INTRODUCTION TO COHERENT SHEAVES ON WEIGHTED PROJECTIVE LINES

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Abstract. These notes provide a description of the abelian categories that arise as categories of coherent sheaves on weighted projective lines. Two different approaches are presented: one is based on a list of axioms and the other yields a description in terms of expansions of abelian categories.

A weighted projective line is obtained from a projective line by inserting finitely many weights. So we describe the category of coherent sheaves on a projective line in some detail, and the insertion of weights amounts to adding simple objects. We call this process 'expansion' and treat it axiomatically. Thus most of these notes are devoted to studying abelian categories, including a brief discussion of tilting theory. We provide many details and have tried to keep the exposition as self-contained as possible.

Contents

Introduction

We begin with a brief description of weighted projective lines and their categories of coherent sheaves.

Let k be an algebraically closed field, let $\mathbb{P}^1_k$ be the projective line over k, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a (possibly empty) collection of distinct closed points of $\mathbb{P}^1_k$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a *weight sequence*, that is, a sequence of positive integers. The triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ is called a *weighted projective line*. Geigle and Lenzing [10] have associated to each weighted projective line a category $\operatorname{coh} \mathbb{X}$ of coherent sheaves on $\mathbb{X}$, which is the quotient category of the category of finitely generated $\mathbf{L}(\mathbf{p})$-graded $S(\mathbf{p}, \boldsymbol{\lambda})$-modules, modulo the Serre subcategory of finite length modules. Here $\mathbf{L}(\mathbf{p})$ is the rank 1 additive group

$$\mathbf{L}(\mathbf{p}) = \langle \vec{x}_1, \ldots, \vec{x}_n, \vec{c} \mid p_1\vec{x}_1 = \cdots = p_n\vec{x}_n = \vec{c} \rangle,$$

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and

$$S(\mathbf{p},\boldsymbol{\lambda}) = k[u,v,x_1,\dots,x_n]/(x_i^{p_i} + \lambda_{i1}u - \lambda_{i0}v),$$

with grading $\deg u = \deg v = \vec{c}$ and $\deg x_i = \vec{x}_i$, where $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ in $\mathbb{P}^1_k$. Geigle and Lenzing showed that $\operatorname{coh}\mathbb{X}$ is a hereditary abelian category with finite dimensional Hom and Ext spaces. The free module $S(\mathbf{p},\boldsymbol{\lambda})$ yields a structure sheaf $\mathcal{O}$, and shifting the grading gives twists $E(\vec{x})$ for any sheaf E and $\vec{x} \in \mathbf{L}(\mathbf{p})$.

Every sheaf is the direct sum of a torsion-free sheaf and a finite length sheaf. A torsion-free sheaf has a finite filtration by line bundles, that is, sheaves of the form $\mathcal{O}(\vec{x})$. The finite length sheaves are easily described as follows. There are simple sheaves S_x ($x \in \mathbb{P}^1_k \smallsetminus \boldsymbol{\lambda}$) and S_{ij} ($1 \le i \le n$, $1 \le j \le p_i$) satisfying for any $r \in \mathbb{Z}$ that $\operatorname{Hom}(\mathcal{O}(r\vec{c}), S_{ij}) \neq 0$ if and only if $j = 1$, and the only extensions between them are

$$\operatorname{Ext}^1(S_x, S_x) = k, \quad \operatorname{Ext}^1(S_{ij}, S_{ij'}) = k \quad (j' \equiv j-1 \,(\operatorname{mod}\, p_i)).$$

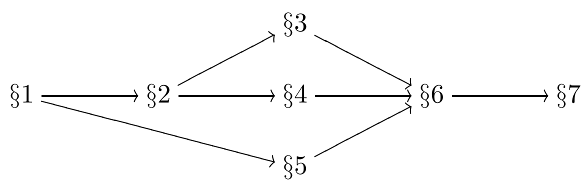
For each simple sheaf S and $l > 0$ there is a unique sheaf with length l and top S, which is *uniserial*, meaning that it has a unique composition series. These are all the finite length indecomposable sheaves.

Categories of the form $\operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X}$ play a special role in the study of abelian categories. This follows from a theorem of Happel [13] which we now explain. Consider a connected hereditary abelian category $\mathcal{A}$ that is k-linear with finite dimensional Hom and Ext spaces. Suppose in addition that $\mathcal{A}$ admits a tilting object, that is some object T with $\operatorname{Ext}^1_{\mathcal{A}}(T,T) = 0$ such that $\operatorname{Hom}_{\mathcal{A}}(T,A) = 0$ and $\operatorname{Ext}^1_{\mathcal{A}}(T,A) = 0$ imply $A = 0$. Thus the functor $\operatorname{Hom}_{\mathcal{A}}(T,-)\colon \mathcal{A} \to \operatorname{mod}\Lambda$ into the category of modules over the endomorphism algebra $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$ induces an equivalence

$$\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$$

of derived categories. There are two important classes of such hereditary abelian categories admitting a tilting object: module categories over path algebras of finite connected quivers without oriented cycles, and categories of coherent sheaves on weighted projective lines. Happel's theorem then states that there are no further classes. More precisely, an abelian category $\mathcal{A}$ as above is, up to a derived equivalence, either of the form $\operatorname{mod} k\Gamma$ for some finite connected quiver Γ without oriented cycles or of the form $\operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X}$.

The following treatment of coherent sheaves on weighted projective lines is based on a list of axioms (extending the list in Happel's theorem) which we postpone until §6. Before that we discuss in some detail the necessary background material: abelian categories, derived categories, tilting theory, expansions of abelian categories, and coherent sheaves on $\mathbb{P}^1_k$.



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Most of the material presented here is taken from the existing literature. An exception is §4, where the concept of an 'expansion of abelian categories' is introduced. The previously unpublished proof of Theorem 1.7.1 is due to Yu Ye.

1. Abelian categories

1.1. **Additive and abelian categories.** A category $\mathcal{A}$ is *additive* if every finite family of objects has a product, each morphism set $\operatorname{Hom}_{\mathcal{A}}(A,B)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(A,B)\times\operatorname{Hom}_{\mathcal{A}}(B,C)\longrightarrow\operatorname{Hom}_{\mathcal{A}}(A,C)$$

are bilinear. Given a finite number of objects $A_1,\dots,A_r$ of an additive category $\mathcal{A}$, there exists a *direct sum* $A_1\oplus\cdots\oplus A_r$, which is by definition an object A together with morphisms $\iota_i\colon A_i\to A$ and $\pi_i\colon A\to A_i$ for $1\le i\le r$ such that $\sum_{i=1}^r\iota_i\pi_i=\mathrm{id}_A$, $\pi_i\iota_i=\mathrm{id}_{A_i}$, and $\pi_j\iota_i=0$ for all $i\neq j$. Note that the morphisms ι_i and π_i induce isomorphisms

$$\coprod_{i=1}^r A_i\cong\bigoplus_{i=1}^r A_i\cong\prod_{i=1}^r A_i.$$

Given any object A in $\mathcal{A}$, we denote by $\operatorname{add}A$ the full subcategory of $\mathcal{A}$ consisting of all finite direct sums of copies of A and their direct summands.

A *decomposition* $\mathcal{A}=\mathcal{A}_1\amalg\mathcal{A}_2$ of an additive category $\mathcal{A}$ is a pair of full additive subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ such that each object in $\mathcal{A}$ is a direct sum of two objects from $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\operatorname{Hom}_{\mathcal{A}}(A_1,A_2)=0=\operatorname{Hom}_{\mathcal{A}}(A_2,A_1)$ for all $A_1\in\mathcal{A}_1$ and $A_2\in\mathcal{A}_2$. An additive category $\mathcal{A}$ is *connected* if it admits no proper decomposition $\mathcal{A}=\mathcal{A}_1\amalg\mathcal{A}_2$.

A functor $F\colon\mathcal{A}\to\mathcal{B}$ between additive categories is *additive* if the induced map $\operatorname{Hom}_{\mathcal{A}}(A,B)\to\operatorname{Hom}_{\mathcal{B}}(FA,FB)$ is linear for each pair of objects A,B in $\mathcal{A}$. The *kernel* $\operatorname{Ker}F$ of an additive functor $F\colon\mathcal{A}\to\mathcal{B}$ is by definition the full subcategory of $\mathcal{A}$ formed by all objects A such that $FA=0$. The *essential image* $\operatorname{Im}F$ of $F\colon\mathcal{A}\to\mathcal{B}$ is the full subcategory of $\mathcal{B}$ formed by all objects B such that B is isomorphic to FA for some A in $\mathcal{A}$.

An additive category $\mathcal{A}$ is *abelian* if every morphism $\phi\colon A\to B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{array}{ccccccc}
\operatorname{Ker}\phi & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \operatorname{Coker}\phi\\
 & & \downarrow & & \uparrow & & \\
 & & \operatorname{Coker}\phi' & \xrightarrow{\bar\phi} & \operatorname{Ker}\phi'' & &
\end{array}$$

of ϕ induces an isomorphism $\bar\phi$.

Given an abelian category $\mathcal{A}$, a finite sequence of morphisms

$$A_1\xrightarrow{\phi_1}A_2\xrightarrow{\phi_2}\cdots\xrightarrow{\phi_n}A_{n+1}$$

in $\mathcal{A}$ is *exact* if $\operatorname{Im}\phi_i=\operatorname{Ker}\phi_{i+1}$ for all $1\le i<n$. An additive functor $F\colon\mathcal{A}\to\mathcal{B}$ between abelian categories is *exact* if F sends each exact sequence in $\mathcal{A}$ to an exact sequence in $\mathcal{B}$.

Example 1.1.1. (1) Let Λ be a right noetherian ring. The category $\operatorname{mod}\Lambda$ of finitely generated right modules over Λ is an abelian category.

(2) Let k be a field and Γ a quiver. The category $\operatorname{rep}(\Gamma, k)$ of finite dimensional k-linear representations of Γ is an abelian category.

Conventions. Throughout, all categories are supposed to be *skeletally small*, unless otherwise stated. This means that the isomorphism classes of objects form a set. Subcategories are usually full subcategories and closed under isomorphisms. Functors between additive categories are always assumed to be additive. The composition of morphisms is written from right to left, and modules over a ring are usually right modules.

1.2. **Serre subcategories and quotient categories.** Let $\mathcal{A}$ be an abelian category. A non-empty full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called a *Serre subcategory* provided that $\mathcal{C}$ is closed under taking subobjects, quotients and extensions. This means that for every exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, the object A belongs to $\mathcal{C}$ if and only if A' and A'' belong to $\mathcal{C}$.

Example 1.2.1. The kernel of an exact functor $\mathcal{A} \to \mathcal{B}$ between abelian categories is a Serre subcategory of $\mathcal{A}$.

Given a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$, the *quotient category* $\mathcal{A}/\mathcal{C}$ of $\mathcal{A}$ with respect to $\mathcal{C}$ is defined as follows. The objects in $\mathcal{A}/\mathcal{C}$ are the objects in $\mathcal{A}$. Given two objects A, B in $\mathcal{A}$, there is for each pair of subobjects $A' \subseteq A$ and $B' \subseteq B$ an induced map $\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}}(A', B/B')$. The pairs (A', B') such that both A/A' and B' lie in $\mathcal{C}$ form a directed set, and one obtains a direct system of abelian groups $\operatorname{Hom}_{\mathcal{A}}(A', B/B')$. We define

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, B) = \operatorname*{colim}_{(A', B')} \operatorname{Hom}_{\mathcal{A}}(A', B/B')$$

and the composition of morphisms in $\mathcal{A}$ induces the composition in $\mathcal{A}/\mathcal{C}$.[1]

The *quotient functor* $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is by definition the identity on objects. The functor takes a morphism in $\operatorname{Hom}_{\mathcal{A}}(A, B)$ to its image under the canonical map $\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, B)$.

Lemma 1.2.2. *Each morphism $A \to B$ in $\mathcal{A}/\mathcal{C}$ is of the form*

$$A \xrightarrow{(Q\iota)^{-1}} A' \xrightarrow{Q\phi} B/B' \xrightarrow{(Q\pi)^{-1}} B \tag{1.2.1}$$

for some pair (A', B') of subobjects with A/A' and B' in $\mathcal{C}$ and some morphism $\phi\colon A' \to B/B'$ in $\mathcal{A}$, where $\iota\colon A' \to A$ and $\pi\colon B \to B/B'$ denote the canonical morphisms in $\mathcal{A}$.

Proof. For each morphism $A \to B$ in $\mathcal{A}/\mathcal{C}$, there is by definition a pair (A', B') of subobjects and a morphism $\phi\colon A' \to B/B'$ in $\mathcal{A}$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle Q\iota}\uparrow & & \downarrow{\scriptstyle Q\pi} \\ A' & \xrightarrow{Q\phi} & B/B' \end{array}$$

Now observe that for each object C in $\mathcal{A}$ the inclusion $\iota\colon A' \to A$ induces a bijection $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, C) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A', C)$. Thus $Q\iota$ is invertible. Analogously, one shows that $Q\pi$ is invertible. $\square$

[1] One needs to verify that $\mathcal{A}/\mathcal{C}$ is a category, in particular that the composition of morphisms is associative. This requires some work; see [8, 9].

The following result summarizes the basic properties of a quotient category and the corresponding quotient functor.

Proposition 1.2.3. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory.*

(1) *The category $\mathcal{A}/\mathcal{C}$ is abelian and the quotient functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact with kernel $\operatorname{Ker} Q = \mathcal{C}$.*
(2) *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. If $\mathcal{C} \subseteq \operatorname{Ker} F$, then there is a unique functor $\bar{F}\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$ such that $F = \bar{F}Q$. Moreover, the functor $\bar{F}$ is exact.*

Proof. (1) It follows from the construction that the morphism sets of the quotient category $\mathcal{A}/\mathcal{C}$ are abelian groups. Also, the quotient functor induces linear maps between the morphism sets and it preserves finite direct sums. Thus the quotient category and the quotient functor are both additive.

The quotient functor sends a morphism in $\mathcal{A}$ to the zero morphism if and only if its image belongs to $\mathcal{C}$. Thus $\operatorname{Ker} Q = \mathcal{C}$.

Let $\psi = (Q\pi)^{-1}Q\phi(Q\iota)^{-1}$ be a morphism in $\mathcal{A}/\mathcal{C}$ as in (1.2.1). Denote by $\iota'\colon \operatorname{Ker}\phi \to A'$ the kernel and by $\pi'\colon B/B' \to \operatorname{Coker}\phi$ the cokernel of ϕ in $\mathcal{A}$. Then the kernel of ψ is $Q(\iota\iota')\colon \operatorname{Ker}\phi \to A$, whereas the cokernel of ψ is $Q(\pi'\pi)\colon B \to \operatorname{Coker}\phi$. It follows that the category $\mathcal{A}/\mathcal{C}$ is abelian and that the quotient functor preserves kernels and cokernels.

(2) The functor $\bar{F}\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$ takes an object A to FA and a morphism of the form $(Q\pi)^{-1}Q\phi(Q\iota)^{-1}$ as in (1.2.1) to $(F\pi)^{-1}F\phi(F\iota)^{-1}$. Note that $F\iota$ and $F\pi$ are isomorphisms in $\mathcal{B}$, since F is exact and $\mathcal{C} \subseteq \operatorname{Ker} F$.

The functor $\bar{F}$ is additive and the description of (co)kernels in (1) shows that $\bar{F}$ preserves (co)kernels. Thus $\bar{F}$ is exact. □

The quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ is the universal functor that inverts the class $S(\mathcal{C})$ of morphisms σ in $\mathcal{A}$ with $\operatorname{Ker}\sigma$ and $\operatorname{Coker}\sigma$ in $\mathcal{C}$. More precisely, for any class S of morphisms in $\mathcal{A}$, there exists a universal functor $P\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$ such that

(1) the morphism $P\sigma$ is invertible for every $\sigma \in S$, and
(2) every functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $F\sigma$ is invertible for each $\sigma \in S$ admits a unique functor $\bar{F}\colon \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $F = \bar{F}P$.

The category $\mathcal{A}[S^{-1}]$ is the *localization* of $\mathcal{A}$ with respect to S and is unique up to a unique isomorphism; see [9, I.1].

Lemma 1.2.4. *Let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. The quotient functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is the universal functor that inverts all morphisms in $S(\mathcal{C})$. Therefore*

$$\mathcal{A}[S(\mathcal{C})^{-1}] = \mathcal{A}/\mathcal{C}.$$

Proof. From Proposition 1.2.3 it follows that Q inverts all morphisms in $S(\mathcal{C})$. Now let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor such that $F\sigma$ is invertible for each $\sigma \in S(\mathcal{C})$. Then for each pair A, B of objects in $\mathcal{A}$ and each pair of subobjects $A' \subseteq A$ and $B' \subseteq B$ with A/A' and B' in $\mathcal{C}$, the map $\operatorname{Hom}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{\mathcal{B}}(FA,FB)$ factors through the canonical map $\operatorname{Hom}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{\mathcal{A}}(A', B/B')$. Thus there are induced maps $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{B}}(FA,FB)$ which induce a unique functor $\bar{F}\colon A/\mathcal{C} \to \mathcal{B}$ such that $F = \bar{F}Q$. It follows that $\mathcal{A}[S(\mathcal{C})^{-1}] = \mathcal{A}/\mathcal{C}$. □

Example 1.2.5. (1) Let Λ be a commutative noetherian ring and $\Lambda_{\mathfrak{p}}$ the localization with respect to a prime ideal $\mathfrak{p}$. The localization functor $T\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda_{\mathfrak{p}}$ sending a Λ-module M to $M_{\mathfrak{p}} = M \otimes_{\Lambda} \Lambda_{\mathfrak{p}}$ is exact and induces an equivalence

$\operatorname{mod}\Lambda/\operatorname{Ker}T \xrightarrow{\sim} \operatorname{mod}\Lambda_{\mathfrak{p}}$. Roughly speaking, restriction of scalars along the morphism $\Lambda \to \Lambda_{\mathfrak{p}}$ yields a quasi-inverse.

(2) Let Λ be a right noetherian ring and $e^2 = e \in \Lambda$ an idempotent. The functor $T\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ sending a Λ-module M to $Me = M \otimes_\Lambda \Lambda e$ is exact. The kernel $\operatorname{Ker}T$ identifies with $\operatorname{mod}\Lambda/\Lambda e\Lambda$ and T induces an equivalence $\operatorname{mod}\Lambda/\operatorname{Ker}T \xrightarrow{\sim} \operatorname{mod} e\Lambda e$. The functor $\operatorname{Hom}_{e\Lambda e}(\Lambda e, -)$ yields a quasi-inverse.

(3) Let Λ be a right artinian ring. Given a set $S_1, \dots, S_n$ of simple Λ-modules, the Λ-modules M having a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ with each factor M_i/M_{i-1} isomorphic to one of the simples $S_1, \dots, S_n$ form a Serre subcategory of $\operatorname{mod}\Lambda$. Moreover, each Serre subcategory of $\operatorname{mod}\Lambda$ arises in this way and is therefore of the form $\operatorname{mod}\Lambda/\Lambda e\Lambda$ for some idempotent $e \in \Lambda$.

1.3. **Properties of quotient categories.** We collect some further properties of abelian quotient categories.

Lemma 1.3.1. *Let $\mathcal{A}$ be an abelian category that is not supposed to be skeletally small, and let $\mathcal{C}$ be a Serre subcategory. Then the following are equivalent:*

1. *The category $\mathcal{A}$ is skeletally small.*
2. *The categories $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are skeletally small. In addition, $\operatorname{Ext}^1_{\mathcal{A}}(A, C)$ and $\operatorname{Ext}^1_{\mathcal{A}}(C, A)$ are sets for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$.*

Proof. One direction is clear. So suppose that $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are skeletally small, and that extensions with objects in $\mathcal{C}$ form sets. First observe that the morphisms in $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, B)$ form a set for each pair of objects A, B. Here one uses that the subobjects $A' \subseteq A$ with A' or A/A' in $\mathcal{C}$ form, up to isomorphism, a set, since $\mathcal{C}$ is skeletally small. Next observe that for each object A in $\mathcal{A}$, there is only a set of isomorphism classes of objects B with $A \cong B$ in $\mathcal{A}/\mathcal{C}$. This follows from the fact that each isomorphism $A \to B$ in $\mathcal{A}/\mathcal{C}$ is represented by a chain

$$A \hookleftarrow A' \twoheadrightarrow I \rightarrowtail B/B' \twoheadleftarrow B$$

of epis and monos in $\mathcal{A}$ with kernel and cokernel in $\mathcal{C}$; see Lemma 1.2.2. Here one uses that $\mathcal{C}$ is skeletally small and that extensions with objects in $\mathcal{C}$ form sets. From this it follows that the isomorphism classes of objects in $\mathcal{A}$ form a set, since the quotient $\mathcal{A}/\mathcal{C}$ has this property. □

The following example gives an abelian category $\mathcal{A}$ with a Serre subcategory $\mathcal{C}$ such that $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are skeletally small but $\mathcal{A}$ itself is not.

