm{coh}\,(\mathbb{X}_{i-1})$ which are concentrated in λ_i. Then $\mathbb{X}_t$ is isomorphic to the weighted projective line corresponding to the above weight data $p_1, \ldots, p_t$ and parameter data $\lambda_1, \ldots, \lambda_t$.

4.2 Parabolic structure

The base field may again be arbitrary. As in the preceding subsection, we start with a category $\mathcal{H}$ of coherent sheaves on an exceptional curve $\mathbb{X}$. In the notation of the previous section a torsion sheaf in $\overline{\mathcal{H}}$ is exactly a p-cycle $\bar{E} = E_0 \xrightarrow{x_0} E_1 \to \cdots \to E_{p-1} \xrightarrow{x_{p-1}} E_0(x)$ where each E_i is a torsion sheaf. Moreover $\bar{E}$ is a bundle in $\bar{\mathcal{H}}$ if and only if each E_i is a bundle in $\mathcal{H}$, and then each x_i is a monomorphism since each $x_{E_i} : E_i \to E_i(x)$ is a monomorphism. We denote by $\mathcal{S}_x$ the semi-simple category consisting of all finite direct sums of simple sheaves in $\mathcal{H}$ concentrated at x.

In the bundle case we may hence interpret the p-cycle $\bar{E}$ as a filtration

$$E_0 \subseteq E_1 \cdots \subseteq E_{p-1} \subseteq E_0(x),$$

equivalently as a filtration

$$0 = E_0/E_0 \subseteq E_1/E_0 \subseteq \cdots \subseteq E_{p-1}/E_0 \subseteq E_0(x)/E_0$$

of the *fibre* $E_x = E_0(x)/E_0$ of E at x given by the $\mathcal{S}_x$-universal extension

$$0 \to E_0 \xrightarrow{x_{E_0}} E_0(x) \to E_x \to 0.$$

The fibre E_x is a member of the semisimple category $\mathcal{S}_x$ so may be viewed as an r-tuple of finite dimensional vector spaces over a finite skew field extension D of k. Here, D is isomorphic to the endomorphism algebra of any simple sheaf from $\mathcal{H}$ concentrated in x and r denotes the number of such sheaves.

Theorem 4.4 *For any exceptional curve $\mathbb{X}$ and any point x of $\mathbb{X}$ the two concepts of p-cycles of vector bundles, where the p-cycle is concentrated in x, and of a quasi-parabolic structure of filtration length p at x (see [58] for the definition) agree.*

PROOF. Straightforward from the above discussion. □

In comparison, the concept of p-cycles seems to be advantageous because it applies to torsion sheaves as well, and thus always yields an abelian category.

5 Characterization of exceptional curves

In [41] we have associated with each canonical algebra Λ (in the sense of [52]) a connected abelian k-category $\mathcal{H}(\Lambda)$ defining an exceptional curve $\mathbb{X}$ that parametrizes the "central" separating tubular family of $\mathrm{mod}(\Lambda)$.

Theorem 5.1 *Let k be a field. For an abelian k-category $\mathcal{H}$ the following assertions are equivalent:*

a. $\mathcal{H}$ is equivalent to a category of coherent sheaves on an exceptional curve.

b. $\mathcal{H}$ has the form $\mathcal{H}(\Lambda)$ for a canonical algebra Λ.

c. $\mathcal{H}$ is equivalent to the category of coherent sheaves on a curve $\mathbb{X}$, arising from a homogeneous exceptional curve $\mathbb{Y}$ by insertion of weights.

PROOF. "$c \Rightarrow a$" see Theorem 4.3. "$a \Rightarrow b$" see Section 2.5. "$b \Rightarrow c$" Let $\mathcal{S}$ be a system of exceptional sheaves collecting for each $x \in \mathbb{X}$ all simple sheaves concentrated in x except one. The right perpendicular category $\mathcal{H}' = \mathcal{S}^{\perp}$ then is easily seen to be equivalent to a category of coherent sheaves on a homogeneous curve $\mathbb{Y}$. We fix a tilting bundle for $\mathbb{Y}$, and extend it by the argument from the proof of Theorem 4.3 to a tilting object T on $\mathcal{H}$ and by the same argument realize T as a tilting sheaf on the curve $\overline{\mathbb{Y}}$ arising from $\mathbb{Y}$ by a suitable insertion of weights. There results a rank preserving equivalence of the derived categories $\mathrm{D}^b(\mathrm{coh}(\mathbb{X}))$ and $\mathrm{D}^b(\mathrm{coh}(\overline{\mathbb{Y}}))$ inducing an equivalence between $\mathrm{coh}(\mathbb{X})$ and $\mathrm{coh}(\overline{\mathbb{Y}})$. □

6 Graded factorial domains of dimension two

Let k be an algebraically closed field. For any sequence $p = (p_1, \dots, p_t)$ we consider the rank one abelian group $\mathbb{L}(p)$ on generators $\vec{x}_1, \dots, \vec{x}_t$ with relations $p_1\vec{x}_1 = \cdots = p_t\vec{x}_t$. For each sequence $\lambda = (\lambda_3, \dots, \lambda_t)$ of pairwise distinct non-zero elements of k the algebra

$$S(p, \lambda) = k[X_1, \dots, X_t]/I,$$

where the ideal I is generated by the regular sequence

$$X_i^{p_i} - (X_2^{p_2} - \lambda_i X_1^{p_1}),\ i = 3, \dots, t,$$

is $\mathbb{L}(p)$-graded by giving the class x_i of X_i degree $\vec{x}_i$. It is easily checked that this algebra is graded factorial [21].