Example 1.3.2. Let k be a field and Γ a quiver with set of vertices $\{1, 2\}$ and a proper class of arrows $1 \to 2$. Each arrow of Γ corresponds to a canonical element of $\operatorname{Ext}^1(S_1, S_2)$, where S_i denotes the simple representation supported at the vertex i. These extensions are linearly independent and yield pairwise non-isomorphic two-dimensional representations. The functor $T\colon \operatorname{rep}(\Gamma, k) \to \operatorname{mod} k$ sending a representation of Γ to the corresponding vector space at vertex 1 induces an equivalence $\operatorname{rep}(\Gamma, k)/\operatorname{Ker}T \xrightarrow{\sim} \operatorname{mod} k$, and $\operatorname{Ker}T$ is equivalent to $\operatorname{mod} k$.

An abelian category is called *noetherian* if each of its objects is *noetherian* (i.e. satisfies the ascending chain condition on subobjects).

Lemma 1.3.3. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory. Then the following are equivalent:*

1. *The category $\mathcal{A}$ is noetherian.*
2. *The categories $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are noetherian, and each object in $\mathcal{A}$ has a largest subobject that belongs to $\mathcal{C}$.*

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{A}$ is noetherian. Then $\mathcal{C}$ is noetherian. Also, $\mathcal{A}/\mathcal{C}$ is noetherian because each ascending chain of subobjects in $\mathcal{A}/\mathcal{C}$ can be represented by an ascending chain of subobjects in $\mathcal{A}$; see Lemma 1.2.2. Noetherianess implies that each non-empty set of subobjects has a maximal element. In particular, each object has a subobject that is maximal among all subobjects belonging to $\mathcal{C}$.

(2) $\Rightarrow$ (1): Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ be an ascending chain of subobjects in $\mathcal{A}$. Using that $\mathcal{A}/\mathcal{C}$ is noetherian, there exists some integer n such that A_m/A_n belongs to $\mathcal{C}$ for all $m > n$. Let $\bar{A}$ be the maximal subobject of A/A_n belonging to $\mathcal{C}$. Then the chain $A_{n+1}/A_n \subseteq A_{n+2}/A_n \subseteq \cdots \subseteq \bar{A}$ becomes stationary since $\mathcal{C}$ is noetherian. It follows that the original chain of subobjects of A becomes stationary. Thus A is noetherian. □

Next we give an example of an abelian category $\mathcal{A}$ with a Serre subcategory $\mathcal{C}$ such that $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are noetherian but $\mathcal{A}$ itself is not.

Example 1.3.4. The ring $\Lambda = \left[\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$ is well known to be left but not right noetherian. Consider the abelian category $\operatorname{mod}\Lambda$ of finitely presented (right) Λ-modules and the functor $T\colon \operatorname{mod}\Lambda \to \operatorname{mod}\mathbb{Q}$ sending a Λ-module M to Me with $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $\operatorname{Ker} T$ is equivalent to $\operatorname{mod}\mathbb{Z}$ and T induces an equivalence $\operatorname{mod}\Lambda/\operatorname{Ker} T \xrightarrow{\sim} \operatorname{mod}\mathbb{Q}$.

The next lemma provides the analogue of a Noether isomorphism for abelian categories. The proof is straightforward.

Lemma 1.3.5. *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_1, \mathcal{A}_2$ a pair of Serre subcategories such that $\mathcal{A}_2 \subseteq \mathcal{A}_1$. Then the following holds:*

1. *The inclusion $\mathcal{A}_1 \to \mathcal{A}$ identifies $\mathcal{A}_1/\mathcal{A}_2$ with a Serre subcategory of $\mathcal{A}/\mathcal{A}_2$.*
2. *The quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{A}_2$ induces an isomorphism*
$$\mathcal{A}/\mathcal{A}_1 \xrightarrow{\sim} (\mathcal{A}/\mathcal{A}_2)/(\mathcal{A}_1/\mathcal{A}_2).$$ □

Recall that a non-zero object S of an abelian category is *simple* if S has no proper subobject $0 \neq U \subsetneq S$. The next lemma says that a quotient functor preserves this property if the object is not annihilated. The proof is straightforward.

Lemma 1.3.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory. If S is a simple object not belonging to $\mathcal{C}$, then S is simple in $\mathcal{A}/\mathcal{C}$ and the quotient functor induces an isomorphism* $\operatorname{End}_{\mathcal{A}}(S) \xrightarrow{\sim} \operatorname{End}_{\mathcal{A}/\mathcal{C}}(S)$. □

1.4. **Perpendicular categories.**

Let $\mathcal{A}$ be an abelian category. In some cases, the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ with respect to a Serre subcategory $\mathcal{C}$ admits a right adjoint. Then the perpendicular category $\mathcal{C}^\perp$ provides another description of the quotient category $\mathcal{A}/\mathcal{C}$.

For any class $\mathcal{C}$ of objects in $\mathcal{A}$, its *perpendicular categories* are by definition the full subcategories
$$\mathcal{C}^\perp = \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C, A) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(C, A) \text{ for all } C \in \mathcal{C}\},$$
$${}^\perp\mathcal{C} = \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(A, C) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(A, C) \text{ for all } C \in \mathcal{C}\}.$$

Lemma 1.4.1. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory. Then the following are equivalent for an object B in $\mathcal{A}$:*

1. *The object B belongs to $\mathcal{C}^\perp$.*
2. *The quotient functor induces a bijection* $\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, B)$ *for every object A in $\mathcal{A}$.*

(3) *The map* $\operatorname{Hom}_{\mathcal{A}}(\sigma, B)$ *is bijective for every morphism* σ *in* $\mathcal{A}$ *with* $\operatorname{Ker}\sigma$ *and* $\operatorname{Coker}\sigma$ *in* $\mathcal{C}$*.*

Proof. (1) $\Rightarrow$ (2): For each pair of subobjects $A' \subseteq A$ and $B' \subseteq B$ such that both A/A' and B' lie in $\mathcal{C}$, the map $\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}}(A', B/B')$ is bijective. Thus the quotient functor induces a bijection $\operatorname{Hom}_{\mathcal{A}}(A, B) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(A, B)$.

(2) $\Rightarrow$ (3): The quotient functor sends a morphism σ in $\mathcal{A}$ with $\operatorname{Ker}\sigma$ and $\operatorname{Coker}\sigma$ in $\mathcal{C}$ to an isomorphism in $\mathcal{A}/\mathcal{C}$. Thus $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(\sigma, B)$ is bijective. Using the bijections in (2) it follows that $\operatorname{Hom}_{\mathcal{A}}(\sigma, B)$ is bijective.

(3) $\Rightarrow$ (1): Let C be an object in $\mathcal{C}$. Then $\sigma\colon 0 \to C$ induces a bijection $\operatorname{Hom}_{\mathcal{A}}(\sigma, B)$. Thus $\operatorname{Hom}_{\mathcal{A}}(C, B) = 0$. Let $\xi\colon 0 \to B \to E \to C \to 0$ an exact sequence in $\mathcal{A}$. The morphism $\sigma\colon B \to E$ induces a bijection $\operatorname{Hom}_{\mathcal{A}}(\sigma, B)$, and therefore ξ splits. Thus $\operatorname{Ext}^1_{\mathcal{A}}(C, B) = 0$. □

We need the following elementary lemma about pairs of adjoint functors.

Lemma 1.4.2. *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *and* $G\colon \mathcal{B} \to \mathcal{A}$ *be a pair of functors such that* G *is a right adjoint of* F*. Denote by* $S(F)$ *the class of morphisms* σ *in* $\mathcal{A}$ *such that* $F\sigma$ *is invertible. Then the following are equivalent:*

(1) *The functor* F *induces an equivalence* $\mathcal{A}[S(F)^{-1}] \xrightarrow{\sim} \mathcal{B}$*.*
(2) *The functor* G *is fully faithful.*
(3) *The adjunction morphism* $F(GA) \to A$ *is invertible for each* $A \in \mathcal{B}$*.*

Proof. See [9, I.1.3]. □

The next result provides a useful criterion for an exact functor to be a quotient functor. Moreover, it describes the right adjoint of a quotient functor.

Proposition 1.4.3. *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be an exact functor between abelian categories and suppose that* F *admits a right adjoint* $G\colon \mathcal{B} \to \mathcal{A}$*. Then the following are equivalent:*

(1) *The functor* F *induces an equivalence* $\mathcal{A}/\operatorname{Ker}F \xrightarrow{\sim} \mathcal{B}$*.*
(2) *The functor* F *induces an equivalence* $(\operatorname{Ker}F)^{\perp} \xrightarrow{\sim} \mathcal{B}$*.*
(3) *The functor* G *induces an equivalence* $\mathcal{B} \xrightarrow{\sim} (\operatorname{Ker}F)^{\perp}$*.*
(4) *The functor* G *is fully faithful.*

Moreover, in that case $(\operatorname{Ker}F)^{\perp} = \operatorname{Im}G$ *and* $\operatorname{Ker}F = {}^{\perp}(\operatorname{Im}G)$*.*

Proof. Let $S(F)$ denote the class of morphisms σ in $\mathcal{A}$ such that $F\sigma$ is invertible. Then it follows from Lemma 1.2.4 that $\mathcal{A}[S(F)^{-1}] = \mathcal{A}/\operatorname{Ker}F$.

(1) $\Rightarrow$ (2): The functor F induces a full and faithful functor $(\operatorname{Ker}F)^{\perp} \to \mathcal{B}$ by Lemma 1.4.1. For each $B \in \mathcal{B}$, we have $F(GB) \cong B$ by Lemma 1.4.2, and GB belongs to $(\operatorname{Ker}F)^{\perp}$ by Lemma 1.4.1. Thus F induces an equivalence $(\operatorname{Ker}F)^{\perp} \xrightarrow{\sim} \mathcal{B}$.

(2) $\Rightarrow$ (3): Note that $\operatorname{Im}G \subseteq (\operatorname{Ker}F)^{\perp}$. Thus G induces a functor $\mathcal{B} \to (\operatorname{Ker}F)^{\perp}$ which is a right adjoint of the equivalence $(\operatorname{Ker}F)^{\perp} \to \mathcal{B}$. Now one uses that an adjoint of an equivalence is again an equivalence.

(3) $\Rightarrow$ (4): An equivalence is fully faithful.

(4) $\Rightarrow$ (1): Use Lemma 1.4.2.

Observe that (3) implies $(\operatorname{Ker}F)^{\perp} = \operatorname{Im}G$. In particular, $\operatorname{Ker}F \subseteq {}^{\perp}(\operatorname{Im}G)$. The other inclusion follows from the isomorphism $\operatorname{Hom}_{\mathcal{B}}(FA, B) \cong \operatorname{Hom}_{\mathcal{A}}(A, GB)$. □

The next result characterizes the fact that the quotient functor admits a right adjoint.

Proposition 1.4.4. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a Serre subcategory. Then the following are equivalent:*

(1) *The quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint $\mathcal{A}/\mathcal{C} \to \mathcal{A}$.*
(2) *Every object A in $\mathcal{A}$ fits into an exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow \bar{A} \longrightarrow A'' \longrightarrow 0 \tag{1.4.1}$$

such that $A', A'' \in \mathcal{C}$ and $\bar{A} \in \mathcal{C}^{\perp}$.
(3) *The quotient functor induces an equivalence $\mathcal{C}^{\perp} \xrightarrow{\sim} \mathcal{A}/\mathcal{C}$.*

In that case the functor $\mathcal{A} \to \mathcal{C}$ sending A to A' is a right adjoint of the inclusion $\mathcal{C} \to \mathcal{A}$, and the functor $\mathcal{A} \to \mathcal{C}^{\perp}$ sending A to $\bar{A}$ is a left adjoint of the inclusion $\mathcal{C}^{\perp} \to \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): We apply Proposition 1.4.3. Suppose that the quotient functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint G. The functor F inverts the adjunction morphism $\eta_A\colon A \to G(FA) = \bar{A}$, since $FG \cong \mathrm{Id}_{\mathcal{A}/\mathcal{C}}$ by Lemma 1.4.2. The exactness of F then implies that $A' = \operatorname{Ker} \eta_A$ and $A'' = \operatorname{Coker} \eta_A$ belong to $\mathcal{C}$. The object $\bar{A}$ belongs to $\operatorname{Im} G = \mathcal{C}^{\perp}$ by construction.

(2) $\Rightarrow$ (3): The quotient functor induces a fully faithful functor $\mathcal{C}^{\perp} \to \mathcal{A}/\mathcal{C}$ by Lemma 1.4.1. This functor is an equivalence, because each object A in $\mathcal{A}/\mathcal{C}$ is isomorphic to one in its image via the isomorphism $A \xrightarrow{\sim} \bar{A}$.

(3) $\Rightarrow$ (1): Choose a quasi-inverse $G\colon \mathcal{A}/\mathcal{C} \to \mathcal{C}^{\perp}$ of the equivalence $\mathcal{C}^{\perp} \to \mathcal{A} \xrightarrow{F} \mathcal{A}/\mathcal{C}$. For each A in $\mathcal{A}$ and B in $\mathcal{A}/\mathcal{C}$, there are bijections

$$\operatorname{Hom}_{\mathcal{A}}(A, GB) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(FA, F(GB)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(FA, B).$$

The first map is bijective by Lemma 1.4.1 and the second is bijective because $FG \cong \mathrm{Id}_{\mathcal{A}/\mathcal{C}}$. Thus G is right adjoint to the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$.

For any C in $\mathcal{C}$, the induced map $\operatorname{Hom}_{\mathcal{A}}(C, A') \to \operatorname{Hom}_{\mathcal{A}}(C, A)$ is bijective. Therefore sending A to A' provides a right adjoint of the inclusion $\mathcal{C} \to \mathcal{A}$. On the other hand, for any B in $\mathcal{C}^{\perp}$, the induced map $\operatorname{Hom}_{\mathcal{A}}(\bar{A}, B) \to \operatorname{Hom}_{\mathcal{A}}(A, B)$ is bijective. Therefore sending A to $\bar{A}$ provides a left adjoint of the inclusion $\mathcal{C}^{\perp} \to \mathcal{A}$. ◻

Remark 1.4.5. The objects A' and A'' occurring in (1.4.1) represent certain functors defined on $\mathcal{C}$. We have

$$\operatorname{Hom}_{\mathcal{A}}(-, A)|_{\mathcal{C}} \cong \operatorname{Hom}_{\mathcal{C}}(-, A') \quad \text{and} \quad \operatorname{Ext}^1_{\mathcal{A}}(-, A/A')|_{\mathcal{C}} \cong \operatorname{Hom}_{\mathcal{C}}(-, A''),$$

where A' is viewed as a subobject of A.

1.5. **Global dimension.** Let $\mathcal{A}$ be an abelian category. For a pair of objects A, B and $n \geq 1$, let $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$ denote the group of extensions in the sense of Yoneda. Set $\operatorname{Ext}^0_{\mathcal{A}}(A, B) = \operatorname{Hom}_{\mathcal{A}}(A, B)$ and $\operatorname{Ext}^n_{\mathcal{A}}(A, B) = 0$ for $n < 0$. Note that there are composition maps

$$\operatorname{Ext}^n_{\mathcal{A}}(A, B) \times \operatorname{Ext}^m_{\mathcal{A}}(B, C) \longrightarrow \operatorname{Ext}^{n+m}_{\mathcal{A}}(A, C)$$

for all $n, m \in \mathbb{Z}$. The *projective dimension* of an object A is by definition

$$\operatorname{proj.dim} A = \inf\{n \geq 0 \mid \operatorname{Ext}^{n+1}_{\mathcal{A}}(A, -) = 0\}.$$

Dually, one defines the *injective dimension* $\operatorname{inj.dim} A$.

Lemma 1.5.1. *Let $\mathcal{A}$ be an abelian category. For A in $\mathcal{A}$ and $n > 0$ the following are equivalent:*

(1) $\operatorname{Ext}^n_{\mathcal{A}}(A, -)$ *is right exact.*
(2) $\operatorname{Ext}^{n+1}_{\mathcal{A}}(A, -) = 0$.

(3) $\operatorname{Ext}^m_{\mathcal{A}}(A,-)=0$ *for all* $m>n$.
(4) $\operatorname{proj.dim} A \le n$.

Proof. It suffices to show that (1) and (2) are equivalent; the rest is straightforward. We use the long exact sequence for $\operatorname{Ext}^*_{\mathcal{A}}(A,-)$.

(1) $\Rightarrow$ (2): Fix an element $\xi \in \operatorname{Ext}^{n+1}_{\mathcal{A}}(A,B)$ which is represented by an exact sequence

$$0 \longrightarrow B \longrightarrow E_{n+1} \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow A \longrightarrow 0.$$

Let C be the image of $E_{n+1}\to E_n$ and write $\xi=\xi''\xi'$ as the composite of extensions $\xi'\in \operatorname{Ext}^n_{\mathcal{A}}(A,C)$ and $\xi''\in \operatorname{Ext}^1_{\mathcal{A}}(C,B)$. For the connecting morphism $\delta\colon \operatorname{Ext}^n_{\mathcal{A}}(A,C)\to \operatorname{Ext}^{n+1}_{\mathcal{A}}(A,B)$ induced by ξ'', we have $\delta=0$ since $\operatorname{Ext}^n_{\mathcal{A}}(A,-)$ is right exact. Thus $\xi=\delta(\xi')=0$.

(2) $\Rightarrow$ (1): Clear. □

The *global dimension* of $\mathcal{A}$ is by definition the smallest integer $n\ge 0$ such that $\operatorname{Ext}^{n+1}_{\mathcal{A}}(-,-)=0$. As usual, the dimension is infinite if such a number n does not exist. We denote this dimension by $\operatorname{gl.dim}\mathcal{A}$ and observe that it is equal to $\sup\{\operatorname{proj.dim} A \mid A\in\mathcal{A}\}$ and $\sup\{\operatorname{inj.dim} A \mid A\in\mathcal{A}\}$. The category $\mathcal{A}$ is called *hereditary* provided that $\operatorname{Ext}^2_{\mathcal{A}}(-,-)=0$.

For a right noetherian ring Λ, the global dimension of the module category $\operatorname{mod}\Lambda$ is called the (right) *global dimension* of Λ and denoted by $\operatorname{gl.dim}\Lambda$.

Example 1.5.2. (1) Let Λ be the ring of integers $\mathbb{Z}$ or the polynomial ring $k[x]$ over a field k. Then $\operatorname{mod}\Lambda$ is hereditary. More generally, $\operatorname{mod}\Lambda$ is hereditary if Λ is a Dedekind domain.

(2) For a field k and a quiver Γ, the category of representations $\operatorname{rep}(\Gamma,k)$ is hereditary.

(3) Let $\mathcal{A}$ be a hereditary abelian category and $\mathcal{C}$ a Serre subcategory. Then $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are again hereditary.

1.6. **Length categories.** Let $\mathcal{A}$ be an abelian category. An object A of $\mathcal{A}$ has *finite length* if there exists a finite chain of subobjects

$$0=A_0\subseteq A_1\subseteq\cdots\subseteq A_{n-1}\subseteq A_n=A$$

such that each quotient A_i/A_{i-1} is a simple object. Such a chain is called a *composition series* of A. A composition series is not necessarily unique but its length is an invariant of A by the Jordan-Hölder theorem; it is called the *length* of A and is denoted by $\ell(A)$. Note that an object has finite length if and only if it is both *artinian* (i.e. satisfies the descending chain condition on subobjects) and *noetherian* (i.e. satisfies the ascending chain condition on subobjects).

Every object of finite length decomposes essentially uniquely into a finite direct sum of indecomposable objects with local endomorphism rings. This follows from the Krull-Remak-Schmidt theorem.

The objects of finite length form a Serre subcategory of $\mathcal{A}$ which is denoted by $\mathcal{A}_0$. The abelian category $\mathcal{A}$ is called a *length category* if $\mathcal{A}=\mathcal{A}_0$.

Let $\mathcal{A}$ be a length category. The *Ext-quiver* or *Gabriel quiver* of $\mathcal{A}$ is a valued quiver $\Sigma=\Sigma(\mathcal{A})$ which is defined as follows. The set Σ_0 of vertices is a fixed set of representatives of the isomorphism classes of simple objects in $\mathcal{A}$. For a simple object S, let $\Delta(S)$ denote its endomorphism ring, which is a division ring. Observe that $\operatorname{Ext}^1_{\mathcal{A}}(S,T)$ carries a natural $\Delta(T)$-$\Delta(S)$-bimodule structure for each pair S,T in Σ_0. There is an arrow $S\to T$ with valuation $\delta_{S,T}=(s,t)$ in Σ if $\operatorname{Ext}^1_{\mathcal{A}}(S,T)\neq 0$ with $s=\dim_{\Delta(S)}\operatorname{Ext}^1_{\mathcal{A}}(S,T)$ and $t=\dim_{\Delta(T)^{\mathrm{op}}}\operatorname{Ext}^1_{\mathcal{A}}(S,T)$. We write $\delta_{S,T}=(0,0)$ if $\operatorname{Ext}^1_{\mathcal{A}}(S,T)=0$.

The following observation is easily proved.

Lemma 1.6.1. *A length category is connected if and only if its Ext-quiver is connected.* □

Example 1.6.2. (1) Let Λ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Then $(\operatorname{mod}\Lambda)_0$ equals the category of $\mathfrak{m}$-torsion modules and $\Lambda/\mathfrak{m}$ is the unique simple Λ-module.

(2) For a field k and a quiver Γ, the category of representations $\operatorname{rep}(\Gamma,k)$ is a length category. Suppose that Γ has no oriented cycle and no pair of parallel arrows. Then the Ext-quiver of $\operatorname{rep}(\Gamma,k)$ is isomorphic to Γ, with valuation $(1,1)$ for each arrow.

1.7. **Uniserial categories.** Let $\mathcal{A}$ be a length category. An object A is *uniserial* provided it has a unique composition series. Note that any non-zero uniserial object is indecomposable. Moreover, subobjects and quotient objects of uniserial objects are uniserial. The length category $\mathcal{A}$ is called *uniserial* provided that each indecomposable object is uniserial.

The following result characterizes uniserial categories in terms of their Ext-quivers.

Theorem 1.7.1 (Gabriel)**.** *A length category $\mathcal{A}$ is uniserial if and only if for each simple object S, we have*

$$\sum_{S'\in\Sigma_0}\dim_{\Delta(S')}\operatorname{Ext}^1_{\mathcal{A}}(S',S)\le 1 \text{ and } \sum_{S'\in\Sigma_0}\dim_{\Delta(S')^{\mathrm{op}}}\operatorname{Ext}^1_{\mathcal{A}}(S,S')\le 1. \tag{1.7.1}$$

The proof of this result requires some preparations and we begin with some notation. Let A be any object in $\mathcal{A}$. We denote by $\operatorname{rad} A$ the intersection of all its maximal subobjects and let $\operatorname{top} A = A/\operatorname{rad} A$. Analogously, $\operatorname{soc} A$ denotes the sum of all simple subobjects of A.

Lemma 1.7.2. *Let $\mathcal{A}$ be a length category and suppose that* (1.7.1) *holds for each simple object S. Let $\xi\colon 0\to A\to E\to S\to 0$ be a non-split extension such that A is uniserial and S is simple. Then we have the following:*

1. *Every epimorphism $A\to B\neq 0$ induces an isomorphism*
$$\operatorname{Ext}^1_{\mathcal{A}}(S,A)\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{A}}(S,B).$$
2. *The object E is uniserial.*
3. *Given any non-split extension ξ' in $\operatorname{Ext}^1_{\mathcal{A}}(S,A)$, there exists an isomorphism $\tau\colon A\to A$ such that $\xi'=\tau\xi$.*

Proof. (1) It is sufficient to consider the case $B=\operatorname{top} A$. Moreover, it is sufficient to show that the induced map $\operatorname{Ext}^1_{\mathcal{A}}(S,A)\to\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{top} A)$ is a monomorphism, since $\dim_{\Delta(S)}\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{top} A)\le 1$ by (1.7.1). We use the long exact sequence which is obtained from the short exact sequence $0\to A'\to A\to\operatorname{top} A\to 0$ by applying $\operatorname{Hom}_{\mathcal{A}}(S,-)$.

If $\operatorname{Ext}^1_{\mathcal{A}}(S,A')=0$, then the induced map $\operatorname{Ext}^1_{\mathcal{A}}(S,A)\to\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{top} A)$ is a monomorphism. Now assume that $\operatorname{Ext}^1_{\mathcal{A}}(S,A')\neq 0$. Using induction on the length, we have that $\operatorname{Ext}^1_{\mathcal{A}}(S,A')\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{top} A')\neq 0$. Next observe that $\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{top} A,\operatorname{top} A')\neq 0$ since A is uniserial. Thus (1.7.1) implies $S\cong\operatorname{top} A$, and a dimension argument shows that the connecting morphism $\operatorname{Hom}_{\mathcal{A}}(S,\operatorname{top} A)\to\operatorname{Ext}^1_{\mathcal{A}}(S,A')$ is an isomorphism. Thus from the long exact sequence we infer that the natural map $\operatorname{Ext}^1_{\mathcal{A}}(S,A)\to\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{top} A)$ is a monomorphism.

(2) It suffices to show that every proper subobject $U \subsetneq E$ is contained in A. Otherwise we have an induced extension $0 \to A \cap U \to U \to S \to 0$. Thus the inclusion $A \cap U \to A$ induces a non-zero map $\operatorname{Ext}^1_{\mathcal{A}}(S, A \cap U) \to \operatorname{Ext}^1_{\mathcal{A}}(S, A)$. Composing this with the isomorphism $\operatorname{Ext}^1_{\mathcal{A}}(S, A) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(S, A/A \cap U)$ from (1) gives a non-zero map $\operatorname{Ext}^1_{\mathcal{A}}(S, A \cap U) \to \operatorname{Ext}^1_{\mathcal{A}}(S, A/A \cap U)$ which is induced by the composite $A \cap U \to A \to A/A \cap U$. This is impossible.

(3) We use induction on the length of A. The case $\ell(A) = 1$ follows from the equality $\dim_{\Delta(A)^{\mathrm{op}}} \operatorname{Ext}^1_{\mathcal{A}}(S, A) = 1$. If $\ell(A) > 1$, choose a maximal subobject $A' \subsetneq A$ and let $\bar{A} = A/A'$. It follows from (1) that the canonical morphism $\pi \colon A \to \bar{A}$ induces an isomorphism $\operatorname{Ext}^1_{\mathcal{A}}(S, A) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(S, \bar{A})$ taking ξ to $\pi\xi$. There is an isomorphism $\bar{\tau} \colon \bar{A} \to \bar{A}$ such that $\pi\xi' = \bar{\tau}(\pi\xi)$ since $\ell(\bar{A}) = 1$. We claim that $\bar{\tau}$ extends to an isomorphism $\tau \colon A \to A$ satisfying $\pi\tau = \bar{\tau}\pi$. This implies $\pi\xi' = \bar{\tau}\pi\xi = \pi\tau\xi$, and therefore $\xi' = \tau\xi$. Thus it remains to construct τ. To this end consider the non-split extension $\mu \colon 0 \to A' \to A \to \bar{A} \to 0$. The induction hypothesis yields an isomorphism $\tau' \colon A' \to A'$ such that $\tau'(\mu\bar{\tau}) = \mu$ since $\ell(A') < \ell(A)$. This gives the isomorphism τ satisfying $\pi\tau = \bar{\tau}\pi$. □

Lemma 1.7.3. *Let $\mathcal{A}$ be a length category and suppose that* (1.7.1) *holds for each simple object S. For two uniserial objects A and B the following are equivalent:*

1. $A \cong B$.
2. $\operatorname{top} A \cong \operatorname{top} B$ *and* $\ell(A) = \ell(B)$.
3. $\operatorname{soc} A \cong \operatorname{soc} B$ *and* $\ell(A) = \ell(B)$.

Proof. The condition (1.7.1) is self-dual. Thus it suffices to show the equivalence (1) ⇔ (2). This equivalence follows from Lemma 1.7.2 using induction on the length $\ell(A)$. □

Proof of Theorem 1.7.1. Suppose first that $\mathcal{A}$ is uniserial. Choose a simple object S and assume that
$$\sum_{S' \in \Sigma_0} \dim_{\Delta(S')^{\mathrm{op}}} \operatorname{Ext}^1_{\mathcal{A}}(S, S') \geq 2.$$
Then there exists an extension $\xi \colon 0 \to S' \oplus S'' \to E \to S \to 0$ with $S', S'' \in \Sigma_0$ such that for each non-zero morphism $\theta \colon S' \oplus S'' \to T$ with $T \in \Sigma_0$, the induced extension $\theta\xi$ does not split. It is not difficult to check that E is indecomposable and has at least two different composition series. Thus E is not uniserial which is a contradiction.

Now assume that (1.7.1) holds for each simple object S and fix an indecomposable object A. We show by induction on $\ell(A)$ that A is uniserial. The case $\ell(A) = 1$ is clear. Thus we choose an exact sequence $\xi \colon 0 \to A' \to A \to S \to 0$ with S simple and fix a decomposition $A' = \bigoplus_{i=1}^{l} A_i$ into indecomposable objects. Note that each A_i is uniserial by our hypothesis. If $l = 1$, then A is uniserial by Lemma 1.7.2. Now assume $l > 1$. Denote by ξ_i the pushout of ξ along the projection $A' \to A_i$. Note that $\xi_i \neq 0$; otherwise A_i is isomorphic to a direct summand of A. Therefore $\operatorname{Ext}^1_{\mathcal{A}}(S, A_i) \neq 0$ for all i, and Lemma 1.7.2 implies $\operatorname{top} A_i \cong \operatorname{top} A_1$ for all i. Assume that $\ell(A_1) \geq \ell(A_2)$. Then we have an epimorphism $\pi \colon A_1 \to A_2$ by Lemma 1.7.3 which induces an isomorphism $\operatorname{Ext}^1_{\mathcal{A}}(S, A_1) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(S, A_2)$ by Lemma 1.7.2. Moreover, there exists an isomorphism $\tau \colon A_2 \to A_2$ such that $\pi\xi_1 = \tau\xi_2$. Consider the morphism $\phi \colon A' \to A_2$ with $\phi_1 = \pi$, $\phi_2 = -\tau$ and $\phi_i = 0$ for $2 < i \leq l$. We have $\phi\xi = 0$ by construction, and therefore A_2 is isomorphic to a direct summand of A. This is a contradiction. Thus A is uniserial. □

Let $\mathcal{A}$ be a uniserial category. Choose a complete set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$ and denote it by $\operatorname{ind}\mathcal{A}$. An object in $\operatorname{ind}\mathcal{A}$ with top S and length n is denoted by $S^{[n]}$. Analogously, we write $S_{[n]}$ for an object in $\operatorname{ind}\mathcal{A}$ with socle S and length n.

For each simple object S, we have a chain of monomorphisms

$$S = S_{[1]} \rightarrowtail S_{[2]} \rightarrowtail S_{[3]} \rightarrowtail \cdots$$

which is either finite or infinite. Dually, there is a chain of epimorphisms

$$\cdots \twoheadrightarrow S^{[3]} \twoheadrightarrow S^{[2]} \twoheadrightarrow S^{[1]} = S.$$

A morphism $\phi\colon A \to B$ in an additive category is called *irreducible* if ϕ is neither a split monomorphism nor a split epimorphism and if for any factorisation $\phi = \phi''\phi'$ the morphism ϕ' is a split monomorphism or ϕ'' is a split epimorphism.

Lemma 1.7.4. *Let $\mathcal{A}$ be a uniserial category. For a morphism $\phi\colon A \to B$ between indecomposable objects, the following are equivalent:*

1. *The morphism ϕ is irreducible.*
2. *The object $\operatorname{Ker}\phi \oplus \operatorname{Coker}\phi$ is simple.*
3. *There exists a simple object S and an integer n such that ϕ is, up to isomorphism, of the form $S_{[n]} \rightarrowtail S_{[n+1]}$ or $S^{[n+1]} \twoheadrightarrow S^{[n]}$.*

Proof. (1) $\Rightarrow$ (2): An irreducible morphism is either a monomorphism or an epimorphism. It suffices to discuss the case that ϕ is an epimorphism; the other case is dual. If $\ell(\operatorname{Ker}\phi) > 1$ and $S \subseteq \operatorname{Ker}\phi$ is a simple subobject, then ϕ can be written as composite $A \to A/S \to B$ of two proper epimorphisms. This is a contradiction, and therefore $\operatorname{Ker}\phi \oplus \operatorname{Coker}\phi$ is simple.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): It suffices to consider the morphism $\phi\colon S^{[n+1]} \twoheadrightarrow S^{[n]}$; the dual argument works for $S_{[n]} \rightarrowtail S_{[n+1]}$. Let $S^{[n+1]} \xrightarrow{\alpha} X \xrightarrow{\beta} S^{[n]}$ be a factorization and fix a decomposition $X = \bigoplus_i X_i$ into indecomposable objects. Then $\beta_{i_0}\alpha_{i_0}$ is an epimorphism for at least one index i_0. It follows from Lemma 1.7.3 that $X_{i_0} = S^{[m]}$ for some $m \geq n$. If $m = n$, then β_{i_0} is an isomorphism, and therefore β is a split epimorphism. Otherwise, we obtain a factorization $S^{[n+1]} \xrightarrow{\alpha_{i_0}} X_{i_0} \xrightarrow{\beta'_{i_0}} S^{[n+1]} \twoheadrightarrow S^{[n]}$ of the epimorphism $\beta_{i_0}\alpha_{i_0}$. It follows that $\beta'_{i_0}\alpha_{i_0}$ is an epimorphism and hence an isomorphism. Thus α is a split monomorphism. ☐

Remark 1.7.5. Let $\mathcal{A}$ be a uniserial category and S a simple object. Suppose there is a bound n such that $\ell(A) \leq n$ for each indecomposable object with $\operatorname{soc} A \cong S$. Then each indecomposable object A of length n with $\operatorname{soc} A \cong S$ is injective, since $\operatorname{Ext}^1_{\mathcal{A}}(T, A) = 0$ for every simple object T in $\mathcal{A}$ by Lemma 1.7.2.

Example 1.7.6. (1) Let Λ be a Dedekind domain. The finitely generated torsion modules over Λ form a uniserial category. We denote this category by $\operatorname{mod}_0 \Lambda$ because it coincides with the category $(\operatorname{mod}\Lambda)_0$ of finite length objects of $\operatorname{mod}\Lambda$. Let $\operatorname{Spec}\Lambda$ denote the set of prime ideals. The functor which takes a Λ-module M to the family of localizations $(M_{\mathfrak{p}})_{\mathfrak{p}\in\operatorname{Spec}\Lambda}$ induces an equivalence

$$\operatorname{mod}_0 \Lambda \xrightarrow{\sim} \coprod_{0\neq\mathfrak{p}\in\operatorname{Spec}\Lambda} \operatorname{mod}_0(\Lambda_{\mathfrak{p}}).$$

(2) Let k be a field and $P \in k[x]$ an irreducible polynomial. For each $n > 0$, the finitely generated $k[x]/(P^n)$-modules form a uniserial category with a unique simple object.

(3) Let k be a field and Γ a quiver having no oriented cycle. The category of representations $\operatorname{rep}(\Gamma, k)$ is uniserial if and only if for each vertex x of Γ, there is at most one arrow starting at x and at most one arrow ending at x.

1.8. **Serre duality.** Let k be a commutative ring. A category $\mathcal{A}$ is *k-linear* if each morphism set is a k-module and the composition maps are k-bilinear. A functor between k-linear categories is *k-linear* provided that the induced maps between the morphism sets are k-linear. A k-linear category is *Hom-finite* if each morphism set is a k-module of finite length. Suppose now that $\mathcal{A}$ is a k-linear abelian category. Then the extension groups are naturally modules over k, and $\mathcal{A}$ is called *Ext-finite* if $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$ is of finite length over k for all A, B in $\mathcal{A}$ and $n \geq 0$.

Fix a field k and a Hom-finite k-linear abelian category $\mathcal{A}$. The category $\mathcal{A}$ is said to satisfy *Serre duality*[2] if there exists an equivalence $\tau\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ with functorial k-linear isomorphisms

$$D \operatorname{Ext}^1_{\mathcal{A}}(A, B) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(B, \tau A)$$

for all A, B in $\mathcal{A}$, where $D = \operatorname{Hom}_k(-, k)$ denotes the standard k-duality. The functor τ is called *Serre functor* or *Auslander-Reiten translation*. Note that a Serre functor is k-linear and essentially unique provided it exists; this follows from Yoneda's lemma.

Recall that $\mathcal{A}_0$ denotes the full subcategory consisting of all finite length objects in $\mathcal{A}$. Denote by $\mathcal{A}_+$ the full subcategory consisting of all objects A in $\mathcal{A}$ satisfying $\operatorname{Hom}_{\mathcal{A}}(A_0, A) = 0$ for all A_0 in $\mathcal{A}_0$.

Proposition 1.8.1. *Let $\mathcal{A}$ be a Hom-finite k-linear abelian category and suppose $\mathcal{A}$ admits a Serre functor τ. Then the following holds:*

(1) *The category $\mathcal{A}$ is hereditary.*
(2) *The category $\mathcal{A}$ has no non-zero projective or injective objects.*
(3) *A noetherian object A has a unique maximal subobject A_0 of finite length. Moreover, A_0 is a direct summand of A and A/A_0 belongs to $\mathcal{A}_+$.*
(4) *For each indecomposable object A in $\mathcal{A}$, there is an almost split sequence*[3] *$0 \to \tau A \to E \to A \to 0$.*
(5) *For each object A in $\mathcal{A}$, we have $A^{\perp} = {}^{\perp}\tau A$.*

Proof. (1) For each object A, the functor $\operatorname{Ext}^1_{\mathcal{A}}(A, -)$ is right exact. Thus the category $\mathcal{A}$ is hereditary by Lemma 1.5.1.

(2) Let A be a projective object. Then $\operatorname{Hom}_{\mathcal{A}}(-, \tau A) \cong D \operatorname{Ext}^1_{\mathcal{A}}(A, -) = 0$. Thus $\tau A = 0$ and therefore $A = 0$. The dual argument works for injective objects.

(3) Choose a maximal subobject A_0 of finite length. Then A/A_0 belongs to $\mathcal{A}_+$, and every finite length subobject of A is contained in A_0. In particular, A_0 is unique. We have $\operatorname{Ext}^1_{\mathcal{A}}(A/A_0, A_0) = 0$ by Serre duality, and therefore A_0 is a direct summand of A.

(4) Let A be an indecomposable object. The endomorphism ring $\operatorname{End}_{\mathcal{A}}(A)$ is local and we denote by $\mathfrak{m}$ its maximal ideal. Choose any non-zero k-linear map $\omega\colon \operatorname{End}_{\mathcal{A}}(A) \to k$ such that ω vanishes on $\mathfrak{m}$. The map ω corresponds via Serre duality to a non-split short exact sequence $\xi\colon 0 \to \tau A \to E \to A \to 0$. We claim that ξ is an almost split sequence. For this one needs to show that each morphism $\alpha\colon A' \to A$ factors through the morphism $E \to A$, provided that α is not a split epimorphism. Thus one needs to show that $\xi\alpha = 0$. The element $\xi\alpha$ corresponds

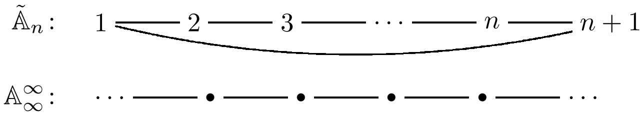
[2]This is the appropriate notion of Serre duality for hereditary abelian categories. Higher dimensional analogues involving $D \operatorname{Ext}^n_{\mathcal{A}}(A, -)$ appear in algebraic geometry; see also §3.4.

[3]For the notion of an almost split sequence, we refer to [1].

via Serre duality to $\omega\alpha$ which sends $\phi \in \operatorname{Hom}_{\mathcal{A}}(A, A')$ to $\omega(\alpha\phi)$. Thus $\xi\alpha = 0$, since $\alpha\phi$ belongs to $\mathfrak{m}$.

(5) This is clear from the definitions. □

Next observe that a Serre functor τ on $\mathcal{A}$ restricts to a Serre functor on the subcategory $\mathcal{A}_0$ of finite length objects. The following result describes the structure of a length category with Serre duality. Let us recall the shape of the relevant diagrams.



Proposition 1.8.2. *Let $\mathcal{A}$ be a Hom-finite k-linear length category and suppose $\mathcal{A}$ admits a Serre functor τ. Then $\mathcal{A}$ is uniserial. The category $\mathcal{A}$ admits a unique decomposition $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ into connected uniserial categories with Serre duality, where the index set equals the set of τ-orbits of simple objects in $\mathcal{A}$. The Ext-quiver of each $\mathcal{A}_i$ is either of type $\mathbb{A}_\infty^\infty$ (with linear orientation) or of type $\tilde{\mathbb{A}}_n$ (with cyclic orientation).*

Proof. We apply the criterion of Theorem 1.7.1 to show that $\mathcal{A}$ is uniserial. To this end fix a simple object S. Then $\operatorname{Ext}^1_{\mathcal{A}}(S, S') \cong D \operatorname{Hom}_{\mathcal{A}}(S', \tau S) \neq 0$ for some $S' \in \Sigma_0$ if and only if $S' \cong \tau S$. Moreover, $\dim_{\Delta(S)} \operatorname{Ext}^1_{\mathcal{A}}(S, \tau S) = 1$. Thus the category $\mathcal{A}$ is uniserial.