Theorem 6.1 (Kussin [32], Mori [45]) *Let k be an algebraically closed field. An affine k-algebra S of Krull dimension two, positively graded by an ordered rank one abelian group H is graded-factorial if and only if, as a graded algebra, it is isomorphic to the $\mathbb{L}(p)$-graded algebra $S(p, \lambda)$ for some choice of positive integers $p_1, \dots, p_t$ and pairwise distinct non-zero scalars $\lambda_3, \dots, \lambda_t$ from k.* □

The result, in its full generality, is due to Kussin [32], who treated the group-graded case. The particular case of a positive $\mathbb{Z}$-grading is known for some time and due to Mori [45]. Note for this that $\mathbb{L}(p)$ is torsion-free, i.e. $\mathbb{L}(p) \cong \mathbb{Z}$, if and only if $p_1, \dots, p_t$ are pairwise coprime.

Combined with the characterization of the exceptional curves over an algebraically closed field as the weighted projective lines [36] this yields:

Proposition 6.2 ([36]) *Let k be an algebraically closed field. For a small abelian connected k-category $\mathcal{H}$ the following assertions are equivalent:*

a. $\mathcal{H}$ is equivalent to the category of coherent sheaves on an exceptional curve.

b. $\mathcal{H}$ has the form $\mathrm{mod}^H(S)/\mathrm{mod}_0^H(S)$ for a graded factorial affine k-algebra S of dimension two, graded by a rank one abelian group H.

Proof. In view of [36] a category of type a is equivalent to a category of coherent sheaves on a weighted projective line. In view of the preceding theorem these are exactly those of type b. □

In geometric terms, graded factoriality of S is (under mild restrictions for S) equivalent to the fact that all line bundles on the associated projective space arise from the structure sheaf by a grading shift.

7 Examples and Applications

It has some tradition to invoke linear algebra methods in the classification of vector bundles for suitable projective varieties. A beautiful result due to Beilinson [9] states that the derived categories of coherent sheaves on the projective n-space and of finite dimensional representations of a certain finite dimensional algebra are equivalent as triangulated categories. This method underlies most applications in this section, even where derived categories are not explicitly mentioned.

7.1 Rational surface singularities

Example 7.1 ([22]) *Let k be an algebraically closed field, and let $\mathbb{X}$ be a weighted projective line of weight type $(2,3,5)$. Then there exists a tilting sheaf T on $\mathbb{X}$ such that $\Lambda = \mathrm{End}(T)$ is the path algebra over k of the extended Dynkin quiver of type $\widetilde{\mathbb{E}}_8$*

$$\begin{array}{ccccccccccccccc} & & & & \circ & & & & & & & & & & \\ & & & & \downarrow & & & & & & & & & & \\ \circ & \rightarrow & \circ & \rightarrow & \bullet & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ & \leftarrow & \circ \end{array}$$

Moreover, if L is a line bundle on $\mathbb{X}$ (resp. a projective Λ-module of rank one) and τ denotes the Auslander-Reiten translation for $\mathrm{coh}(\mathbb{X})$ *(resp. for* $\mathrm{mod}(\Lambda)$*), then there is an isomorphism of graded algebras*

$$\bigoplus_{n\geq 0} \mathrm{Hom}(L, \tau^{-n}L) \cong k[X,Y,Z]/(X^2+Y^3+Z^5)$$

where the grading of the algebra on the right is given by $\deg(x,y,z) = (15,10,6)$.

For the base field of complex numbers the equation $X^2+Y^3+Z^5$ describes the rational surface singularity of Dynkin type $\mathbb{E}_8$.

The correspondence between rational surface singularities and the associated small preprojective algebra, illustrated by the preceding example, is valid in general. The attached table is taken from [22].

Theorem 7.2 ([22]) *For any Dynkin diagram $\Delta = (p_1,p_2,p_3)$ let Λ be the path algebra of a quiver of extended Dynkin type $\widetilde{\Delta}$, P a projective Λ-module of rank one, and Π_Δ the preprojective algebra $\Pi(\Lambda,P)$. Then Π_Δ has the form $k[x,y,z] = k[X,Y,Z]/(f_\Delta)$, where the degrees of the generators x,y,z, their degrees, and the relation f_Δ can be seen from the following table:*