The structure of the Ext-quiver of $\mathcal{A}$ follows from the condition (1.7.1). The Serre functor acts on Σ_0 and the set of τ-orbits $I = \Sigma_0/\tau$ is the index set of the decomposition $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ into connected components; see Lemma 1.6.1. The Ext-quiver of $\mathcal{A}_i$ is of type $\mathbb{A}_\infty^\infty$ if the corresponding τ-orbit is infinite. Otherwise, the Ext-quiver of $\mathcal{A}_i$ is of type $\tilde{\mathbb{A}}_n$ where $n+1$ equals the cardinality of the τ-orbit. □

Let $\mathcal{A}$ be a Hom-finite k-linear length category and suppose $\mathcal{A}$ admits a Serre functor. Then a complete set of representatives of the isomorphism classes of indecomposable objects of $\mathcal{A}$ is given by $\{S^{[n]} \mid S \in \Sigma_0, n \geq 1\}$ and also by $\{S_{[n]} \mid S \in \Sigma_0, n \geq 1\}$; see Remark 1.7.5.

Example 1.8.3. Let k be a field and Γ a quiver of extended Dynkin type $\tilde{\mathbb{A}}_n$ with cyclic orientation. Denote by $\mathcal{A} = \operatorname{rep}_0(\Gamma, k)$ the full subcategory of $\operatorname{rep}(\Gamma, k)$ consisting of all nilpotent representations. Then $\mathcal{A}$ satisfies Serre duality and the Ext-quiver of $\mathcal{A}$ is isomorphic to Γ, with valuation $(1,1)$ for each arrow. Note that the Serre functor on $\mathcal{A}$ has order $n+1$ and every simple object has endomorphism algebra k. In fact, a connected Hom-finite k-linear length category with Serre duality satisfying these properties is equivalent to $\operatorname{rep}_0(\Gamma, k)$.

2. Derived categories

To each abelian category is associated its derived category. This section provides a brief introduction. We present the definition and discuss two cases where one has a convenient description: If the abelian category is hereditary, then each complex is isomorphic to its cohomology. On the other hand, if there are enough projective objects, then one can compute morphisms in the derived category by passing to the homotopy category of projective objects.

2.1. **Categories of complexes.** Let $\mathcal{A}$ be an additive category. A *cochain complex* in $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots$$

such that $d^n d^{n-1} = 0$ for all $n \in \mathbb{Z}$. We denote by $\mathbf{C}(\mathcal{A})$ the category of cochain complexes, where a morphism $\phi\colon X \to Y$ between cochain complexes consists of morphisms $\phi^n\colon X^n \to Y^n$ with $d_Y^n \phi^n = \phi^{n+1} d_X^n$ for all $n \in \mathbb{Z}$.

A *chain complex* in $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Any chain complex may be viewed as a cochain complex by changing its indices, and vice versa. Thus we often confuse both concepts and simply use the term *complex*.

A morphism $\phi\colon X \to Y$ between complexes is *null-homotopic* if there are morphisms $\rho^n\colon X^n \to Y^{n-1}$ such that $\phi^n = d_Y^{n-1}\rho^n + \rho^{n+1} d_X^n$ for all $n \in \mathbb{Z}$. The null-homotopic morphisms form an *ideal* $\mathcal{N}$ in $\mathbf{C}(\mathcal{A})$, that is, for each pair X, Y of complexes a subgroup

$$\mathcal{N}(X,Y) \subseteq \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(X,Y)$$

such that any composition $\psi\phi$ of morphisms in $\mathbf{C}(\mathcal{A})$ belongs to $\mathcal{N}$ if ϕ or ψ belongs to $\mathcal{N}$. The *homotopy category* $\mathbf{K}(\mathcal{A})$ is the quotient of $\mathbf{C}(\mathcal{A})$ with respect to this ideal. Thus

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) = \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(X,Y)/\mathcal{N}(X,Y)$$

for every pair of complexes X, Y.

Now let $\mathcal{A}$ be an abelian category. The *cohomology* of a complex X in degree n is by definition $H^n X = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$, and each morphism $\phi\colon X \to Y$ of complexes induces a morphism $H^n\phi\colon H^n X \to H^n Y$. The morphism ϕ is a *quasi-isomorphism* if $H^n\phi$ is an isomorphism for all $n \in \mathbb{Z}$. Note that two morphisms $\phi, \psi\colon X \to Y$ induce the same morphism $H^n\phi = H^n\psi$, if $\phi - \psi$ is null-homotopic.

The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathbf{K}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. To be precise, one defines

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})[\mathrm{qis}^{-1}]$$

as the localization of $\mathbf{K}(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. The full subcategory consisting of objects that are isomorphic to a complex X such that $X^n = 0$ for almost all $n \in \mathbb{Z}$ is denoted by $\mathbf{D}^b(\mathcal{A})$.

An object A in $\mathcal{A}$ is identified with the complex

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree zero, and this complex is also denoted by A. Given any complex X in $\mathcal{A}$ and $p \in \mathbb{Z}$, we denote by $X[p]$ the shifted complex with

$$X[p]^n = X^{n+p} \quad \text{and} \quad d_{X[p]}^n = (-1)^p d_X^{n+p}.$$

This operation induces an isomorphism $\mathbf{D}(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathcal{A})$ and is called *shift*.

The derived category $\mathbf{D}(\mathcal{A})$ is an additive category with some additional structure: it is a triangulated category in the sense of Verdier [25]. For instance, any exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ induces an exact triangle $A \to B \to C \to A[1]$ in $\mathbf{D}(\mathcal{A})$.

Given two abelian categories $\mathcal{A}$ and $\mathcal{A}'$, a functor $F\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}')$ is by definition a *derived equivalence* if it is an *equivalence of triangulated categories*, that is, F is an equivalence, there is a functorial isomorphism $(FX)[1] \cong F(X[1])$ for each X in $\mathbf{D}(\mathcal{A})$, and F preserves exact triangles.

The following statement justifies the study of derived categories.

Proposition 2.1.1. *Let A, B be objects in $\mathcal{A}$. Then*

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[n]) \quad \textit{for all} \quad n \in \mathbb{Z}.$$

Proof. For the case that $\mathcal{A}$ has enough injectives or enough projectives, see [26, Corollary 10.7.5]. For the general case, see [25, III.3]. □

2.2. **Hereditary abelian categories.** Let $\mathcal{A}$ be a hereditary abelian category, that is, $\mathrm{Ext}^2_{\mathcal{A}}(-,-)$ vanishes. In this case, there is an explicit description of all objects and morphisms in $\mathbf{D}^b(\mathcal{A})$ via the ones in $\mathcal{A}$. Every complex X is completely determined by its cohomology because there is an isomorphism between X and the following complex with trivial differential.

$$\cdots \longrightarrow H^{n-1}X \xrightarrow{0} H^nX \xrightarrow{0} H^{n+1}X \longrightarrow \cdots$$

To construct this isomorphism, note that the vanishing of $\mathrm{Ext}^2_{\mathcal{A}}(H^nX, -)$ implies the existence of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^nX & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Im}\, d^{n-1} & \longrightarrow & \mathrm{Ker}\, d^n & \longrightarrow & H^nX & \longrightarrow & 0
\end{array}$$

with exact rows. We obtain the following commutative diagram.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^nX & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

The vertical morphisms yield two morphisms in $\mathbf{D}^b(\mathcal{A})$. The upper one is a quasi-isomorphism, and the lower one induces a cohomology isomorphism in degree n. This yields for each $n \in \mathbb{Z}$ a morphism $(H^nX)[-n] \to X$ in $\mathbf{D}^b(\mathcal{A})$ and therefore the following description of X.

Lemma 2.2.1. *Let $\mathcal{A}$ be a hereditary abelian category and X a complex in $\mathcal{A}$. In $\mathbf{D}^b(\mathcal{A})$ there is a (non-canonical) isomorphism*

$$\coprod_{n \in \mathbb{Z}} (H^nX)[-n] \xrightarrow{\sim} X.$$

Proof. The morphism is a quasi-isomorphism by construction. □

For an abelian category $\mathcal{A}$ one defines its *repetitive category* $\bigsqcup_{n\in\mathbb{Z}} \mathcal{A}[n]$ as the additive closure of the union of disjoint copies $\mathcal{A}[n]$ of $\mathcal{A}$ with morphisms

$$\mathrm{Hom}(A,B) = \mathrm{Ext}^{q-p}_{\mathcal{A}}(A,B) \quad \text{for} \quad A \in \mathcal{A}[p],\, B \in \mathcal{A}[q]$$

and composition given by the Yoneda product of extensions. It follows from Proposition 2.1.1 that the family of functors $\mathcal{A}[n] \to \mathbf{D}^b(\mathcal{A})$ $(n \in \mathbb{Z})$ sending an object A to $A[n]$ induces a fully faithful functor

$$\bigsqcup_{n \in \mathbb{Z}} \mathcal{A}[n] \longrightarrow \mathbf{D}^b(\mathcal{A}).$$

Corollary 2.2.2. *The canonical functor $\bigsqcup_{n\in\mathbb{Z}} \mathcal{A}[n] \to \mathbf{D}^b(\mathcal{A})$ is an equivalence for any hereditary abelian category $\mathcal{A}$.* □

2.3. **Abelian categories with enough projectives.** We describe the derived category of an abelian category $\mathcal{A}$ in terms of its projective objects. The crucial observation is the following.

Lemma 2.3.1. *Let X, Y be a pair of complexes in $\mathcal{A}$. Suppose that each X^n is projective and $X^n = 0$ for $n \gg 0$. Then the map $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y)$ is bijective.*

Proof. See for example [26, Corollary 10.4.7]. □

Let $\operatorname{Proj}\mathcal{A}$ denote the full subcategory of $\mathcal{A}$ consisting of all objects that are projective. Denote by $\mathbf{K}^{-,b}(\operatorname{Proj}\mathcal{A})$ the full subcategory of complexes X in $\mathbf{K}(\operatorname{Proj}\mathcal{A})$ such that $X^n = 0$ for $n \gg 0$ and $H^nX = 0$ for almost all $n \in \mathbb{Z}$. One says that $\mathcal{A}$ has *enough projectives* if each object in $\mathcal{A}$ is the quotient of some projective object.

Proposition 2.3.2. *Let $\mathcal{A}$ be an abelian category having enough projectives. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ induces an equivalence*

$$\mathbf{K}^{-,b}(\operatorname{Proj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Proof. The functor $F\colon \mathbf{K}^{-,b}(\operatorname{Proj}\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is by definition the identity on objects, and F is fully faithful by Lemma 2.3.1. It is clear that each object in the image of F is isomorphic to one in $\mathbf{D}^b(\mathcal{A})$. To show that each complex X in $\mathbf{D}^b(\mathcal{A})$ is isomorphic to one in the image of F, we use induction on

$$\ell(X) = \operatorname{card}\{n \in \mathbb{Z} \mid X^n \neq 0\}.$$

Note that each bounded complex $X \neq 0$ fits into an exact triangle $X' \to X'' \to X \to X'[1]$ such that $\ell(X') = 1$ and $\ell(X'') = \ell(X) - 1$. If $\ell(X) = 1$, say $X^n \neq 0$, then $X \cong F(P[-n])$ where P denotes a projective resolution of X^n. Such a resolution exists since $\mathcal{A}$ has enough projectives. If $\ell(X) > 1$, then the induction hypothesis implies that the morphism $X' \to X''$ is up to isomorphism of the form $F\phi$ for some morphism $\phi\colon P' \to P''$ in $\mathbf{K}^{-,b}(\operatorname{Proj}\mathcal{A})$. Completing the morphism ϕ to an exact triangle $P' \to P'' \to P \to P'[1]$ shows that X belongs to $\operatorname{Im} F$ since $X \cong FP$. □

3. Tilting theory

Tilting provides a method to relate a category of coherent sheaves to a category of linear representations. For instance, a result of Beilinson [2] establishes for the category $\operatorname{coh}\mathbb{P}^n_k$ of coherent sheaves on the projective n-space over a field k an equivalence of derived categories

$$\mathbf{R}\operatorname{Hom}(T,-)\colon \mathbf{D}^b(\operatorname{coh}\mathbb{P}^n_k) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\operatorname{End}(T))$$

via a tilting object T in $\operatorname{coh}\mathbb{P}^n_k$.[4]

In this section let k be a field and $\mathcal{A}$ a k-linear abelian category that is Ext-finite. We show that each tilting object T in $\mathcal{A}$ provides an equivalence of derived categories

$$\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T,-)\colon \mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\operatorname{End}_{\mathcal{A}}(T))$$

as in the example above. The principal reference for this result is [14], even though the proof given here is somewhat more direct, avoiding the formalism of torsion pairs and t-structures.

[4] Except for $n = 1$, the object $T = \mathcal{O}(0) \oplus \cdots \oplus \mathcal{O}(n)$ is not a tilting object in the strict sense of these notes; see Proposition 5.8.1.

3.1. **Tilting objects.** Fix an object T in $\mathcal{A}$. The object T is called *tilting object*, provided that

1. $\operatorname{proj.dim} T \leq 1$,
2. $\operatorname{Ext}^1_{\mathcal{A}}(T,T)=0$, and
3. $\operatorname{Hom}_{\mathcal{A}}(T,A)=0=\operatorname{Ext}^1_{\mathcal{A}}(T,A)$ implies $A=0$ for each object A in $\mathcal{A}$.

A morphism $T'\to A$ in $\mathcal{A}$ is called *right T-approximation* of A if it induces an epimorphism $\operatorname{Hom}_{\mathcal{A}}(T,T')\to\operatorname{Hom}_{\mathcal{A}}(T,A)$ and T' belongs to $\operatorname{add} T$. An exact sequence $0\to A\to B\to T'\to 0$ is called *universal T-extension* of A if it induces an epimorphism $\operatorname{Hom}_{\mathcal{A}}(T,T')\to\operatorname{Ext}^1_{\mathcal{A}}(T,A)$ and T' belongs to $\operatorname{add} T$. Such approximations and extensions exist for all A in $\mathcal{A}$, since $\mathcal{A}$ is Ext-finite. Finally, set

$$\mathcal{T}(T)=\{A\in\mathcal{A}\mid \operatorname{Ext}^1_{\mathcal{A}}(T,A)=0\}.$$

Lemma 3.1.1. *Let $T\in\mathcal{A}$ be a tilting object. Then the following holds:*

1. *Let $\pi\colon T'\to A$ be a right T-approximation. Then $\operatorname{Ker}\pi$ is in $\mathcal{T}(T)$, $\operatorname{Hom}_{\mathcal{A}}(T,\operatorname{Coker}\pi)=0$, and $\operatorname{Ext}^1_{\mathcal{A}}(T,A)\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{A}}(T,\operatorname{Coker}\pi)$.*
2. *Let $0\to A\to B\to T'\to 0$ be a universal T-extension. Then $B\in\mathcal{T}(T)$.*
3. *The objects in $\mathcal{T}(T)$ are precisely the factor objects of objects in $\operatorname{add} T$.*

Proof. (1) Write the sequence $0\to A'\to T'\xrightarrow{\pi} A\to A''\to 0$ as composite of two exact sequences $0\to A'\to T'\to\bar{A}\to 0$ and $0\to\bar{A}\to A\to A''\to 0$. Then apply $\operatorname{Hom}_{\mathcal{A}}(T,-)$ to both sequences.

(2) Apply $\operatorname{Hom}_{\mathcal{A}}(T,-)$ to the sequence $0\to A\to B\to T'\to 0$.

(3) Clearly, each factor of an object in $\operatorname{add} T$ belongs to $\mathcal{T}(T)$. For the other implication one uses (1). □

3.2. **A derived equivalence.** Let T be an object in $\mathcal{A}$ and $\Lambda=\operatorname{End}_{\mathcal{A}}(T)$. We consider the functor

$$\operatorname{Hom}_{\mathcal{A}}(T,-)\colon\mathcal{A}\longrightarrow\operatorname{mod}\Lambda.$$

This functor induces an equivalence $\operatorname{add} T\xrightarrow{\sim}\operatorname{proj}\Lambda$ and admits a left adjoint

$$-\otimes_\Lambda T\colon\ \operatorname{mod}\Lambda\longrightarrow\mathcal{A}.$$

Given a Λ-module M with projective presentation $P_1\to P_0\to M\to 0$, the cokernel of the corresponding morphism $T_1\to T_0$ in $\operatorname{add} T$ is by definition $M\otimes_\Lambda T$. For $i>0$, denote by

$$\operatorname{Tor}_i^\Lambda(-,T)\colon\ \operatorname{mod}\Lambda\longrightarrow\mathcal{A}$$

the i-th left derived functor of $-\otimes_\Lambda T$ and set

$$\mathcal{Y}(T)=\{M\in\operatorname{mod}\Lambda\mid\operatorname{Tor}_1^\Lambda(M,T)=0\}.$$

Lemma 3.2.1. *Let $T\in\mathcal{A}$ be a tilting object. Then $\operatorname{Tor}_i^\Lambda(-,T)=0$ for $i>1$.*

Proof. Let $M\in\operatorname{mod}\Lambda$ and choose a projective resolution

$$\cdots\longrightarrow P_2\xrightarrow{\delta_2}P_1\xrightarrow{\delta_1}P_0\xrightarrow{\delta_0}M\longrightarrow 0.$$

Apply $-\otimes_\Lambda T$ and set $Z_i=\operatorname{Ker}(\delta_i\otimes_\Lambda T)$. The induced morphism $\bar{\delta}_{i+1}\colon P_{i+1}\otimes_\Lambda T\to Z_i$ is a right T-approximation for $i>0$, and therefore Z_i belongs to $\mathcal{T}(T)$ for $i>1$ by Lemma 3.1.1. Thus $\bar{\delta}_{i+1}$ is an epimorphism for $i>1$, and this implies $\operatorname{Tor}_i^\Lambda(-,T)=0$. □

Lemma 3.2.2. *Let $T\in\mathcal{A}$ be a tilting object. Then $\operatorname{Hom}_{\mathcal{A}}(T,-)$ and $-\otimes_\Lambda T$ restrict to equivalences between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.*

Proof. Fix objects $A \in \mathcal{T}(T)$ and $M \in \mathcal{Y}(T)$. We need to show that the adjunction morphisms

$$\operatorname{Hom}_{\mathcal{A}}(T, A) \otimes_\Lambda T \xrightarrow{\theta_A} A \quad \text{and} \quad M \xrightarrow{\eta_M} \operatorname{Hom}_{\mathcal{A}}(T, M \otimes_\Lambda T)$$

are invertible.

Choose an exact sequence

$$\xi\colon \quad \cdots \longrightarrow T_2 \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} A \longrightarrow 0$$

such that the induced morphism $T_i \to \operatorname{Im} \delta_i$ is a right T-approximation of $\operatorname{Im} \delta_i$ for each $i \geq 0$. Such a sequence exists by Lemma 3.1.1.

The functor $\operatorname{Hom}_{\mathcal{A}}(T, -)$ sends the sequence ξ to a projective resolution of the Λ-module $\operatorname{Hom}_{\mathcal{A}}(T, A)$. Applying then $- \otimes_\Lambda T$ gives back ξ, that is, the adjunction morphism θ_A is invertible. Moreover, $\operatorname{Hom}_{\mathcal{A}}(T, A)$ belongs to $\mathcal{Y}(T)$.

Now choose an exact sequence $\pi\colon 0 \to M' \to P \to M \to 0$ such that P is projective. Note that M' belongs to $\mathcal{Y}(T)$ by Lemma 3.2.1. The sequence $\pi \otimes_\Lambda T$ is exact since $M \in \mathcal{Y}(T)$, and the sequence $\operatorname{Hom}_{\mathcal{A}}(T, \pi \otimes_\Lambda T)$ is exact since $M' \otimes_\Lambda T$ belongs to $\mathcal{T}(T)$. Thus there is the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_{M'}} & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle \eta_M} & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(T, M' \otimes_\Lambda T) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(T, P \otimes_\Lambda T) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(T, M \otimes_\Lambda T) & \longrightarrow & 0
\end{array}$$

The morphism η_P is an isomorphism and it follows that η_M is an epimorphism for all M in $\mathcal{Y}(T)$. In particular, $\eta_{M'}$ is an epimorphism. Using the snake lemma, it follows that η_M is an isomorphism. □

Let A be an object in $\mathcal{A}$. An $\operatorname{add} T$*-resolution* of A is by definition a complex

$$Q\colon \quad \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

together with a quasi-isomorphism $Q \to A$ such that each Q_n belongs to $\operatorname{add} T$.

Lemma 3.2.3. *Let $T \in \mathcal{A}$ be a tilting object and $Q \to A$ an $\operatorname{add} T$-resolution of an object $A \in \mathcal{A}$. Then*

$$H^n \operatorname{Hom}_{\mathcal{A}}(Q, B) \cong \operatorname{Ext}^n_{\mathcal{A}}(A, B)$$

for all $B \in \mathcal{T}(T)$ and $n \geq 0$.

Proof. Use induction on n and dimension shifting. □

Lemma 3.2.4. *Let $T \in \mathcal{A}$ be a tilting object. Then the functor $- \otimes_\Lambda T$ induces an isomorphism*

$$\operatorname{Ext}^n_\Lambda(M, N) \xrightarrow{\sim} \operatorname{Ext}^n_{\mathcal{A}}(M \otimes_\Lambda T, N \otimes_\Lambda T)$$

for all M, N in $\mathcal{Y}(T)$ and $n \geq 0$.

Proof. Choose a projective resolution $P \to M$ of M. Note that $N \cong \operatorname{Hom}_{\mathcal{A}}(T, N \otimes_\Lambda T)$ by Lemma 3.2.2, since N belongs to $\mathcal{Y}(T)$. Then we obtain the following sequence of isomorphisms.

$$\begin{aligned}
\operatorname{Ext}^n_\Lambda(M, N) &\cong H^n \operatorname{Hom}_\Lambda(P, N) \\
&\cong H^n \operatorname{Hom}_\Lambda(P, \operatorname{Hom}_{\mathcal{A}}(T, N \otimes_\Lambda T)) \\
&\cong H^n \operatorname{Hom}_{\mathcal{A}}(P \otimes_\Lambda T, N \otimes_\Lambda T) \\
&\cong \operatorname{Ext}^n_{\mathcal{A}}(M \otimes_\Lambda T, N \otimes_\Lambda T)
\end{aligned}$$

The last isomorphism follows from Lemma 3.2.3, since $P \otimes_\Lambda T \to M \otimes_\Lambda T$ is an $\operatorname{add} T$-resolution. □

For a tilting object T in $\mathcal{A}$, let us define the derived functor

$$- \otimes^{\mathbf{L}}_\Lambda T \colon \mathbf{D}^b(\operatorname{mod} \Lambda) \xrightarrow{\sim} \mathbf{K}^{-,b}(\operatorname{proj} \Lambda) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

by taking a complex P of projective Λ-modules with bounded cohomology to $P \otimes_\Lambda T$; see Proposition 2.3.2. The cohomology of $P \otimes_\Lambda T$ is bounded, since $\operatorname{Tor}_i^\Lambda(-,T) = 0$ for $i > 1$ by Lemma 3.2.1.

Theorem 3.2.5 (Happel-Reiten-Smalø)**.** *Let $\mathcal{A}$ be a k-linear abelian category that is Ext-finite. Let T be a tilting object in $\mathcal{A}$ and $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$. Then the functor*

$$- \otimes^{\mathbf{L}}_\Lambda T \colon \mathbf{D}^b(\operatorname{mod} \Lambda) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

is an equivalence of triangulated categories and its right adjoint $\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T,-)$ is a quasi-inverse.

We do not give the formal definition of the derived functor $\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T,-)$; all we use is the fact that it is a right adjoint of $- \otimes^{\mathbf{L}}_\Lambda T$.

Proof. Set $F_T = - \otimes^{\mathbf{L}}_\Lambda T$. We identify objects in $\operatorname{mod}\Lambda$ and $\mathcal{A}$ with complexes that are concentrated in degree zero. For instance, $F_T M = M \otimes_\Lambda T$ for each M in $\mathcal{Y}(T)$.

We need to show that for each pair of complexes X, Y in $\operatorname{mod}\Lambda$, the induced map

$$\phi_{X,Y} \colon \operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(X,Y) \longrightarrow \operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(F_T X, F_T Y)$$

is bijective. Set

$$\ell(X) = \operatorname{card}\{n \in \mathbb{Z} \mid X_n \neq 0\}$$

and note that each bounded complex $X \neq 0$ fits into an exact triangle $X' \to X \to X'' \to X'[1]$ such that $\ell(X') = \ell(X) - 1$ and $\ell(X'') = 1$.

Using the five lemma and induction on $\ell(X) + \ell(Y)$, one shows that $\phi_{X,Y}$ is bijective. The case $\ell(X) = \ell(Y) = 1$ follows from Lemma 3.2.4. To be precise, one uses that each Λ-module M fits into an exact sequence $0 \to M' \to P \to M \to 0$ with M', P in $\mathcal{Y}(T)$, which yields an exact triangle $M' \to P \to M \to M'[1]$ in $\mathbf{D}^b(\operatorname{mod}\Lambda)$.

Next we show that each object in $\mathbf{D}^b(\mathcal{A})$ is isomorphic to one in the image of F_T. In fact, it suffices to show that each object in $\mathcal{A}$ belongs to the essential image $\operatorname{Im} F_T$, since $\operatorname{Im} F_T$ is a triangulated subcategory and $\mathbf{D}^b(\mathcal{A})$ is generated (as a triangulated category) by the objects from $\mathcal{A}$.

It follows from Lemma 3.1.1 that each object A in $\mathcal{A}$ fits into an exact triangle $A \to B \to C \to A[1]$ with B, C in $\mathcal{T}(T)$. On the other hand, each A in $\mathcal{T}(T)$ belongs to $\operatorname{Im} F_T$, since $A \cong F_T(\operatorname{Hom}_{\mathcal{A}}(T,A))$ by Lemma 3.2.2. □

Example 3.2.6. (1) Let T, T' be two objects in $\mathcal{A}$ with $\operatorname{add} T = \operatorname{add} T'$. Then T is a tilting object if and only if T' is a tilting object.

(2) Let k be a field and Λ a finite dimensional k-algebra. Then any free Λ-module of finite rank is a tilting object in $\operatorname{mod}\Lambda$.

(3) Let k be a field and $\Lambda = k\Gamma$ the path algebra of a finite quiver Γ without oriented cycles. For each vertex $i \in \Gamma_0$ let e_i denote the corresponding idempotent. Fix a vertex i_0 which is not a sink and consider the following short exact sequence

$$0 \longrightarrow e_{i_0}\Lambda \longrightarrow \bigoplus_{\alpha\colon i_0 \to i} e_i\Lambda \longrightarrow T_{i_0} \longrightarrow 0$$

where the direct sum runs over all arrows starting at i_0 and each morphism $e_{i_0}\Lambda \to e_i\Lambda$ is given by multiplication with the corresponding arrow $\alpha\colon i_0 \to i$. Set $T_i = e_i\Lambda$ for each vertex $i \neq i_0$. Then $T = \bigoplus_{i\in\Gamma_0} T_i$ is a tilting object of $\operatorname{mod}\Lambda$.

3.3. **Grothendieck groups.** Let $\mathcal{A}$ be an abelian category. Denote by $F(\mathcal{A})$ the free abelian group generated by the isomorphism classes of objects in $\mathcal{A}$. Let $F_0(\mathcal{A})$ be the subgroup generated by $[X]-[Y]+[Z]$ for all exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. The *Grothendieck group* $K_0(\mathcal{A})$ of $\mathcal{A}$ is by definition the factor group $F(\mathcal{A})/F_0(\mathcal{A})$.

Lemma 3.3.1. *Let $\mathcal{A}$ be a length category. Then $K_0(\mathcal{A})$ is a free abelian group and the isomorphism classes of simple objects in $\mathcal{A}$ form a basis.*

Proof. Let X be an object in $\mathcal{A}$ and $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ a composition series. Then $[X] = [X_1/X_0] + \cdots + [X_n/X_{n-1}]$ in $K_0(\mathcal{A})$. The Jordan-Hölder theorem implies the uniqueness of this expression. □

Let $\mathcal{T}$ be a triangulated category. Denote by $F(\mathcal{T})$ the free abelian group generated by the isomorphism classes of objects in $\mathcal{T}$. Let $F_0(\mathcal{T})$ be the subgroup generated by $[X]-[Y]+[Z]$ for all exact triangles $X \to Y \to Z \to X[1]$ in $\mathcal{T}$. The *Grothendieck group* $K_0(\mathcal{T})$ of $\mathcal{T}$ is by definition the factor group $F(\mathcal{T})/F_0(\mathcal{T})$.

Lemma 3.3.2. *Let $\mathcal{A}$ be an abelian category. The inclusion $\mathcal{A} \to \mathbf{D}^b(\mathcal{A})$ induces an isomorphism $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathbf{D}^b(\mathcal{A}))$.*

Proof. Each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ induces an exact triangle $X \to Y \to Z \to X[1]$ in $\mathbf{D}^b(\mathcal{A})$. This gives a morphism $K_0(\mathcal{A}) \to K_0(\mathbf{D}^b(\mathcal{A}))$. The inverse map sends the class $[X]$ of a complex X to $\sum_{n\in\mathbb{Z}}(-1)^n[H^nX]$. □

3.4. **Serre duality.** Let k be a field and Λ a finite dimensional k-algebra. We denote by $D = \operatorname{Hom}_k(-,k)$ the standard k-duality.

The *Nakayama functor* $\nu = D\operatorname{Hom}_\Lambda(-,\Lambda)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ identifies the category of projective Λ-modules with the category of injective Λ-modules. Note that

$$D\operatorname{Hom}_\Lambda(P,-) \cong \operatorname{Hom}_\Lambda(-,\nu P)$$

for every finitely generated projective Λ-module P, since both functors are left exact and agree on Λ. This isomorphism induces for every bounded complex X of finitely generated projective Λ-modules a sequence of natural isomorphisms

$$\begin{aligned} D\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(X,-) &\cong D\operatorname{Hom}_{\mathbf{K}^b(\operatorname{mod}\Lambda)}(X,-) \\ &\cong \operatorname{Hom}_{\mathbf{K}^b(\operatorname{mod}\Lambda)}(-,\nu X) \\ &\cong \operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(-,\nu X), \end{aligned} \tag{3.4.1}$$

where the first and the last isomorphism follow from Lemma 2.3.1.

A Hom-finite k-linear triangulated category $\mathcal{T}$ is said to satisfy *Serre duality* if there exists an equivalence $\tau\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ of triangulated categories with functorial k-linear isomorphisms

$$D\operatorname{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(Y,\tau X)$$

for all X, Y in $\mathcal{T}$. The functor τ is called a *Serre functor.* Note that a Serre functor is k-linear and essentially unique provided it exists; this follows from Yoneda's lemma.

Proposition 3.4.1. *Let Λ be a finite dimensional k-algebra. Then $\mathbf{D}^b(\operatorname{mod}\Lambda)$ satisfies Serre duality if and only if Λ has finite global dimension.*

Proof. If the global dimension of Λ is finite, then every bounded complex in $\operatorname{mod}\Lambda$ is quasi-isomorphic to a bounded complex of finitely generated projective Λ-modules. Thus Serre duality for $\mathbf{D}^b(\operatorname{mod}\Lambda)$ follows from the isomorphism (3.4.1). The converse follows immediately from Lemmas 3.4.2 and 3.4.3 below. □

Lemma 3.4.2. *Let $\mathcal{A}$ be an abelian category and $X, Y \in \mathbf{D}^b(\mathcal{A})$. Then the following holds:*

(1) $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, Y[n]) = 0$ *for* $n \ll 0$.
(2) $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, Y[n]) = 0$ *for* $n \gg 0$, *if $\mathcal{A}$ has finite global dimension.*

Proof. Use induction on $\ell(X) + \ell(Y)$, where $\ell(Z) = \operatorname{card}\{n \in \mathbb{Z} \mid Z^n \neq 0\}$ for any complex Z. The case $\ell(X) = 1 = \ell(Y)$ is clear, since $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(A, B[n]) \cong \operatorname{Ext}^n_{\mathcal{A}}(A, B)$ for all objects A, B in $\mathcal{A}$; see Proposition 2.1.1. □

Lemma 3.4.3. *Let Λ be a finite dimensional algebra and $S_1, \dots, S_r$ a set of representatives of the isomorphism classes of simple Λ-modules. Then* $\operatorname{gl.dim}\Lambda \leq n$ *if and only if* $\operatorname{Ext}^{n+1}_{\Lambda}(S_i, S_j) = 0$ *for all i, j.*

Proof. Use that each Λ-module M has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = M$ such that M_i/M_{i-1} is semisimple for all i. □

Lemma 3.4.4. *Let $\mathcal{A}$ be an Ext-finite k-linear abelian category and suppose there exists a tilting object T. If $\mathcal{A}$ has finite global dimension, then* $\operatorname{End}_{\mathcal{A}}(T)$ *has finite global dimension.*

Proof. Let $\Lambda = \operatorname{End}_{\Lambda}(T)$. The functor $\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T, -)$ provides an equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$ of triangulated categories by Theorem 3.2.5. Thus we have $\operatorname{Ext}^n_{\Lambda}(S, T) = 0$ for $n \gg 0$ and each pair S, T of simple Λ-modules by Lemma 3.4.2. It follows from Lemma 3.4.3 that the global dimension of Λ is finite. □

Proposition 3.4.5. *Let $\mathcal{A}$ be a k-linear abelian category that is Ext-finite and admits a tilting object. Then the following are equivalent:*

(1) *The category $\mathcal{A}$ is hereditary and has no non-zero projective object.*
(2) *The category $\mathcal{A}$ satisfies Serre duality.*

Proof. (1) ⇒ (2): Let T be the tilting object and $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$. Then Λ has finite global dimension by Lemma 3.4.4, and therefore $\mathbf{D}^b(\operatorname{mod}\Lambda)$ has Serre duality by Proposition 3.4.1. There is an equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$ by Theorem 3.2.5, and this yields a Serre functor $\nu\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$.

Now let A, B be objects in $\mathcal{A}$ and view them as complexes concentrated in degree zero. Then

$$D\operatorname{Ext}^1_{\mathcal{A}}(A, B) \cong \operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(B, \nu A[-1]),$$

and it remains to show that $H^i(\nu A[-1])$ for all $i \neq 0$. Any complex X in $\mathcal{A}$ is quasi-isomorphic to $\coprod_{i\in\mathbb{Z}}(H^iX)[-i]$ since $\mathcal{A}$ is hereditary; see Lemma 2.2.1. Assume that A is indecomposable. Then $\nu A[-1] \cong \bar{A}[d]$ for some $d \in \mathbb{Z}$ and some object $\bar{A}$ in $\mathcal{A}$. We claim that $d = 0$. First observe that $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(B, \bar{A}[d]) \neq 0$ for some object B, since A is non-projective. Thus $d = 0$ or $d = 1$. The case $d = 1$ is impossible since $\operatorname{Ext}^2_{\mathcal{A}}(A, -) = 0$. Thus $\nu A[-1]$ is concentrated in degree zero.

(2) ⇒ (1): See Proposition 1.8.1. □

3.5. **The Euler form.** Let k be a field and $\mathcal{A}$ a k-linear abelian category. Suppose that $\mathcal{A}$ is Ext-finite and of finite global dimension. The *Euler form* associated to $\mathcal{A}$ is by definition the bilinear form $K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$ with

$$\langle [A], [B] \rangle = \sum_{n \geq 0} (-1)^n \dim_k \operatorname{Ext}^n_{\mathcal{A}}(A, B).$$

Suppose that $K_0(\mathcal{A})$ is a free abelian group of finite rank and fix a basis $b_1, \ldots, b_r$. The *discriminant* of the Euler form is then by definition the determinant of the matrix $(\langle b_i, b_j \rangle)_{i,j}$ and we denote it by $\operatorname{disc}\langle -, - \rangle$. Note that this value does not depend on the choice of the basis since the matrix is defined over $\mathbb{Z}$.

Lemma 3.5.1. *Let Λ be a finite dimensional k-algebra of finite global dimension. Then the Euler form associated to* $\operatorname{mod} \Lambda$ *is non-degenerate.*

Proof. Set $\mathcal{A} = \operatorname{mod} \Lambda$. Let $S_1, \ldots, S_r$ be representatives of the isomorphism classes of simple Λ-modules and choose a projective cover $P_i \to S_i$ for each i. Then $[P_1], \ldots, [P_r]$ form a basis of $K_0(\mathcal{A})$, since each S_i has a finite projective resolution. Let $x = \sum_i \alpha_i [P_i]$ be a non-zero element of $K_0(\mathcal{A})$ and pick an index j such that $\alpha_j \neq 0$. Then $\langle x, [S_j] \rangle = \alpha_j \dim_k \operatorname{Hom}_\Lambda(P_j, S_j) \neq 0$. Thus $\langle -, - \rangle$ is non-degenerate. $\square$

Let $\mathcal{T}$ be a k-linear triangulated category. Suppose that $\mathcal{T}$ is Hom-finite and that $\operatorname{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ for each pair of objects X, Y and $|n| \gg 0$. The *Euler form* associated to $\mathcal{T}$ is by definition the bilinear form $K_0(\mathcal{T}) \times K_0(\mathcal{T}) \to \mathbb{Z}$ with

$$\langle [X], [Y] \rangle = \sum_{n \in \mathbb{Z}} (-1)^n \dim_k \operatorname{Hom}_{\mathcal{T}}(X, Y[n]).$$

It is clear from these definitions that the isomorphism $\phi \colon K_0(\mathcal{A}) \to K_0(\mathbf{D}^b(\mathcal{A}))$ from Lemma 3.3.2 is an *isometry*, that is,

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \text{for all} \quad x, y \in K_0(\mathcal{A}).$$

Now let $\mathcal{A}$ and $\mathcal{B}$ be k-linear abelian categories that are Ext-finite and of finite global dimension. A k-linear equivalence $F \colon \mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{B})$ of triangulated categories induces an isometry $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{B})$ which is defined by the commutativity of the following diagram.

$$\begin{array}{ccc} K_0(\mathcal{A}) & \xrightarrow{\quad\sim\quad} & K_0(\mathcal{B}) \\ \downarrow{\scriptstyle \phi_{\mathcal{A}}} & & \downarrow{\scriptstyle \phi_{\mathcal{B}}} \\ K_0(\mathbf{D}^b(\mathcal{A})) & \xrightarrow{K_0(F)} & K_0(\mathbf{D}^b(\mathcal{B})) \end{array}$$

This has the following consequence.

Proposition 3.5.2. *Let $\mathcal{A}$ be a k-linear abelian category that is Ext-finite and of finite global dimension. Suppose that $\mathcal{A}$ has a tilting object. Then the Grothendieck group $K_0(\mathcal{A})$ is free of finite rank and the Euler form associated to $\mathcal{A}$ is non-degenerate.*

Proof. We identify $K_0(\mathcal{A})$ with $K_0(\operatorname{mod} \Lambda)$, where $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$ for a tilting object T in $\mathcal{A}$. Then the Grothendieck group $K_0(\mathcal{A})$ is free of finite rank by Lemma 3.3.1, and the Euler form is non-degenerate by Lemma 3.5.1. $\square$

The following examples show that the existence of a tilting object is an essential assumption for the Euler form to be non-degenerate.

Example 3.5.3. Let $\mathcal{A}$ be a k-linear length category that is Hom-finite and satisfies Serre duality. Suppose that the Grothendieck group has finite rank, and let $S_1, \dots, S_n$ be a representative set of the simple objects. Then $\langle x, [S_i]\rangle = 0$ for $x = \sum_j [S_j]$ and all i. Thus the Euler form is degenerate.

Example 3.5.4. Let E be a smooth elliptic curve over some algebraically closed field. Then the category of coherent sheaves on E is hereditary and satisfies Serre duality, but the corresponding Euler form is degenerate. In fact, any two simple sheaves S, T satisfy $\langle -, [S]\rangle = \langle -, [T]\rangle$, but $[S] \neq [T]$ if S and T are concentrated in different points of E. Indeed, $[S] \neq [T]$ follows from [15, Chap. II, Exercise 6.11] and the fact that the set of closed points is naturally identified with a subset of the Picard group [15, Chap. IV, Example 1.3.7].

Next we collect some further properties of the Grothendieck group and its Euler form.

Lemma 3.5.5. *Let $\mathcal{A}$ be a k-linear abelian category that is hereditary, Ext-finite, and has a non-degenerate Euler form. Suppose also that $[A] \neq 0$ for each non-zero object A in $\mathcal{A}$. Then an object T is a tilting object if and only if* $\operatorname{Ext}^1_{\mathcal{A}}(T,T) = 0$ *and the classes of the indecomposable direct summands of T generate $K_0(\mathcal{A})$.*

Proof. Suppose first that T is a tilting object with $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$. The isomorphism $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\operatorname{mod}\Lambda)$ identifies (the classes of) the indecomposable direct summands of T with the indecomposable projective Λ-modules. Now one uses that the indecomposable projective Λ-modules generate $K_0(\operatorname{mod}\Lambda)$; see the proof of Lemma 3.5.1

Conversely, suppose that $\operatorname{Ext}^1_{\mathcal{A}}(T,T) = 0$ and that the indecomposable direct summands of T generate $K_0(\mathcal{A})$. Then there exists for any non-zero object A in $\mathcal{A}$ some indecomposable direct summand T' of T such that $\langle [T'], [A]\rangle \neq 0$. Thus $\operatorname{Ext}^*_{\mathcal{A}}(T, A) \neq 0$, and it follows that T is a tilting object. □

A full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is called *exact abelian* if $\mathcal{B}$ is an abelian category and the inclusion functor is exact.