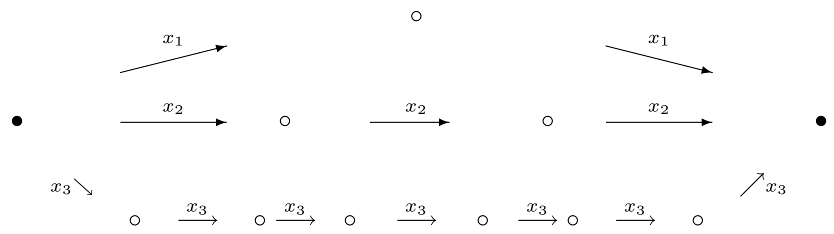
Dynkin type Δ	generators (x,y,z)	$\mathbb{Z}$-degrees	relations f_Δ
(p,q)	$(x_0\,x_1, x_1^{p+q}, x_0^{p+q})$	$(1,p,q)$	$X^{p+q} - Y\,Z$
$(2,2,2l)$	$(x_2^2, x_0^2, x_0\,x_1\,x_2)$	$(2,l,l+1)$	$Z^2 + X(Y^2 + Y\,X^l)$
$(2,2,2l+1)$	$(x_2^2, x_0\,x_1, x_0^2\,x_2)$	$(2,2l+1,2l+2)$	$Z^2 + X(Y^2 + Z\,X^l)$
$(2,3,3)$	$(x_0, x_1\,x_2, x_1^3)$	$(3,4,6)$	$Z^2 + Y^3 + X^2\,Z$
$(2,3,4)$	$(x_1, x_2^2, x_0\,x_2)$	$(4,6,9)$	$Z^2 + Y^3 + X^3\,Y$
$(2,3,5)$	(x_2, x_1, x_0)	$(6,10,15)$	$X^2 + Y^3 + X^5$

□

Lists for characteristic zero, usually give the cases $(2,3,3)$ and $(2,2,n)$ in a different form $X^4+Y^3+Z^2$ resp. $X(Y^2-X^n)+Z^2$, then equivalent to the one given here.

7.2 Algebras of automorphic forms

Example 7.3 ([34]) *Let $\mathbb{X}$ be the weighted projective line over k of weight type $(2,3,7)$. There exists a tilting sheaf T on $\mathbb{X}$ such that $\Lambda = \mathrm{End}(T)$ is the canonical algebra of type $(2,3,7)$ given by the quiver*



with relations $x_1^2 + x_2^3 + x_3^7 = 0$.

If L is a line bundle on $\mathbb{X}$ (resp. an indecomposable rank one module over Λ not lying in the preprojective component), and τ denotes the Auslander-Reiten translation for $\mathrm{coh}(\mathbb{X})$ *(resp. for* $\mathrm{mod}(\Lambda)$*), then we get an isomorphism*

$$\bigoplus_{n\geq 0} \mathrm{Hom}(L, \tau^{n}L) \cong k[X_1,X_2,X_3]/({X_1}^2+{X_2}^3+{X_3}^7)$$

of $\mathbb{Z}$-graded algebras, where the generators $x_1=[X_1]$, $x_2=[X_2]$, $x_3=[X_3]$ for the algebra on the right hand side are homogeneous of degree $21,14,6$, respectively.

For the base field of complex numbers, this algebra is isomorphic to the algebra of entire automorphic forms with respect to an action of the triangle group $G = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2 = \sigma_2^3 = \sigma_3^7 \rangle$ on the upper complex half plane $\mathbb{H}_+$. In fact, this is a special case of a more general setting, treated in [34] in more detail:

It is classical that a Fuchsian group of the first kind, $G = \langle \sigma_1, \ldots, \sigma_t | \sigma_1^{p_1} = \cdots = \sigma_t^{p_t} = 1 = \sigma_1 \cdots \sigma_t \rangle$, satisfying the extra condition $(t-2) - \sum_{i=1}^{t} \frac{1}{p_i} > 0$, can be realized as a discrete group G of automorphisms acting discontinuously on $\mathbb{H}_+$ such that

1. the quotient $\mathbb{H}_+/G$ is isomorphic to the Riemann sphere $\mathbb{P}_1(\mathbb{C})$,
2. only for finitely many orbits $\lambda_1, \ldots, \lambda_t \in \mathbb{P}_1(\mathbb{C})$ the corresponding stabilizer groups are non-trivial and then cyclic of finite order $p_1, \ldots, p_t$, respectively.

The above data $p = (p_1, \ldots, p_t)$ and $\lambda = (\lambda_1, \ldots, \lambda_t)$ define a weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$ and an $\mathbb{L}(p)$-graded factorial algebra $S(p, \lambda)$ as in Section 6. It is easily seen that the character group G^* of G is finite, and moreover that $\mathbb{L}(p)$ can be identified with a subgroup H of $\mathbb{Q} \times G^*$. Then for each pair (a, χ) the $\mathbb{C}$-space $A_{a,\chi}$ of χ-automorphic forms of degree a is finite-dimensional and multiplication of automorphic forms then defines a graded algebra $A = \bigoplus_{(a,\chi) \in \mathbb{Q} \times G^*} A_{a,\chi}$ (see [44, 47] for further details).

Theorem 7.4 ([44, 47, 34]) *The algebra A of automorphic forms on $\mathbb{H}_+$ with respect to G is naturally H-graded, and is isomorphic to the $\mathbb{L}(p)$-graded algebra $S(p, \lambda)$.*

Restricting the grading to the subgroup $\mathbb{Z} \times \{0\}$ yields the algebra R of entire automorphic forms, which is isomorphic to the algebra $\mathbb{A}(\tau; L) = \bigoplus_{n \geq 0} \mathrm{Hom}(L, \tau^n L)$ associated with the Auslander-Reiten translation. Here, L may be chosen as a line bundle on the weighted projective line $\mathbb{X}$ associated with $S(p, \lambda)$ or as a rank one non-preprojective module over the canonical algebra Λ associated with $\mathbb{X}$. □

Note that the algebra $R = \mathbb{A}(\tau; L)$ can be formed over any algebraically closed field k; it is always commutative, affine over k and Cohen-Macaulay of Krull dimension two. Exactly for the minimal wild canonical algebras $(2,3,7)$, $(2,4,5)$, $(3,3,4)$ and their close "neighbours" given in the table below, the algebra R can be generated by three homogeneous elements. In this case R has the form

$$R = k[x, y, z] = k[X, Y, Z]/(F),$$

where the relation F, the degree-triple $\deg(x, y, z)$, and $\deg F$ are displayed in the table below which is taken from [34].