Lemma 3.5.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ an exact abelian subcategory such that the inclusion admits an exact left adjoint. Let $\mathcal{C} = {}^{\perp}\mathcal{B}$. Then $K_0(\mathcal{A}) = K_0(\mathcal{B}) \oplus K_0'(\mathcal{C})$, where $K_0'(\mathcal{C})$ denotes the image of the canonical map $K_0(\mathcal{C}) \to K_0(\mathcal{A})$.*

Let $i \colon \mathcal{B} \to \mathcal{A}$ be the inclusion and i_λ its left adjoint. Observe that $\mathcal{C} = \operatorname{Ker} i_\lambda$ is a Serre subcategory of $\mathcal{A}$ by Proposition 1.4.3. Thus the inclusion $\mathcal{C} \to \mathcal{A}$ induces a linear map $K_0(\mathcal{C}) \to K_0(\mathcal{A})$.

Proof. We have $i_\lambda i \cong \operatorname{Id}_{\mathcal{B}}$ and therefore $K_0(i)$ identifies $K_0(\mathcal{B})$ with a direct summand of $K_0(\mathcal{A})$. The kernel of $K_0(i_\lambda)$ equals $K_0'(\mathcal{C})$, since there is an exact sequence $0 \to A' \to A \to i i_\lambda A \to A'' \to 0$ for each object A in $\mathcal{A}$ with A', A'' in $\mathcal{C}$; see Proposition 1.4.4. □

The following lemma describes more specifically the term $K_0'(\mathcal{C})$ in the decomposition $K_0(\mathcal{A}) \cong K_0(\mathcal{B}) \oplus K_0'(\mathcal{C})$.

Lemma 3.5.7. *Let $\mathcal{A}$ be a k-linear abelian category that is Ext-finite and of finite global dimension. Suppose that $K_0(\mathcal{A})$ is free of finite rank. Let $\mathcal{B}$ be an exact abelian subcategory such that the inclusion admits an exact left adjoint and ${}^{\perp}\mathcal{B}$ is equivalent to* $\operatorname{mod}\Delta$ *for some division ring Δ. Then $K_0(\mathcal{A}) \cong K_0(\mathcal{B}) \oplus \mathbb{Z}$ and*

$$\operatorname{disc}\langle -,-\rangle_{\mathcal{A}} = \dim_k \Delta \cdot \operatorname{disc}\langle -,-\rangle_{\mathcal{B}}.$$

Proof. Let $i\colon \mathcal{B} \to \mathcal{A}$ be the inclusion and denote by i_λ its left adjoint. Then ${}^\perp\mathcal{B} = \operatorname{Ker} i_\lambda = \operatorname{add} S$ for some simple object S with $\Delta \cong \operatorname{End}_\mathcal{A}(S)$; see Proposition 1.4.3. Lemma 3.5.6 implies that $K_0(\mathcal{A}) = K_0(\mathcal{B}) \oplus \mathbb{Z}[S]$. Observe that $n[S] \neq 0$ for $n \neq 0$ in $\mathbb{Z}$, since $\langle n[S], [S]\rangle \neq 0$. Thus $K_0(\mathcal{A}) \cong K_0(\mathcal{B}) \oplus \mathbb{Z}$. The formula for $\operatorname{disc}\langle -,-\rangle_\mathcal{A}$ follows since $\langle [S], [B]\rangle = 0$ for every object B in $\mathcal{B}$. □

4. Expansions of abelian categories

In this section we introduce the concept of expansion and contraction for abelian categories.[5] Roughly speaking, an expansion is a fully faithful and exact functor $\mathcal{B} \to \mathcal{A}$ between abelian categories that admits an exact left adjoint and an exact right adjoint. In addition one requires the existence of simple objects S_λ and S_ρ in $\mathcal{A}$ such that $S_\lambda^\perp = \mathcal{B} = {}^\perp S_\rho$, where $\mathcal{B}$ is viewed as a full subcategory of $\mathcal{A}$. In fact, these simple objects are related by an exact sequence $0 \to S_\rho \to S \to S_\lambda \to 0$ in $\mathcal{A}$ such that S is a simple object in $\mathcal{B}$. In terms of the Ext-quivers of $\mathcal{A}$ and $\mathcal{B}$, the expansion $\mathcal{B} \to \mathcal{A}$ turns the vertex S into an arrow $S_\lambda \to S_\rho$. On the other hand, $\mathcal{B}$ is a contraction of $\mathcal{A}$ in the sense that the left adjoint of $\mathcal{B} \to \mathcal{A}$ identifies S_ρ with S, whereas the right adjoint identifies S_λ with S.

In the following we use the term 'expansion' but there are interesting situations where 'contraction' yields a more appropriate point of view. So one should think of a process having two directions that are opposite to each other.

Further material about expansions can be found in [6].

4.1. **Left and right expansions.** Let $\mathcal{A}$ be an abelian category. Recall that a full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called exact abelian if $\mathcal{B}$ is an abelian category and the inclusion functor is exact.

Now let $i\colon \mathcal{B} \to \mathcal{A}$ be a fully faithful and exact functor between abelian categories. It is convenient to identify $\mathcal{B}$ with the essential image of i, which means that $\mathcal{B}$ is an exact abelian subcategory of $\mathcal{A}$. We call the functor i a *left expansion* if the following conditions are satisfied:

(1) The functor $\mathcal{B} \to \mathcal{A}$ admits an exact left adjoint.
(2) The category ${}^\perp\mathcal{B}$ is equivalent to $\operatorname{mod}\Delta$ for some division ring Δ.
(3) $\operatorname{Ext}^2_\mathcal{A}(A, B) = 0$ for all $A, B \in {}^\perp\mathcal{B}$.

The functor $\mathcal{B} \to \mathcal{A}$ is called *right expansion* if the dual conditions are satisfied.

Lemma 4.1.1. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a left expansion of abelian categories. Denote by i_λ its left adjoint and set $\mathcal{C} = \operatorname{Ker} i_\lambda$.*

(1) *The category $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ satisfying $\mathcal{C} = {}^\perp\mathcal{B}$ and $\mathcal{C}^\perp = \mathcal{B}$.*
(2) *The composite $\mathcal{B} \xrightarrow{i} \mathcal{A} \xrightarrow{\text{can}} \mathcal{A}/\mathcal{C}$ is an equivalence and the left adjoint i_λ induces a quasi-inverse $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{B}$.*
(3) *$\operatorname{Ext}^n_\mathcal{B}(i_\lambda A, B) \cong \operatorname{Ext}^n_\mathcal{A}(A, iB)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $n \geq 0$.*

Proof. Part (1) and (2) follow from Proposition 1.4.3. It remains to prove (3). The case $n = 0$ is clear. For $n \geq 1$, the isomorphism sends a class $[\xi]$ in $\operatorname{Ext}^n_\mathcal{B}(i_\lambda A, B)$ to $[(i\xi).\eta_A]$ in $\operatorname{Ext}^n_\mathcal{A}(A, iB)$, where $\eta_A\colon A \to ii_\lambda(A)$ is the unit of the adjoint pair and $(i\xi).\eta_A$ denotes the pullback of $i\xi$ along η_A. □

An object S in $\mathcal{A}$ is called *localizable* if the following conditions are satisfied:

(1) The object S is simple.
(2) $\operatorname{Hom}_\mathcal{A}(S, A)$ and $\operatorname{Ext}^1_\mathcal{A}(S, A)$ are of finite length over $\operatorname{End}_\mathcal{A}(S)$ for all $A \in \mathcal{A}$.

[5] The authors are indebted to Claus Michael Ringel for suggesting the terms 'expansion' and 'contraction'.

(3) $\operatorname{Ext}^1_{\mathcal{A}}(S,S)=0$ and $\operatorname{Ext}^2_{\mathcal{A}}(S,A)=0$ for all $A\in\mathcal{A}$.

An object S is *colocalizable* if the dual conditions are satisfied.

Lemma 4.1.2. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ an exact abelian subcategory. Then the following are equivalent:*

(1) *The inclusion $\mathcal{B}\to\mathcal{A}$ is a left expansion.*
(2) *There exists a localizable object $S\in\mathcal{A}$ such that $S^\perp=\mathcal{B}$.*

Proof. (1) ⇒ (2): Let S be an indecomposable object in ${}^\perp\mathcal{B}$. Then S is a simple object and $\operatorname{Ext}^1_{\mathcal{A}}(S,S)=0$ since ${}^\perp\mathcal{B}=\operatorname{add}S$ is semisimple. For each object A in $\mathcal{A}$, we use the natural exact sequence (1.4.1)

$$0\longrightarrow A'\longrightarrow A\xrightarrow{\eta_A}\bar{A}\longrightarrow A''\longrightarrow 0$$

with $A',A''\in{}^\perp\mathcal{B}$ and $\bar{A}\in\mathcal{B}$. This sequence induces the following isomorphisms.

$$\operatorname{Hom}_{\mathcal{A}}(S,A')\xrightarrow{\sim}\operatorname{Hom}_{\mathcal{A}}(S,A)$$
$$\operatorname{Ext}^1_{\mathcal{A}}(S,A)\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{A}}(S,\operatorname{Im}\eta_A)\xleftarrow{\sim}\operatorname{Hom}_{\mathcal{A}}(S,A'')$$

Here we use the condition $\operatorname{Ext}^2_{\mathcal{A}}(S,S)=0$. It follows that $\operatorname{Hom}_{\mathcal{A}}(S,A)$ and $\operatorname{Ext}^1_{\mathcal{A}}(S,A)$ are of finite length over $\operatorname{End}_{\mathcal{A}}(S)$. Now observe that the functor sending A to $\operatorname{Hom}_{\mathcal{A}}(S,A'')$ is right exact. Thus $\operatorname{Ext}^2_{\mathcal{A}}(S,A)=0$ by Lemma 1.5.1. Finally, $S^\perp=\mathcal{B}$ follows from Proposition 1.4.3.

(2) ⇒ (1): A left adjoint i_λ of the inclusion $\mathcal{B}\to\mathcal{A}$ is constructed as follows. Fix an object A in $\mathcal{A}$. There exists an exact sequence $0\to A\to B\to S^n\to 0$ for some $n\geq 0$ such that $\operatorname{Ext}^1_{\mathcal{A}}(S,B)=0$ since $\operatorname{Ext}^1_{\mathcal{A}}(S,A)$ is of finite length over $\operatorname{End}_{\mathcal{A}}(S)$. Now choose a morphism $S^m\to B$ such that the induced map $\operatorname{Hom}_{\mathcal{A}}(S,S^m)\to\operatorname{Hom}_{\mathcal{A}}(S,B)$ is surjective and let $\bar{A}$ be its cokernel. It is easily checked that the composite $A\to B\to\bar{A}$ is the universal morphism into $S^\perp$. Thus we define $i_\lambda A=\bar{A}$.

Next observe that the kernel and cokernel of the adjunction morphism $A\to i_\lambda A$ belong to $\mathcal{C}=\operatorname{add}S$ for each object A in $\mathcal{A}$. Moreover, $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ since S is simple and $\operatorname{Ext}^1_{\mathcal{A}}(S,S)=0$. Thus we can apply Proposition 1.4.4 and infer that the composite $\mathcal{A}\xrightarrow{i_\lambda}\mathcal{C}^\perp\xrightarrow{\sim}\mathcal{A}/\mathcal{C}$ is the quotient functor with respect to $\mathcal{C}$. Therefore the left adjoint i_λ is exact. We have ${}^\perp\mathcal{B}=\mathcal{C}$ by Proposition 1.4.3, and $\operatorname{Hom}_{\mathcal{A}}(S,-)$ induces an equivalence $\mathcal{C}\xrightarrow{\sim}\operatorname{mod}\operatorname{End}_{\mathcal{A}}(S)$. Thus the inclusion $\mathcal{B}\to\mathcal{A}$ is a left expansion. □

4.2. **Expansions of abelian categories.** A fully faithful and exact functor $\mathcal{B}\to\mathcal{A}$ between abelian categories is by definition an *expansion* of abelian categories if the functor is a left and a right expansion.

Let $i\colon\mathcal{B}\to\mathcal{A}$ be an expansion of abelian categories. Then we identify $\mathcal{B}$ with the essential image of i. We denote by i_λ the left adjoint of i and by i_ρ the right adjoint of i. We choose an indecomposable object S_λ in ${}^\perp\mathcal{B}$ and an indecomposable object S_ρ in $\mathcal{B}^\perp$. Thus ${}^\perp\mathcal{B}=\operatorname{add}S_\lambda$ and $\mathcal{B}^\perp=\operatorname{add}S_\rho$. Finally, set $\bar{S}=i_\lambda(S_\rho)$.

An expansion $i\colon\mathcal{B}\to\mathcal{A}$ is called *split* if $\mathcal{B}^\perp={}^\perp\mathcal{B}$. If the expansion is non-split, then the exact sequences (1.4.1) for S_λ and S_ρ are of the form

$$0\to S_\rho\to ii_\lambda(S_\rho)\to S_\lambda^l\to 0\quad\text{and}\quad 0\to S_\rho^r\to ii_\rho(S_\lambda)\to S_\lambda\to 0 \tag{4.2.1}$$

for some integers $l,r\geq 1$. In Lemma 4.2.2, we see that $l=1=r$.

Lemma 4.2.1. *Let $\mathcal{B}\to\mathcal{A}$ be an expansion of abelian categories. Then the following are equivalent:*

(1) *The expansion $\mathcal{B}\to\mathcal{A}$ is split.*

(2) *$\mathcal{A} = \mathcal{B} \amalg \mathcal{C}$ for some Serre subcategory $\mathcal{C}$ of $\mathcal{A}$.*
(3) *$\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2): Take $\mathcal{C} = {}^{\perp}\mathcal{B} = \mathcal{B}^{\perp}$.

(2) $\Rightarrow$ (3): An object $A \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$ for all $B \in \mathcal{C}$. Thus $\mathcal{B}$ is closed under taking quotients and extensions. The dual argument shows that $\mathcal{B}$ is closed under taking subobjects.

(3) $\Rightarrow$ (1): If the expansion is non-split, then the sequences in (4.2.1) show that $\mathcal{B}$ is not a Serre subcategory. $\square$

Lemma 4.2.2. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) *The object $\bar{S} = i_\lambda(S_\rho)$ is a simple object in $\mathcal{B}$ and isomorphic to $i_\rho(S_\lambda)$.*
(2) *The functor i_λ induces an equivalence $\mathcal{B}^{\perp} \xrightarrow{\sim} \mathrm{add}\, \bar{S}$.*
(3) *The functor i_ρ induces an equivalence ${}^{\perp}\mathcal{B} \xrightarrow{\sim} \mathrm{add}\, \bar{S}$.*

Proof. (1) Let $\phi\colon i_\lambda(S_\rho) \to A$ be a non-zero morphism in $\mathcal{B}$. Adjunction takes this to a monomorphism $S_\rho \to A$ in $\mathcal{A}$ since S_ρ is simple. Applying i_λ gives back a morphism which is isomorphic to ϕ. This is a monomorphism since i_λ is exact. Thus $i_\lambda(S_\rho)$ is simple.

Now apply i_ρ to the first and i_λ to the second sequence in (4.2.1). Note that by adjunction $i_\lambda i \cong \mathrm{Id}_{\mathcal{B}} \cong i_\rho i$. Then we have

$$i_\lambda(S_\rho) \cong i_\rho(S_\lambda)^l \text{ and } i_\rho(S_\lambda) \cong i_\lambda(S_\rho)^r.$$

This implies $l = 1 = r$ and therefore $i_\lambda(S_\rho) \cong i_\rho(S_\lambda)$.

(2) We have a sequence of isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(S_\rho, S_\rho) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(S_\rho, i i_\lambda(S_\rho)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(i_\lambda(S_\rho), i_\lambda(S_\rho))$$

which takes a morphism ϕ to $i_\lambda \phi$. Thus i_λ induces an equivalence $\mathrm{add}\, S_\rho \xrightarrow{\sim} \mathrm{add}\, i_\lambda(S_\rho)$.

(3) Follows from (2) by duality. $\square$

An expansion $\mathcal{B} \to \mathcal{A}$ of abelian categories determines a division ring Δ such that ${}^{\perp}\mathcal{B}$ and $\mathcal{B}^{\perp}$ are equivalent to $\mathrm{mod}\, \Delta$; we call Δ the *associated division ring*.

Fix an expansion $i\colon \mathcal{B} \to \mathcal{A}$ with associated division ring Δ. We identify the perpendicular categories of $\mathcal{B}$ with $\mathrm{mod}\, \Delta$ via the equivalences ${}^{\perp}\mathcal{B} \xrightarrow{\sim} \mathrm{mod}\, \Delta \xleftarrow{\sim} \mathcal{B}^{\perp}$. There are inclusions $j\colon {}^{\perp}\mathcal{B} \to \mathcal{A}$ and $k\colon \mathcal{B}^{\perp} \to \mathcal{A}$ with adjoints j_ρ and k_λ. These functors yield the following diagram.

$$\mathcal{B} \;\begin{array}{c} \xleftarrow{\;i_\lambda\;} \\ \xrightarrow{\;i\;} \\ \xleftarrow[\;i_\rho\;]{} \end{array}\; \mathcal{A} \;\begin{array}{c} \xleftarrow{\;j\;} \\ \xrightarrow{\;j_\rho\;} \\ \xrightarrow{\;k_\lambda\;} \\ \xleftarrow[\;k\;]{} \end{array}\; \mathrm{mod}\, \Delta$$

Note that this diagram induces a recollement of triangulated categories [3]:

$$\mathbf{D}^b(\mathcal{B}) \;\begin{array}{c} \xleftarrow{\;\mathbf{D}^b(i_\lambda)\;} \\ \xrightarrow{\;\mathbf{D}^b(i)\;} \\ \xleftarrow[\;\mathbf{D}^b(i_\rho)\;]{} \end{array}\; \mathbf{D}^b(\mathcal{A}) \;\begin{array}{c} \xleftarrow{\;\mathbf{D}^b(j)\;} \\ \xrightarrow{\;\;} \\ \xleftarrow[\;\mathbf{D}^b(k)\;]{} \end{array}\; \mathbf{D}^b(\mathrm{mod}\, \Delta)$$

Indeed, the labeled functors are part of a recollement, and therefore the right adjoint of $\mathbf{D}^b(j)$ is isomorphic to the left adjoint of $\mathbf{D}^b(k)$, both of which are isomorphic to the quotient functor of $\mathbf{D}^b(\mathcal{A})$ with respect to the triangulated subcategory $\mathbf{D}^b(\mathcal{B})$.

4.3. **Simple objects.** Let $i\colon \mathcal{B} \to \mathcal{A}$ be an expansion. The left adjoint i_λ induces a bijection between the isomorphism classes of simple objects of $\mathcal{A}$ that are different from S_λ, and the isomorphism classes of simple objects of $\mathcal{B}$. On the other hand, all simple objects of $\mathcal{A}$ correspond to simple objects of $\mathcal{B}$ via i. All this is made precise in the next lemma.

Lemma 4.3.1. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be an expansion of abelian categories.*

(1) *If S is a simple object in $\mathcal{B}$ and $S \not\cong \bar{S}$, then iS is simple in $\mathcal{A}$ and $i_\lambda iS \cong S$.*
(2) *There is an exact sequence $0 \to S_\rho \to i\bar{S} \to S_\lambda \to 0$ in $\mathcal{A}$, provided the expansion $\mathcal{B} \to \mathcal{A}$ is non-split.*
(3) *If S is a simple object in $\mathcal{A}$ and $S \not\cong S_\lambda$, then $i_\lambda S$ is simple in $\mathcal{B}$. Moreover, $S \cong ii_\lambda S$ if $S \not\cong S_\rho$.*

Proof. (1) Let $0 \neq U \subseteq iS$ be a subobject. Then $\mathrm{Hom}_{\mathcal{B}}(i_\lambda U, S) \cong \mathrm{Hom}_{\mathcal{A}}(U, iS) \neq 0$ shows that $U \notin \mathrm{Ker}\, i_\lambda$. Thus $i_\lambda U = S$, and therefore iS/U belongs to $\mathrm{Ker}\, i_\lambda = \mathrm{add}\, S_\lambda$. On the other hand, $\mathrm{Hom}_{\mathcal{A}}(iS, S_\lambda) \cong \mathrm{Hom}_{\mathcal{B}}(S, \bar{S}) = 0$. Thus $iS/U = 0$, and it follows that iS is simple. Finally observe that $i_\lambda iA \cong A$ for every object A in $\mathcal{B}$.

(2) Take the exact sequence in (4.2.1).