	$\deg(x,y,z)$	relation F	$\deg F$	
$(2,3,7)$	$(6,14,21)$	$Z^2+Y^3+X^7$	42	
$(2,3,8)$	$(6,8,15)$	$Z^2+X^5+XY^3$	30	
$(2,3,9)$	$(6,8,9)$	$Y^3+XZ^2+X^4$	24	
$(2,4,5)$	$(4,10,15)$	$Z^2+Y^3+X^5Y$	30	
$(2,4,6)$	$(4,6,11)$	$Z^2+X^4Y+XY^3$	22	
$(2,4,7)$	$(4,6,7)$	$Y^3+X^3Y+XZ^2$	18	
$(2,5,5)$	$(4,5,10)$	$Z^2+Y^2Z+X^5$	20	•
$(2,5,6)$	$(4,5,6)$	$XZ^2+Y^2Z+X^4$	16	
$(3,3,4)$	$(3,8,12)$	$Z^2+Y^3+X^4Z$	24	•
$(3,3,5)$	$(3,5,9)$	$Z^2+XY^3+X^3Z$	18	•
$(3,3,6)$	$(3,5,6)$	$Y^3+X^3Z+XZ^2$	15	•
$(3,4,4)$	$(3,4,8)$	$Z^2-Y^2Z+X^4Y$	16	•
$(3,4,5)$	$(3,4,5)$	$X^3Y+XZ^2+Y^2Z$	13	
$(4,4,4)$	$(3,4,4)$	$X^4-YZ^2+Y^2Z$	12	•

For the base field of complex numbers the 14 equations are equivalent to Arnold's 14 exceptional unimodal singularities in the theory of singularities of differentiable maps [1]. In the theory of automorphic forms they occur as the relations of exactly those algebras of entire automorphic forms having three generators [60]. In the rows marked by • the two references quote a different expression for the singularity F, which — for the base field of complex numbers — is equivalent to the above.

7.3 A real exceptional curve

Proposition 7.5 *The $\mathbb{Z}$-graded $\mathbb{R}$-algebra $\mathbb{R}[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2)$, where the generators x_i get degree one, is graded-factorial and defines an exceptional curve $\mathbb{X}$, which is commutative and homogeneous and has a tilting bundle whose endomorphism ring is the tame bimodule algebra*

$$\Lambda = \begin{pmatrix} \mathbb{R} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix}.$$

Moreover, each simple sheaf has endomorphism ring $\mathbb{C}$.

Proof. We fix a line bundle L on $\mathbb{X}$. By graded factoriality any line bundle has the form $L(n)$ for some integer n, moreover the Auslander-Reiten translation is given by the grading shift $F \mapsto F(-1)$. This implies that the middle term of the almost-split sequence $0 \to L \to A \to L(1) \to 0$ is indecomposable. It follows that $L \oplus A$ is a tilting bundle with the required properties. □

The following relates the study of coh($\mathbb{X}$) to the classification of real subspaces of a quaternion vector space (see [15]).

Corollary 7.6 *The category* coh($\mathbb{X}$) *is equivalent to $\mathcal{H}(\Lambda)$, and R as a graded algebra is isomorphic to the preprojective algebra $\Pi(\Lambda, P)$, where P is a projective Λ-module of rank one.* □

7.4 Factoriality derived from representation theory

It is remarkable that classification results from the representation theory of finite dimensional algebras allow to decide on the factoriality of certain complete local rings. We start with a regular local ring R. An R-algebra S, accordingly an S-module M, is called Cohen-Macaulay (CM for short) if it is finitely generated free as a module over R. We denote by CM(S) the category of Cohen-Macaulay modules. If R is additionally complete, a theorem of Auslander [3] states that S is an isolated singularity if and only if CM(S) has almost-split sequences. Here, being an isolated singularity means that each localization of S with respect to any non-maximal prime ideal yields a regular local ring. It is straightforward to define graded analogues of these notions.

Starting with a $\mathbb{Z}$-graded CM-algebra S such that completion $\hat{S}$ is an isolated singularity, Auslander and Reiten have shown [4] that under mild restrictions on S the completion functor $\mathrm{CM}^{\mathbb{Z}}(S) \to \mathrm{CM}(\hat{S})$, $M \mapsto \hat{M}$, preserves indecomposability and almost-split sequences. Moreover, two indecomposable graded CM-modules X, Y do have isomorphic completions if and only if X and Y agree up to a degree-shift, i.e. $Y = X(n)$ for some integer n. Further, the completion $\hat{S}$ is CM-finite, i.e. CM($\hat{S}$) has only finitely many isomorphism classes of indecomposable objects, if and only if S is graded CM-finite, i.e. $\mathrm{CM}^{\mathbb{Z}}(S)$ has only finitely many shift-classes of indecomposable objects. Moreover, in this case the completion functor is dense, i.e. surjective on isomorphism classes.

Theorem 7.7 *Each of the isolated surface singularities*

1. $k[[X, Y, Z]]/(X^2 + Y^3 + Z^5)$, *where k is a field,*
2. $\mathbb{R}[[X, Y, Z]]/(X^2 + Y^2 + Z^2)$

is a factorial domain.

PROOF. Assertion 1 is, for the base field of complex numbers, due to Mumford [46] and to Scheja [56] for the case of an algebraically closed base field of any characteristic. A representation theoretic proof is given below. The line of the argument is the same in both cases.

The path algebra (over k) of the extended Dynkin quiver $\mathbb{E}_8$ has only finitely many preprojective Auslander-Reiten orbits. By means of ([22], Theorem 8.6), see also Example 7.1, this implies that the $\mathbb{Z}$-graded algebra $R = k[X, Y, Z]/(X^2 + Y^3 + Z^5)$, with the grading specified by $\deg(x, y, z) = (15, 10, 6)$ has up to grading shift only a finite number of graded Cohen Macaulay modules, i.e. is graded CM-finite. Since, R is graded factorial, each graded Cohen-Macaulay module over R has the form $R(n)$ for some integer n.

By Auslander and Reiten's completion theorem the completion functor is dense; in particular each rank one CM-module L over $\hat{R}$ is of the form $\hat{M}$, where M is a graded CM-module over R of rank one. By graded factoriality of R, the module M has the form $M = R(n)$ for some integer n, therefore $L \cong \hat{R}$. By a standard argument, this implies that $\hat{R}$ is factorial. The argument is similar with respect to the second algebra. □

7.5 Completion of Cohen-Macaulay modules

Quite some time ago Auslander has raised the question, whether the completion functor is always dense. The answer is no:

Theorem 7.8 *For the algebra $S = \mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^7)$, graded by $\deg(x, y, z) = (21, 14, 6)$ the following holds:*

1. *The surface singularity $\hat{S} = \mathbb{C}[[X, Y, Z]]/(X^2 + Y^3 + Z^7)$ is not factorial.*
2. *The completion functor $\mathrm{CM}^{\mathbb{Z}}(S) \to \mathrm{CM}(\hat{S})$, $M \mapsto \hat{M}$ is not dense.*

PROOF. S as a graded algebra equals the coordinate algebra of the weighted projective line $\mathbb{X}$ over $\mathbb{C}$ of weight type $(2, 3, 7)$. The curve $\mathbb{X}$ has a wild classification problem for its category $\mathrm{vec}(\mathbb{X})$ of vector bundles [40]. Since the category $\mathrm{CM}^{\mathbb{Z}}(S)$ of graded CM-modules and the category $\mathrm{vec}(\mathbb{X})$ of vector bundles over $\mathbb{X}$ are equivalent, with the shift $X \mapsto X(1)$ for $\mathrm{CM}^{\mathbb{Z}}(S)$ corresponding to the Auslander-Reiten translation for $\mathrm{vec}(\mathbb{X})$ [22], it follows from the properties of the completion functor that $\hat{S}$ has infinite CM-type and hence, in view of Theorem 7.7, is not isomorphic to the algebra $R = \mathbb{C}[[X, Y, Z]]/(X^2 + Y^3 + Z^5)$. A result of Brieskorn [12] (see [42] for an extension to algebraically closed base fields of characteristic different from 2, 3 or 5) states that R is the only complete local two-dimensional factorial algebra with residue class field $\mathbb{C}$, and thus $\hat{S}$ is not factorial. Hence there exists a rank one CM-module M over $\hat{S}$ which is not isomorphic to $\hat{S}$. But for the weight type $(2, 3, 7)$ all line bundles over $\mathbb{X}$ lie in the same Auslander-Reiten orbit [40], hence under completion are all mapped to $\hat{S}$. Therefore, M is not in the image of the completion functor. □

We refer to [59] for a different relationship between CM-modules and representation theory.

7.6 Nonisomorphic derived-equivalent curves

We now discuss an interesting weighted variant of the real exceptional curve treated in Section 7.3. Let H be the rank-one abelian group on generators $\vec{x}_1, \vec{x}_2, \vec{x}_3$ with relations $2\vec{x}_1 = 2\vec{x}_2 = \vec{x}_3$.

Example 7.9 (Kussin, 1996) *The $\mathbb{R}$-algebra $\mathbb{R}[X_1, X_2, X_3]/(X_1^4 + X_2^4 + X_3^2)$, H-graded by $\deg(x_i) = \vec{x}_i$, defines an exceptional non-commutative curve $\mathbb{X}$ of genus one having a tilting bundle, such that the following holds.*

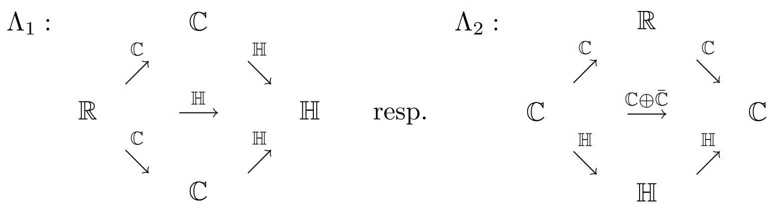
1. *There exist tilting bundles T_1, T_2 whose endomorphism algebras Λ_1, Λ_2 are canonical algebras in the sense of [52] with underlying bimodules of type $(1, 4)$ and $(2, 2)$ respectively.*

2. *There exists a second non-commutative exceptional curve $\mathbb{Y}$ of genus one such that* $\operatorname{coh}(\mathbb{X})$ *and* $\operatorname{coh}(\mathbb{Y})$ *are not equivalent but such their derived categories are.*

Moreover, each simple sheaf on $\mathbb{X}$ has endomorphism ring $\mathbb{C}$ whereas $\mathbb{Y}$ has simple sheaves with endomorphism rings $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.