(3) This is a general fact: A quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ takes each simple object of $\mathcal{A}$ not belonging to $\mathcal{C}$ to a simple object of $\mathcal{A}/\mathcal{C}$; see Lemma 1.3.6. Here, we take $\mathcal{C} = \mathrm{Ker}\, i_\lambda$ and identify i_λ with the corresponding quotient functor.

If $S \not\cong S_\rho$, then $i_\lambda S \not\cong \bar{S}$ and therefore $ii_\lambda S$ is simple by (1). Thus the canonical morphism $S \to ii_\lambda S$ is an isomorphism. □

The Ext-groups of most simple objects in $\mathcal{A}$ can be computed from appropriate Ext-groups in $\mathcal{B}$. This follows from an adjunction formula; see Lemma 4.1.1. The remaining cases are treated in the following lemma.

Lemma 4.3.2. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \mathrm{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho) \cong \mathrm{Hom}_{\mathcal{A}}(S_\rho, S_\rho)$.
(2) $\mathrm{Ext}^n_{\mathcal{B}}(\bar{S}, \bar{S}) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, S_\lambda)$ *for* $n \geq 1$.

Proof. (1) Applying $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, -)$ to the first sequence in (4.2.1) yields the isomorphism $\mathrm{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \mathrm{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho)$. The other isomorphism is dual.

(2) We have

$$\mathrm{Ext}^n_{\mathcal{B}}(i_\lambda(S_\rho), i_\lambda(S_\rho)) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, ii_\lambda(S_\rho)) \cong \mathrm{Ext}^n_{\mathcal{A}}(S_\rho, S_\lambda),$$

where the first isomorphism follows from Lemma 4.1.1 and the second from the first sequence in (4.2.1). □

Proposition 4.3.3. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be an expansion of abelian categories.*

(1) *The functor i and its adjoints i_λ and i_ρ send finite length objects to finite length objects.*
(2) *The restriction $\mathcal{B}_0 \to \mathcal{A}_0$ is an expansion of abelian categories.*
(3) *The induced functor $\mathcal{B}/\mathcal{B}_0 \to \mathcal{A}/\mathcal{A}_0$ is an equivalence.*

Proof. (1) follows from Lemma 4.3.1 and (2) is an immediate consequence.

(3) Let $\mathcal{C} = \mathrm{Ker}\, i_\lambda$. The functor i_λ induces an equivalence $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{B}$. Moreover, $\mathcal{C} \subseteq \mathcal{A}_0$ and i_λ identifies $\mathcal{A}_0/\mathcal{C}$ with $\mathcal{B}_0$ by (1). It follows from Lemma 1.3.5 that i_λ induces an equivalence $\mathcal{A}/\mathcal{A}_0 \xrightarrow{\sim} \mathcal{B}/\mathcal{B}_0$. This is a quasi-inverse for the functor $\mathcal{B}/\mathcal{B}_0 \to \mathcal{A}/\mathcal{A}_0$ induced by i. □

The Ext-quiver $\Sigma(\mathcal{A})$ of $\mathcal{A}$ can be computed explicitly from the Ext-quiver $\Sigma(\mathcal{B})$, and vice versa. The following statement makes this precise.

Proposition 4.3.4. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories. The functor induces a bijection*

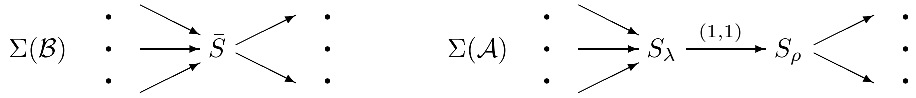
$$\Sigma_0(\mathcal{B}) \smallsetminus \{\bar{S}\} \xrightarrow{\sim} \Sigma_0(\mathcal{A}) \smallsetminus \{S_\lambda, S_\rho\},$$

and for each pair $U, V \in \Sigma_0(\mathcal{B}) \smallsetminus \{\bar{S}\}$ the following identities:

$$\delta_{iU,iV} = \delta_{U,V}, \quad \delta_{iU,S_\lambda} = \delta_{U,\bar{S}}, \quad \delta_{S_\rho,iV} = \delta_{\bar{S},V}, \quad \delta_{S_\rho,S_\lambda} = \delta_{\bar{S},\bar{S}}, \quad \delta_{S_\lambda,S_\rho} = (1,1).$$

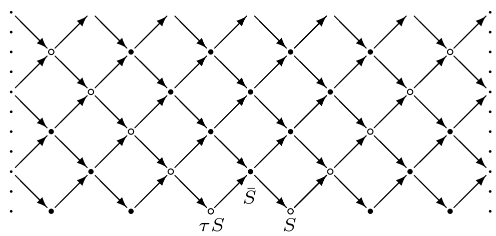
Proof. Combine Lemmas 4.1.1, 4.3.1, and 4.3.2. □

The following diagram shows how $\Sigma(\mathcal{B})$ and $\Sigma(\mathcal{A})$ are related.[6]



Example 4.3.5. (1) Let k be a field and Γ_n a quiver of extended Dynkin type $\tilde{\mathbb{A}}_n$ with cyclic orientation. Consider the category $\mathcal{A} = \operatorname{rep}_0(\Gamma_n, k)$ of all finite dimensional nilpotent representations. Fix a simple object S. This object is (co)localizable and $S^\perp = {}^\perp(\tau S)$. Thus the inclusion $S^\perp \to \operatorname{rep}_0(\Gamma_n, k)$ is an expansion, and $S^\perp$ is equivalent to $\operatorname{rep}_0(\Gamma_{n-1}, k)$.

The following diagram depicts the shape of the *Auslander-Reiten quiver* of $\operatorname{rep}_0(\Gamma_n, k)$. Thus the vertices represent the indecomposable objects, and there is an arrow between two indecomposable objects if and only if there exists an irreducible morphism; see Lemma 1.7.4.



Note that the two dotted lines are identified and that there are $n+1$ simple objects which sit at the bottom ($n = 5$ in this example). The Serre functor τ induces an automorphism of order $n+1$, that is, $\tau^{n+1} = \mathrm{Id}_\mathcal{A}$. The indecomposables not belonging to $S^\perp$ are represented by circles. Thus S and τS disappear, while $\bar{S}$ becomes a simple object in $S^\perp$.

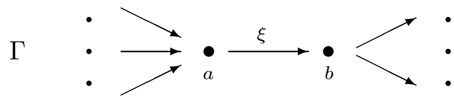
Suppose that $n > 1$. Then $\bar{S}$ is a (co)localizable object of $S^\perp$ and this gives another expansion $\bar{S}^\perp \to S^\perp$. Iterating this formation of perpendicular categories yields a chain $\mathcal{A}^0 \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^{n+1} = \mathcal{A}$ of expansions such that $\mathcal{A}^i$ is equivalent to $\operatorname{rep}_0(\Gamma_i, k)$ for each i. The category $\mathcal{A}^0$ has a unique simple object S_0 which the inclusion $\mathcal{A}^0 \to \mathcal{A}$ sends to $S^{[n+1]}$. The induced map $K_0(\mathcal{A}^0) \to K_0(\mathcal{A})$ sends the class $[S_0]$ to $\sum_{i=0}^n [\tau^i S]$.[7]

(2) Let k be a field and consider a finite quiver Γ without oriented cycles having two vertices a, b that are joined by an arrow $\xi\colon a \to b$ which is the unique arrow

[6]The expansion of the vertex $\bar{S}$ into an arrow linking S_λ with S_ρ justifies the term 'expansion of abelian categories'.

[7]This expansion of the class $[S_0]$ in $K_0(\mathcal{A})$ again explains the term 'expansion of abelian categories'.

starting at a and the unique arrow terminating at b.



We obtain a new quiver Γ' by identifying a and b and removing ξ. Let $\mathcal{A} = \operatorname{rep}(\Gamma, k)$ and consider the full subcategory $\mathcal{B}$ of representations such that ξ is represented by an isomorphism. Note that $\mathcal{B}$ is equivalent to $\operatorname{rep}(\Gamma', k)$.

The simple representations S_a and S_b are (co)localizing objects of $\mathcal{A}$ and they are joined by an almost split sequence $0 \to S_b \to E \to S_a \to 0$. The Auslander-Reiten formulae

$$D\operatorname{Ext}^1_{\mathcal{A}}(-, S_b) \cong \operatorname{Hom}_{\mathcal{A}}(S_a, -) \quad \text{and} \quad D\operatorname{Ext}^1_{\mathcal{A}}(S_a, -) \cong \operatorname{Hom}_{\mathcal{A}}(-, S_b)$$

imply that $S_a^{\perp} = \mathcal{B} = {}^{\perp}S_b$. Thus the inclusion functor $\mathcal{B} \to \mathcal{A}$ is an expansion.

4.4. **An Auslander-Reiten formula.** Given a non-split expansion $\mathcal{B} \to \mathcal{A}$, the corresponding simple objects S_λ and S_ρ in $\mathcal{A}$ are related by an Auslander-Reiten formula.

Proposition 4.4.1. *Let $\mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories and Δ its associated division ring. Then*

$$D\operatorname{Ext}^1_{\mathcal{A}}(-, S_\rho) \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, -) \quad \textit{and} \quad D\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, -) \cong \operatorname{Hom}_{\mathcal{A}}(-, S_\rho),$$

where $D = \operatorname{Hom}_\Delta(-, \Delta)$ denotes the standard duality. In particular, any non-split extension $0 \to S_\rho \to E \to S_\lambda \to$ is an almost split sequence.

Proof. Recall that ${}^{\perp}\mathcal{B} = \operatorname{add} S_\lambda$ and $\mathcal{B}^{\perp} = \operatorname{add} S_\rho$. Fix an object A in $\mathcal{A}$ and consider the corresponding exact sequence (1.4.1)

$$0 \longrightarrow A' \longrightarrow A \longrightarrow \bar{A} \longrightarrow A'' \longrightarrow 0.$$

with A', A'' in ${}^{\perp}\mathcal{B}$ and $\bar{A}$ in $\mathcal{B}$. The morphism $A' \to A$ induces the first and the third isomorphism in the sequence below, while the second isomorphism follows from the isomorphism $\operatorname{Hom}_{\mathcal{A}}(S_\lambda, S_\lambda) \cong \operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, S_\rho)$ in Lemma 4.3.2.

$$D\operatorname{Ext}^1_{\mathcal{A}}(A, S_\rho) \cong D\operatorname{Ext}^1_{\mathcal{A}}(A', S_\rho) \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, A') \cong \operatorname{Hom}_{\mathcal{A}}(S_\lambda, A).$$

The isomorphism $D\operatorname{Ext}^1_{\mathcal{A}}(S_\lambda, -) \cong \operatorname{Hom}_{\mathcal{A}}(-, S_\rho)$ follows from the first by duality.

The argument given in the proof of Proposition 1.8.1 shows that any non-split extension $0 \to S_\rho \to E \to S_\lambda \to$ is an almost split sequence. □

4.5. **Decompositions.** Expansions of abelian categories respect decompositions of abelian categories. The following lemma is a precise formulation of this fact.

Lemma 4.5.1. *Let $i \colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) *If $\mathcal{A} = \mathcal{A}_1 \amalg \mathcal{A}_2$ is a decomposition, then there exists a decomposition $i = \left[\begin{smallmatrix} i_1 & 0 \\ 0 & i_2 \end{smallmatrix}\right] : \mathcal{B}_1 \amalg \mathcal{B}_2 \to \mathcal{A}_1 \amalg \mathcal{A}_2$ such that one of i_1 and i_2 is a non-split expansion and the other is an equivalence.*
(2) *If $\mathcal{B} = \mathcal{B}_1 \amalg \mathcal{B}_2$ is a decomposition, then there exists a decomposition $i = \left[\begin{smallmatrix} i_1 & 0 \\ 0 & i_2 \end{smallmatrix}\right] : \mathcal{B}_1 \amalg \mathcal{B}_2 \to \mathcal{A}_1 \amalg \mathcal{A}_2$ such that one of i_1 and i_2 is a non-split expansion and the other is an equivalence.*
(3) *If $i' \colon \mathcal{B}' \to \mathcal{A}'$ is an equivalence of abelian categories, then $\left[\begin{smallmatrix} i & 0 \\ 0 & i' \end{smallmatrix}\right] : \mathcal{B} \amalg \mathcal{B}' \to \mathcal{A} \amalg \mathcal{A}'$ is a non-split expansion.*

Therefore $\mathcal{A}$ is connected if and only if $\mathcal{B}$ is connected.

Proof. We identify $\mathcal{B}$ with the essential image of i and choose a simple object S in $\mathcal{A}$ with $S^{\perp} = \mathcal{B}$.

(1) The decomposition of $\mathcal{A}$ restricts to a decomposition of $\mathcal{B}$ by taking $\mathcal{B}_\alpha = \mathcal{B} \cap \mathcal{A}_\alpha$ for $\alpha = 1, 2$. Now suppose without loss of generality that S belongs to $\mathcal{A}_1$. Then i_1 is a non-split expansion and i_2 is an equivalence.

(2) Let $\bar{S} = i_\rho S$ and suppose without loss of generality that $\bar{S}$ belongs to $\mathcal{B}_1$. Then $\mathcal{B}_2 \subseteq {}^{\perp}S$ and this yields a decomposition $\mathcal{A} = \mathcal{A}_1 \amalg \mathcal{A}_2$ if we set $\mathcal{A}_2 = \mathcal{B}_2$ and $\mathcal{A}_1 = \mathcal{A}_2^{\perp} = {}^{\perp}\mathcal{A}_2$. It follows that i_1 is a non-split expansion and i_2 is an equivalence.

(3) Clear. □

4.6. **Dimensions.** We compute the global dimension for an expansion of abelian categories.

Lemma 4.6.1. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of abelian categories.*

(1) $\operatorname{gl.dim} \mathcal{A} = \max\{1, \operatorname{gl.dim} \mathcal{B}\}$.
(2) *$\mathcal{A}$ has non-zero projective objects if and only if $\mathcal{B}$ has non-zero projective objects.*

Proof. (1) Use the adjunction formula for $\operatorname{Ext}^n(-,-)$ from Lemma 4.1.1 together with the fact that $\operatorname{proj.dim} A \leq 1$ for all A in ${}^{\perp}\mathcal{B}$. Note also that $\operatorname{Ext}^1_{\mathcal{A}}(-,-) \neq 0$ by Lemma 4.3.2.

(2) We use the general fact that a functor between abelian categories preserves projectivity if it admits an exact right adjoint. Thus i and its left adjoint i_λ preserve projectivity. Note that there are no non-zero projectives in the kernel of i_λ by Lemma 4.3.2. □

Next we discuss Ext-finiteness for expansions of abelian categories.

Lemma 4.6.2. *Let k be a commutative ring and $i\colon \mathcal{B} \to \mathcal{A}$ a non-split expansion of k-linear abelian categories. Then $\mathcal{A}$ is Ext-finite if and only if $\mathcal{B}$ is Ext-finite.*

Proof. We use the adjunction formula for $\operatorname{Ext}^n(-,-)$ from Lemma 4.1.1. Note that these isomorphisms are k-linear since we assume the functor i to be k-linear. It is clear that $\mathcal{B}$ is Ext-finite if $\mathcal{A}$ is Ext-finite. To prove the converse, fix a simple object S in $\mathcal{A}$ such that $S^{\perp} = \mathcal{B}$, and an arbitrary object C in $\mathcal{A}$. Then $\operatorname{End}_{\mathcal{A}}(S) \cong \operatorname{End}_{\mathcal{B}}(\bar{S})$ for some simple object $\bar{S}$ in $\mathcal{B}$; see Lemma 4.2.2. Thus $\operatorname{End}_{\mathcal{A}}(S)$ is of finite length over k, and it follows that $\operatorname{Ext}^n_{\mathcal{A}}(S, C)$ is of finite length over k for all $n \geq 0$ since the object S is localizable; see Lemma 4.1.2. On the other hand, $\operatorname{Ext}^n_{\mathcal{A}}(B, C)$ is of finite length over k for all B in $\mathcal{B}$ by the adjunction formula for $\operatorname{Ext}^n(-,-)$ from Lemma 4.1.1. Now choose A in $\mathcal{A}$ and apply $\operatorname{Ext}^n_{\mathcal{A}}(-, C)$ to the natural exact sequence (1.4.1)

$$0 \longrightarrow A' \longrightarrow A \xrightarrow{\eta_A} \bar{A} \longrightarrow A'' \longrightarrow 0$$

with A', A'' in ${}^{\perp}\mathcal{B} = \operatorname{add} S$ and $\bar{A}$ in $\mathcal{B}$. It follows that $\operatorname{Ext}^n_{\mathcal{A}}(A, C)$ is of finite length over k for all $n \geq 0$. □

5. Coherent sheaves on the projective line

In this section we discuss the category of coherent sheaves on the projective line $\mathbb{P}^1_k$ over an arbitrary base field k. This is a hereditary abelian category with finite dimensional Hom and Ext spaces. Moreover, the category satisfies Serre duality and admits a tilting object. Various structural properties can be derived from these basic facts.

The projective line $\mathbb{P}^1_k$ is covered by two copies of the affine line $\mathbb{A}^1_k$. Using this fact, we identify sheaves on $\mathbb{P}^1_k$ with 'triples', that is, pairs of modules over the polynomial ring $k[x]$ which are glued together by a glueing morphism. In this way, basic properties of sheaves are easily derived from properties of modules over a polynomial ring in one variable.

Every coherent sheaf is the direct sum of a torsion-free sheaf and a finite length sheaf. An indecomposable torsion-free sheaf is a line bundle and an indecomposable finite length sheaf is uniserial. The simple sheaves are parametrized by the closed points of $\mathbb{P}^1_k$, and for each simple sheaf S and $r > 0$ there is a unique sheaf with length r and top S. This yields a complete classification of all indecomposable sheaves.

A classical theorem of Serre identifies the category of coherent sheaves on $\mathbb{P}^1_k$ with the quotient category of the category of finitely generated $\mathbb{Z}$-graded $k[x_0, x_1]$-modules modulo the Serre subcategory of finite length modules. We use the concept of dehomogenization to pass from graded $k[x_0, x_1]$-modules to modules over $k[x]$. In geometric terms, this reflects the passage from the projective to an affine line.

5.1. **Coherent sheaves on $\mathbb{A}^1_k$.** Let k be a field and $\mathbb{A}^1_k$ the affine line over k. The polynomial ring $k[x]$ is the ring of regular functions and the category of coherent sheaves $\operatorname{coh}\mathbb{A}^1_k$ is equivalent to the module category $\operatorname{mod} k[x]$ via the global section functor.

Let $\operatorname{Spec} k[x]$ denote the set of prime ideals of $k[x]$. Note that $k[x]$ is a principal ideal domain. Thus irreducible polynomials correspond to non-zero prime ideals by taking a polynomial P to the ideal (P) generated by P. A *closed point* of $\mathbb{A}^1_k$ is by definition a non-zero prime ideal $\mathfrak{p}$ and the *generic point* is the zero ideal.

The following result describes the category $\operatorname{mod}_0 k[x]$ of torsion modules and the corresponding quotient category.

Proposition 5.1.1. *Let $k[x]$ be the polynomial ring over a field k.*

(1) *The functor which sends a $k[x]$-module M to its family of localizations $(M_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} k[x]}$ induces an equivalence*

$$\operatorname{mod}_0 k[x] \xrightarrow{\sim} \coprod_{0\neq\mathfrak{p}\in\operatorname{Spec} k[x]} \operatorname{mod}_0(k[x]_\mathfrak{p}).$$

(2) *The localization functor $\operatorname{mod} k[x] \to \operatorname{mod} k(x)$ induces an equivalence*

$$\frac{\operatorname{mod} k[x]}{\operatorname{mod}_0 k[x]} \xrightarrow{\sim} \operatorname{mod} k(x).$$

Proof. (1) The assertion follows from standard properties of finitely generated modules over principal ideal domains. Note that the quasi-inverse functor takes a family of modules $(N_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} k[x]}$ to $\bigoplus_\mathfrak{p} N_\mathfrak{p}$.

(2) Set $\mathcal{A} = \operatorname{mod} k[x]/\operatorname{mod}_0 k[x]$. The kernel of the localization functor $T = -\otimes_{k[x]} k(x)$ is the category $\operatorname{mod}_0 k[x]$. Thus T induces a faithful functor $\bar{T}\colon \mathcal{A} \to \operatorname{mod} k(x)$. The functor is dense, since a $k(x)$-module of rank r is isomorphic to $\bar{T}(k[x]^r)$. To show that $\bar{T}$ is full, it suffices to show that $\bar{T}$ induces a surjective map

$$f\colon \operatorname{Hom}_\mathcal{A}(k[x], k[x]) \longrightarrow \operatorname{Hom}_{k(x)}(k(x), k(x)) \cong k(x).$$

Given any non-zero polynomial $P \in k[x]$, let $\mu_P\colon k[x] \to k[x]$ denote the multiplication by P. The kernel and cokernel of μ_P belong to $\operatorname{mod}_0 k[x]$, and therefore μ_P becomes invertible in $\mathcal{A}$. Thus $f(\mu_P^{-1}) = P^{-1}$. It follows that f is surjective. □

5.2. **Coherent sheaves on $\mathbb{P}^1_k$.** Let k be a field and $\mathbb{P}^1_k$ the projective line over k. We view $\mathbb{P}^1_k$ as a scheme and begin with a description of the underlying set of points.