It is therefore not always possible to reconstruct the points of an exceptional curve $\mathbb{X}$ from its derived category $\mathrm{D}^b(\operatorname{coh}(\mathbb{X}))$. By contrast, if $\mathbb{X}$ is an exceptional curve of genus different from one, it is easy to recover $\operatorname{coh}(\mathbb{X})$ from $\mathrm{D}^b(\operatorname{coh}(\mathbb{X}))$. The example sheds some light on recent work of A. Bondal and D. Orlov [10], who reconstruct the the variety $\mathbb{X}$ from its derived category of coherent sheaves for certain classes of (commutative) spaces.

The species of Λ_1, Λ_2, in particular the underlying bimodules ${}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$ and ${}_{\mathbb{C}}\mathbb{C} \oplus \overline{\mathbb{C}}_{\mathbb{C}}$ where the bar indicates a right $\mathbb{C}$-action by conjugation, are depicted below. We do not give the relations.



7.7 Infinite dimensional modules

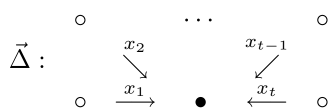
We will be very brief here. Let $\mathcal{H}$ be the category of coherent sheaves over an exceptional curve $\mathbb{X}$, and let T be any tilting sheaf and $\Lambda = \operatorname{End}(T)$. By tilting theory, the functor $\operatorname{Hom}(T, -)$ yields in particular a full embedding from the full subcategory of injective sheaves in $\vec{\mathcal{H}}$ into $\operatorname{Mod}(\Lambda)$. The indecomposable injective (quasi-coherent) sheaves fall in two classes

1. the torsion sheaves, coinciding with the injective envelopes of simple sheaves;
2. a single torsion-free sheaf isomorphic to the injective envelope of any line bundle.

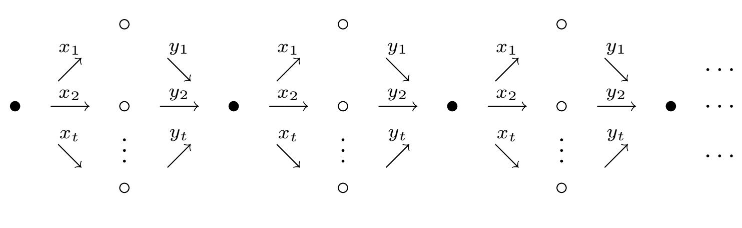
For the special situation where Λ is tame hereditary, the quasicoherent sheaves of type 1 (resp. type 2) yield by application of $\operatorname{Hom}(T, -)$ the Prüfer modules (resp. the indecomposable torsion-free divisible or generic Λ-module) from [50]. For the case of a tubular algebra, corresponding to genus one, we refer to [37].

7.8 Curve attached to a wild hereditary algebra

Assume that Λ is a wild hereditary algebra, for instance the path algebra of the wild quiver ($t \geq 5$)



whose classification problem for indecomposable representations is equivalent to classify the position of t subspaces in a vectorspace. The preprojective component of Λ in terms of generators and relations looks as follows



where $\sum_{i=1}^{t} y_i x_i = 0$ and $x_i y_i = 0$ for $i = 1, \ldots, t$.

The characteristic polynomial of the Coxeter transformation Φ is $(T+1)^{t-1} \cdot (T^2 - (t-2)T + 1)$, so for $t \geq 5$ the spectral radius ρ of Φ is > 1. Note that $x_1, \ldots, x_t$ are elements of the path algebra $\Lambda = k[\vec{\Delta}]$ which forms the zero component of the preprojective algebra $\Pi = \Pi(\Lambda)$ given as $\Pi = \Lambda\langle y_1, \ldots, y_t \rangle$, where $\sum_{i=1}^{t} y_i x_i = 0$, $x_i y_i = 0$ for $i = 1, \ldots, t$ and $\deg(y_i) = 1$. In particular Π is a finitely presented k-algebra.

In accordance with [17, 48] we get for the preprojective algebra Π of any connected wild hereditary algebra

$$\lim_{n \to \infty} \frac{\dim_k \Pi_n}{\rho^n} > 0,$$

hence Π has infinite Gelfand-Kirillov dimension. In particular Π does not satisfy a polynomial identity, and also is not noetherian [6]. It is further shown in [61] that the minimal spectral radius (=growth number) of any wild hereditary algebra is given by the largest real root of $T^{10} + T^9 - T^7 - T^6 - T^5 - T^4 - T^3 + T + 1$ which is about 1.176.

It follows from [33] that the quotient category

$$\mathcal{H} = \frac{\operatorname{mod}^{\mathbb{Z}}(\Pi)}{\operatorname{mod}_0^{\mathbb{Z}}(\Pi)}$$

satisfies all requirements for an exceptional curve from Section 2.5, except noetherianness. Moreover $\mathcal{H}$ has a tilting object with endomorphism algebra Λ, which implies that

$$\mathcal{H} \cong \mathcal{H}(\Lambda) = \mathcal{I}(\Lambda)[-1] \vee \mathcal{P}(\Lambda) \vee \mathcal{R}(\Lambda),$$

where $\mathcal{P}(\Lambda)$, $\mathcal{R}(\Lambda)$, $\mathcal{I}(\Lambda)$ denotes the subcategory of preprojective, regular and injective Λ-modules, respectively. Further $\mathcal{H}$ has no simple and therefore no non-zero noetherian or artinian object. We note that abelianness of $\mathcal{H}(\Lambda)$ can alternatively be derived from [26]. Note that it is open whether $\mathcal{H}$ or $\vec{\mathcal{H}}$ can be obtained by gluing from module categories.

It is tempting to view $\mathcal{H}$ as a candidate for a category of "coherent sheaves on a non-noetherian exceptional curve" $\mathbb{X}$. One could think of its geometric meaning to parametrize the regular components of $\operatorname{mod}(\Lambda)$ which would relate to work of Kerner [31] on the regular components of wild hereditary algebras. But this topic needs further investigation.

References

[1] V. I. Arnold, S. M. Gusejn-Zade, A. N. Varchenko, Singularities of differentiable maps. Volume I, Monographs in Mathematics, Vol. 82. Birkhaeuser, Basel, (1985).

[2] M. Artin, J.J. Zhang, Noncomutative projective schemes. Adv. Math. 109, 228-287 (1994).

[3] M. Auslander, Isolated singularities and existence of almost split sequences, Notes by Luise Unger. Representation theory II, Groups and orders, Proc. 4th Int. Conf., Ottawa 1984, Lect. Notes Math. 1178, 194-242 (1986).

[4] M. Auslander, I. Reiten, Cohen-Macaulay modules for graded Cohen-Macaulay rings and their completions, Commutative algebra, Proc. Microprogram, Berkeley 1989, Publ., Math. Sci. Res. Inst. 15, 21–31 (1989).

[5] M. Auslander, I. Reiten, S. O. Smalø, Representation theory of Artin algebras. Cambridge Studies in Advanced Mathematics. v. 36, Cambridge University Press, Cambridge (1995).

[6] D. Baer, Homological properties of wild hereditary Artin algebras, Representation theory I, Finite dimensional algebras, Proc. 4th Int. Conf., Ottawa 1984, Lect. Notes Math. 1177, 1–12 (1986).

[7] D. Baer, Tilting sheaves in representation theory of algebras, Manuscr. Math. 60, 323-347 (1988).

[8] D. Baer, W. Geigle, H. Lenzing, The preprojective algebra of a tame hereditary Artin algebra, Comm. Algebra 15, 425-457 (1987).

[9] A. A. Beilinson, Coherent sheaves on P_n and problems of linear algebra, Anal. Appl. 12, 214-216 (1979).

[10] A. Bondal, D. Orlov, Reconstruction of a variety from the derived category and groups of automorphisms. Preprint 1996.

[11] A. Braun, C. R. Hajarnavis, Finitely generated P.I. rings of global dimension two, J. Algebra 169, 587-604 (1994).

[12] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Inventiones Math. 4, 336–358 (1968).

[13] W. W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. Lond. Math. Soc. 62, 490-508 (1991).

[14] V. Dlab, C. M. Ringel, Indecomposable representations of graphs and algebras, Mem. Am. Math. Soc. 173 (1976).

[15] V. Dlab, C. M. Ringel, Real subspaces of a quaternion vector space, Can. J. Math. 30, 1228–1242 (1978).

[16] V. Dlab, C. M. Ringel, The preprojective algebra of a modulated graph, Representation theory II, Proc. 2nd int. Conf., Ottawa 1979, Lect. Notes Math. 832, 216-231 (1980).

[17] V. Dlab, C. M. Ringel, Eigenvalues of Coxeter transformations and the Gelfand-Kirillov dimension of the preprojective algebras, Proc. Am. Math. Soc. 83, 228-232 (1981).

[18] P. Gabriel, Des catégories abéliennes, Bull. Soc. math. France 90, 323-448 (1962).

[19] P. Gabriel, Indecomposable representations. II, Sympos. math. 11, 81–104 (1973).

[20] P. Gabriel, A.V. Roiter, Algebra VIII: Representations of finite-dimensional algebras. Encyclopaedia of Mathematical Sciences 73, Springer-Verlag, Berlin (1992).

[21] W. Geigle, H. Lenzing, A class of weighted projective curves arising in representation theory of finite dimensional algebras, Singularities, representations of algebras, and vector bundles, Lect. Notes Math. 1273, 265–297 (1987).

[22] W. Geigle, H. Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144, 273-343 (1991).

[23] I. M. Gelfand, V. A. Ponomarev, Model algebras and representations of graphs, Funct. Anal. Appl. 13, 157–166 (1979).

[24] P.P. Grivel, Catégories derivées et foncteurs derivés, in Algebraic D-modules, Perspectives in Math., vol. 2, Academic Press 1987.

[25] D.Happel, Triangulated categories in the representation theory of finite dimensional algebras, LMS Lecture Note Series **119**, Cambridge 1988.

[26] D. Happel, I. Reiten, S. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Am. Math. Soc. 575 (1996).

[27] D. Happel, C. M. Ringel, Tilted Algebras, Trans. Am. Math. Soc. 274, 399–443 (1982).

[28] R. Hartshorne, Algebraic geometry. Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[29] M. M. Kapranov, On the derived category of coherent sheaves on Grassmann manifolds, Math. USSR, Izv. 24, 183–192 (1985).

[30] M. M. Kapranov, The derived category of coherent sheaves on a quadric, Funct. Anal. Appl. 20, 141-142 (1986).

[31] O. Kerner, Tilting wild algebras, J. London Math. Soc. 39, 29–47 (1989).

[32] D. Kussin, Graded factorial algebras of dimension two, Bull. London Math Soc., to appear.

[33] H. Lenzing, Curve singularities arising from the representation theory of tame hereditary algebras. Representation theory I, Finite dimensional algebras, Proc. 4th Int. Conf., Ottawa 1984, Lect. Notes Math. 1177, 199-231 (1986).

[34] H. Lenzing, Wild Canonical algebras and rings of automorphic forms, V. Dlab and L.L. Scott (eds.), Kluwer Academic Publishers Finite Dimensional Algebras and Related Topics, 191-212 (1994).

[35] H. Lenzing, A K-theoretic study of canonical algebras, CMS Conf. Proc. 18, 433-454 (1996).

[36] H. Lenzing, Hereditary noetherian categories with a tilting complex, Proc. Amer. Math. Soc. 125, 1893-1901 (1997).

[37] H. Lenzing, Generic modules over a tubular algebra, Algebra and Model Theory, Proceedings Essen 1994 and Dresden 1995 (to appear).

[38] H. Lenzing, H. Meltzer, Sheaves on a weighted projective line of genus one, and representations of a tubular algebra, Representations of algebras, Sixth International Conference, Ottawa 1992 CMS Conf. Proc. 14, 313–337 (1993).

[39] H. Lenzing, H. Meltzer, Tilting sheaves and concealed-canonical algebras, Representations of algebras. Seventh International Conference, Cocoyoc (Mexico) 1994. CMS Conf. Proc. 18, 455–473 (1996).

[40] H. Lenzing, J. A. de la Peña, Wild canonical algebras, Math. Z. 224, 403-425 (1997).

[41] H. Lenzing, J. A. de la Peña, Concealed-canonical algebras and algebras with a separating tubular family, Proc. London Math. Soc. (to appear).

[42] J. Lipman, Rational singularities with applications to algebraic surfaces and unique factorization, Publ. Math. Inst. Hautes Études Scientifiques 36, 195–280 (1969).

[43] H. Meltzer, Mutations as derived equivalences for sheaves on weighted projective lines of genus one, Preprint, Paderborn, (1994).

[44] Milnor, John, On the 3-dimensional Brieskorn manifolds M(p,q,r), Knots , Groups, 3-Manif.; Pap. dedic. Mem. R. H. Fox, 175–225 (1975).

[45] S. Mori, Graded factorial domains, Japan. J. Math. 3, 223–238 (1977).

[46] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. Inst. Hautes Études Scientifiques 9, 229–246 (1961).

[47] W. Neumann, Brieskorn complete intersections and automorphic forms, Invent. Math. 42, 285–293 (1977).

[48] J. A. de la Pena, M. Takane, Spectral properties of Coxeter transformations and applications, Arch. Math. 55, 120–134 (1990).

[49] N. Popescu, Abelian categories with applications to rings and modules, Academic Press, London, New York (1973).

[50] C. M. Ringel, Infinite dimensional representations of finite-dimensional hereditary algebras, Symposia Mat. Inst. Alta Mat. 23, 321–412 (1979).

[51] C. M. Ringel, Tame algebras and integral quadratic forms. Lect. Notes Math. 1099 Springer, Berlin-Heidelberg-New York (1984).

[52] C. M. Ringel, The canonical algebras with an appendix by William Crawley-Boevey, Topics in Algebra, Banach Center Publ. 26, 407–432 (1990).

[53] C. M. Ringel, The preprojective algebra of a quiver. Preprint 1997.

[54] A. L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras. Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[55] A. N. Rudakov, ed., Helices and vector bundles: seminaire Rudakov, London Math. Soc. Lect. Note Series 148, Cambridge University Press (1990).

[56] G. Scheja, Einige Beispiele faktorieller lokaler Ringe, Math. Annalen 172, 124–134 (1967).

[57] J.-P. Serre, Faisceaux algébriques cohérents, Ann. of Math., II. Ser. 61, 197-278 (1955).

[58] C. S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque 96 (1982).

[59] D. Simson, Linear representations of partially ordered sets and vector space categories, Gordon and Breach Science Publishers, Brooklyn, NY, (1992).

[60] P. Wagreich, Algebras of automorphic forms with few generators, Trans. Am. Math. Soc. 262, 367–389 (1980).

[61] Ch. Xi, On wild hereditary algebras with small growth numbers, Commun. Algebra 18, 3413–3422 (1990).

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