Let $k[x_0, x_1]$ be the polynomial ring in two variables with the usual $\mathbb{Z}$-grading by total degree. Denote by $\operatorname{Proj} k[x_0, x_1]$ the set of homogeneous prime ideals of $k[x_0, x_1]$ that are different from the unique maximal ideal consisting of positive degree elements. Note that $k[x_0, x_1]$ is a two-dimensional graded factorial domain. Thus homogeneous irreducible polynomials correspond to non-zero homogeneous prime ideals by taking a polynomial P to the ideal (P) generated by P. A *closed point* of $\mathbb{P}^1_k$ is by definition an element $\mathfrak{p} \neq 0$ in $\operatorname{Proj} k[x_0, x_1]$, and the *generic point* is the zero ideal. Using homogeneous coordinates, a *rational point* of $\mathbb{P}^1_k$ is a pair $[\lambda_0 : \lambda_1]$ of elements of k subject to the relation $[\lambda_0 : \lambda_1] = [\alpha\lambda_0 : \alpha\lambda_1]$ for all $\alpha \in k$, $\alpha \neq 0$. We identify each rational point $[\lambda_0 : \lambda_1]$ with the prime ideal $(\lambda_1 x_0 - \lambda_0 x_1)$ of $k[x_0, x_1]$.

Using the identification $y = x_1/x_0$, we cover $\mathbb{P}^1_k$ by two copies $U' = \operatorname{Spec} k[y]$ and $U'' = \operatorname{Spec} k[y^{-1}]$ of the affine line, with $U' \cap U'' = \operatorname{Spec} k[y, y^{-1}]$. More precisely, the morphism $k[x_0, x_1] \to k[y]$ which sends a polynomial P to $P(1, y)$ induces a bijection

$$\operatorname{Proj} k[x_0, x_1] \smallsetminus \{(x_0)\} \xrightarrow{\sim} \operatorname{Spec} k[y];$$

see §5.5 for details. Analogously, the morphism $k[x_0, x_1] \to k[y^{-1}]$ which sends a polynomial P to $P(y^{-1}, 1)$ induces a bijection

$$\operatorname{Proj} k[x_0, x_1] \smallsetminus \{(x_1)\} \xrightarrow{\sim} \operatorname{Spec} k[y^{-1}].$$

Based on the covering $\mathbb{P}^1_k = U' \cup U''$, the category $\operatorname{coh} \mathbb{P}^1_k$ of coherent sheaves admits a description in terms of the following pullback of abelian categories

$$\begin{array}{ccc} \operatorname{coh} \mathbb{P}^1_k & \longrightarrow & \operatorname{coh} U' \\ \downarrow & & \downarrow \\ \operatorname{coh} U'' & \longrightarrow & \operatorname{coh} U' \cap U'' \end{array}$$

where each functor is given by restricting a sheaf to the appropriate open subset; see [8, Chap. VI, Prop. 2]. More concretely, this pullback diagram has, up to equivalence, the form

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \operatorname{mod} k[y] \\ \downarrow & & \downarrow \\ \operatorname{mod} k[y^{-1}] & \longrightarrow & \operatorname{mod} k[y, y^{-1}] \end{array}$$

where the category $\mathcal{A}$ is defined as follows. The objects of $\mathcal{A}$ are triples (M', M'', μ), where M' is a finitely generated $k[y]$-module, M'' is a finitely generated $k[y^{-1}]$-module, and $\mu\colon M'_y \xrightarrow{\sim} M''_{y^{-1}}$ is an isomorphism of $k[y, y^{-1}]$-modules. Here, we use for any R-module M the notation M_x to denote the localization with respect to an element $x \in R$. A morphism from (M', M'', μ) to (N', N'', ν) in $\mathcal{A}$ is a pair (ϕ', ϕ'') of morphisms, where $\phi'\colon M' \to N'$ is $k[y]$-linear and $\phi''\colon M'' \to N''$ is $k[y^{-1}]$-linear such that $\nu\phi'_y = \phi''_{y^{-1}}\mu$.

Given a sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$, we denote for any open subset $U \subseteq \mathbb{P}^1_k$ by $\Gamma(U, \mathcal{F})$ the *sections* over U.

Lemma 5.2.1. *The functor which sends a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$ to the triple $(\Gamma(U', \mathcal{F}), \Gamma(U'', \mathcal{F}), \mathrm{id}_{\Gamma(U' \cap U'', \mathcal{F})})$ gives an equivalence $\operatorname{coh} \mathbb{P}^1_k \xrightarrow{\sim} \mathcal{A}$.*

Proof. The description of a sheaf $\mathcal{F}$ on $\mathbb{P}^1_k = U' \cup U''$ in terms of its restrictions $\mathcal{F}|_{U'}$, $\mathcal{F}|_{U''}$, and $\mathcal{F}|_{U'\cap U''}$ is classical; see [8, Chap. VI, Prop. 2]. Thus it remains to observe that taking global sections identifies $\operatorname{coh} U' = \operatorname{mod} k[y]$, $\operatorname{coh} U'' = \operatorname{mod} k[y^{-1}]$, and $\operatorname{coh} U' \cap U'' = \operatorname{mod} k[y, y^{-1}]$. □

From now on we identify the categories $\operatorname{coh} \mathbb{P}^1_k$ and $\mathcal{A}$ via the above equivalence.

5.3. **Serre's theorem and Serre duality.** Let $\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]$ denote the category of finitely generated $\mathbb{Z}$-graded $k[x_0, x_1]$-modules and let $\operatorname{mod}^{\mathbb{Z}}_0 k[x_0, x_1]$ be the Serre subcategory consisting of all finite length objects.

There is a functor

(5.3.1) $$\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1] \longrightarrow \operatorname{coh} \mathbb{P}^1_k$$

that takes each graded $k[x_0, x_1]$-module M to the triple $((M_{x_0})_0, (M_{x_1})_0, \sigma_M)$, where y acts on the degree zero part of M_{x_0} via the identification $y = x_1/x_0$, the variable y^{-1} acts on the degree zero part of M_{x_1} via the identification $y^{-1} = x_0/x_1$, and the isomorphism σ_M equals the obvious identification $[(M_{x_0})_0]_{x_1/x_0} = [(M_{x_1})_0]_{x_0/x_1}$.

Proposition 5.3.1 (Serre)**.** *The functor* (5.3.1) *induces an equivalence*

$$\frac{\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]}{\operatorname{mod}^{\mathbb{Z}}_0 k[x_0, x_1]} \xrightarrow{\sim} \operatorname{coh} \mathbb{P}^1_k.$$

Proof. We refer to [24] for the proof. It is clear from the definition that the functor (5.3.1) is exact having kernel $\operatorname{mod}^{\mathbb{Z}}_0 k[x_0, x_1]$. This fact yields the induced functor which is faithful by construction. □

For any $n \in \mathbb{Z}$ and $\mathcal{F} = (M', M'', \mu)$ in $\operatorname{coh} \mathbb{P}^1_k$, denote by $\mathcal{F}(n)$ the *twisted sheaf* $(M', M'', \mu^{(n)})$, where $\mu^{(n)}$ is the map μ followed by multiplication with y^{-n}. Given a module M in $\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]$, the *twisted module* $M(n)$ is obtained by shifting the grading, that is, $M(n)_i = M_{i+n}$ for $i \in \mathbb{Z}$. Note that the functor (5.3.1) is compatible with the twist functors defined on $\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]$ and $\operatorname{coh} \mathbb{P}^1_k$.

Proposition 5.3.2 (Serre)**.** *The category* $\operatorname{coh} \mathbb{P}^1_k$ *is a Hom-finite k-linear abelian category satisfying Serre duality. More precisely, there is a functorial k-linear isomorphism*

$$D \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{G}, \mathcal{F}(-2)) \quad \textit{for all} \quad \mathcal{F}, \mathcal{G} \in \operatorname{coh} \mathbb{P}^1_k.$$

Proof. See [15, III.7]. □

5.4. **Locally free and torsion sheaves.** A sheaf (M', M'', μ) in $\operatorname{coh} \mathbb{P}^1_k$ is called *locally free* or *vector bundle* if M' and M'' are free modules over $k[y]$ and $k[y^{-1}]$ respectively. We denote the full subcategory of vector bundles in $\operatorname{coh} \mathbb{P}^1_k$ by $\operatorname{vect} \mathbb{P}^1_k$.

The *structure sheaf* is the sheaf $\mathcal{O} = (k[y], k[y^{-1}], \mathrm{id}_{k[y,y^{-1}]})$. For any pair $m, n \in \mathbb{Z}$, we have a natural bijection

(5.4.1) $$k[x_0, x_1]_{n-m} \xrightarrow{\sim} \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n)).$$

The map sends a homogeneous polynomial P of degree $n - m$ to the morphism (ϕ', ϕ''), where $\phi' \colon k[y] \to k[y]$ is multiplication by $P(1, y)$ and $\phi'' \colon k[y^{-1}] \to k[y^{-1}]$ is multiplication by $P(y^{-1}, 1)$. This bijection is a special case of the next result.

Let $R = k[x_0, x_1]$ and denote by $\operatorname{proj}^{\mathbb{Z}} R$ the category of finitely generated projective $\mathbb{Z}$-graded R-modules. Note that R is a graded local ring since the homogeneous elements of positive degree form the unique maximal homogeneous ideal.

Thus finitely generated projective R-modules are up to isomorphism of the form $R(n_1) \oplus \cdots \oplus R(n_r)$.

Proposition 5.4.1 (Grothendieck)**.** *The functor* (5.3.1) *induces an equivalence*

$$\operatorname{proj}^{\mathbb{Z}} k[x_0, x_1] \xrightarrow{\sim} \operatorname{vect} \mathbb{P}^1_k.$$

In particular, each locally free coherent sheaf on $\mathbb{P}^1_k$ is isomorphic to a sheaf of the form $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$.

Proof. We need to show that the functor (5.3.1) is fully faithful when it is restricted to $\operatorname{proj}^{\mathbb{Z}} R$, where $R = k[x_0, x_1]$. Every finitely generated projective R-module is up to isomorphism of the form $R(n_1) \oplus \cdots \oplus R(n_r)$. Thus it suffices to show that (5.3.1) induces a bijection $\operatorname{Hom}_R(R(m), R(n)) \to \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ for each pair $m, n \in \mathbb{Z}$. But this is clear since the map coincides with the bijection (5.4.1).

It remains to show that the functor is dense, that is, each locally free coherent sheaf is isomorphic to one of the form $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$. For this we refer to [12]. An elementary proof is based on an argument due to Birkhoff [4, 7]. A vector bundle $\mathcal{F} = (M', M'', \mu)$ is basically determined by an invertible matrix over $k[y, y^{-1}]$, which represents the isomorphism μ. Now one uses the fact that such a matrix can be transformed into a diagonal matrix with entries $(y^{-n_1}, \ldots, y^{-n_r})$ by multiplying it with an invertible matrix over $k[y]$ from the right and an invertible matrix over $k[y^{-1}]$ from the left. This yields an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$. □

A sheaf (M', M'', μ) in $\operatorname{coh} \mathbb{P}^1_k$ is called *torsion* if M' and M'' are torsion modules over $k[y]$ and $k[y^{-1}]$ respectively. We denote the full subcategory of torsion sheaves in $\operatorname{coh} \mathbb{P}^1_k$ by $\operatorname{coh}_0 \mathbb{P}^1_k$. Note that each sheaf $\mathcal{F} = (M', M'', \mu)$ admits a unique maximal subobject $\operatorname{tor} \mathcal{F}$ in $\operatorname{coh} \mathbb{P}^1_k$ that is torsion. One obtains $\operatorname{tor} \mathcal{F}$ by taking the torsion parts of M' and M'' respectively. Clearly, $\mathcal{F}/\operatorname{tor} \mathcal{F}$ is locally free.

Proposition 5.4.2. *Each coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$ admits an essentially unique decomposition $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ such that $\mathcal{F}'$ is torsion and $\mathcal{F}''$ is locally free. The torsion sheaves are precisely the objects of finite length in $\operatorname{coh} \mathbb{P}^1_k$.*

Proof. We use elementary facts about finitely generated modules over $k[y]$ and $k[y^{-1}]$ respectively. Take $\mathcal{F}' = \operatorname{tor} \mathcal{F}$ and $\mathcal{F}'' = \mathcal{F}/\operatorname{tor} \mathcal{F}$. Serre duality implies $\operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}') = 0$, since there are no non-zero morphisms from torsion to locally free sheaves. Thus the canonical exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ splits.

Let $\mathcal{F} = (M', M'', \mu)$ be a coherent sheaf. If $\mathcal{F}$ is torsion, then it is a finite length object in $\operatorname{coh} \mathbb{P}^1_k$, since the corresponding torsion modules M' and M'' have finite length. On the other hand, the structure sheaf has infinite length, since it admits factor objects of arbitrary length. Thus each non-zero locally free sheaf is of infinite length, by Proposition 5.4.1. □

5.5. **Dehomogenization.** Let R be a $\mathbb{Z}$-graded commutative ring and $x \in R$ a non-nilpotent element of degree 1. The *dehomogenization* of R with respect to x is the ring $R/(x-1)$.

Lemma 5.5.1. *The canonical morphism $\pi \colon R \to R/(x-1)$ induces an isomorphism $(R_x)_0 \xrightarrow{\sim} R/(x-1)$. Moreover, π induces a bijection between the set of homogeneous (prime) ideals of R modulo which the element x is regular and the set of all (prime) ideals of $R/(x-1)$.*

Proof. The morphism π induces a morphism $S = R_x \to R/(x-1)$ and its kernel is the ideal generated by $x - 1$. Now observe that $S_0 \cong S/(x-1)$, since x is in S a unit of degree 1.

The bijective correspondence between ideals of R and $R/(x-1)$ sends an ideal $\mathfrak{a} \subseteq R$ to $\pi(\mathfrak{a})$, and its inverse sends an ideal $\mathfrak{b} \subseteq (R_x)_0 = R/(x-1)$ to $\mathfrak{b}R_x \cap R$. □

Let us consider the dehomogenization of the polynomial ring $k[x_0, x_1]$ with respect to the variable x_0. Then Lemma 5.5.1 implies an isomorphism

$$k[x_0, x_1]/(x_0 - 1) \cong (k[x_0, x_1]_{x_0})_0 = k[y]$$

via the identification $y = x_1/x_0$. Denote by $\pi\colon k[x_0, x_1] \to k[y]$ the canonical morphism which sends a polynomial P to $P(1, y)$. The morphism π induces a bijection

$$\operatorname{Proj} k[x_0, x_1] \smallsetminus \{(x_0)\} \xrightarrow{\sim} \operatorname{Spec} k[y] \tag{5.5.1}$$

and for any prime ideal $\mathfrak{p} \neq (x_0)$ in $\operatorname{Proj} k[x_0, x_1]$ an isomorphism

$$(k[x_0, x_1]_{\mathfrak{p}})_0 \xrightarrow{\sim} k[y]_{\pi(\mathfrak{p})}. \tag{5.5.2}$$

Here, $k[x_0, x_1]_{\mathfrak{p}}$ denotes the homogeneous localization with respect to $\mathfrak{p}$ which inverts all homogeneous elements not lying in $\mathfrak{p}$.

The following lemma describes the dehomogenization for graded modules over $k[x_0, x_1]$ with respect to x_0. This functor induces an equivalence if one passes to the localization with respect to a prime ideal $\mathfrak{p} \neq (x_0)$.

Lemma 5.5.2. *Let $\mathfrak{p} \neq (x_0)$ be in $\operatorname{Proj} k[x_0, x_1]$. The functor sending a graded $k[x_0, x_1]$-module M to $FM = (M_{x_0})_0$ induces the following commutative diagram of exact functors*

$$\begin{array}{ccccc}
\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1] & & \xrightarrow{\quad F \quad} & & \operatorname{mod} k[y] \\
\downarrow & & & & \downarrow \\
\operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]_{\mathfrak{p}} & \xrightarrow{F'_{\mathfrak{p}}} & \operatorname{mod}(k[x_0, x_1]_{\mathfrak{p}})_0 & \xrightarrow{F''_{\mathfrak{p}}} & \operatorname{mod} k[y]_{\pi(\mathfrak{p})}
\end{array}$$

where the vertical functors are the localization functors with respect to $\mathfrak{p}$ and $\pi(\mathfrak{p})$ respectively. Moreover, the functors $F'_{\mathfrak{p}}$ and $F''_{\mathfrak{p}}$ are equivalences.

Proof. The composite $k[x_0, x_1] \to k[y] \to k[y]_{\pi(\mathfrak{p})}$ induces a morphism $k[x_0, x_1]_{\mathfrak{p}} \to k[y]_{\pi(\mathfrak{p})}$ and therefore F composed with localization at $\pi(\mathfrak{p})$ induces a functor

$$F_{\mathfrak{p}}\colon \operatorname{mod}^{\mathbb{Z}} k[x_0, x_1]_{\mathfrak{p}} \longrightarrow \operatorname{mod} k[y]_{\pi(\mathfrak{p})}.$$

This functor can be written as composite $F''_{\mathfrak{p}} F'_{\mathfrak{p}}$. The first functor $F'_{\mathfrak{p}}$ takes a graded $k[x_0, x_1]_{\mathfrak{p}}$-module to its degree zero part; it is an equivalence since $k[x_0, x_1]_{\mathfrak{p}}$ is strongly graded. The second functor $F''_{\mathfrak{p}}$ is an equivalence thanks to the isomorphism (5.5.2). □

Remark 5.5.3. There are analogous results for the dehomogenization of $k[x_0, x_1]$ with respect to x_1 which is denoted by $k[y^{-1}]$.

Next we describe the category $\operatorname{coh}_0 \mathbb{P}^1_k$ of torsion sheaves more explicitly. Note that $\operatorname{coh}_0 \mathbb{P}^1_k$ is uniserial because it is a length category with Serre duality; see Proposition 1.8.2. The category decomposes into connected abelian categories, and each component has a unique simple object since it is equivalent to the category of finite length modules over a local ring.

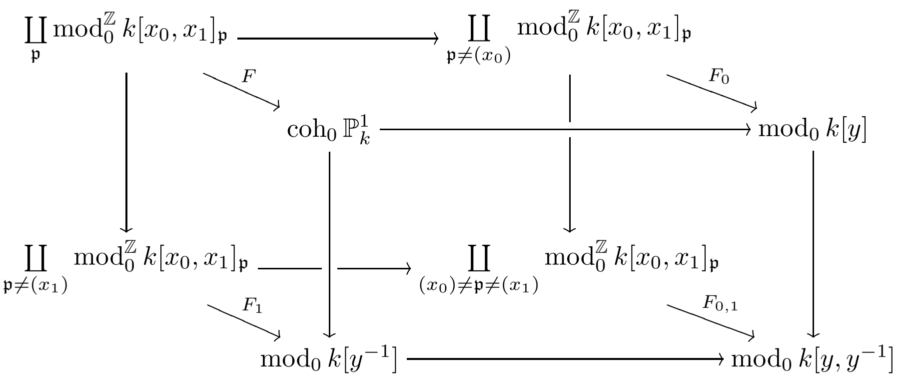
Proposition 5.5.4. (1) *The functor* (5.3.1) *induces an equivalence*

$$\coprod_{0 \neq \mathfrak{p} \in \operatorname{Proj} k[x_0, x_1]} \operatorname{mod}^{\mathbb{Z}}_0 k[x_0, x_1]_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{coh}_0 \mathbb{P}^1_k$$

(2) *The functor taking a sheaf* (M', M'', μ) *to* $M' \otimes_{k[y]} k(y) \cong M'' \otimes_{k[y^{-1}]} k(y^{-1})$ *induces an equivalence*

$$\frac{\operatorname{coh} \mathbb{P}^1_k}{\operatorname{coh}_0 \mathbb{P}^1_k} \xrightarrow{\sim} \operatorname{mod} k(y).$$

Proof. (1) We denote the functor (5.3.1) by F. Dehomogenization with respect to x_0 and x_1, respectively, induces the functors F_0, F_1, and $F_{0,1}$, which make the following diagram commutative.



Here, $\mathfrak{p}$ runs through all non-zero prime ideals in $\operatorname{Proj} k[x_0, x_1]$. Note that the front square and the back square are pullback diagrams of abelian categories. The functors F_0, F_1, and $F_{0,1}$ are equivalences. This follows from Proposition 5.1.1 and Lemma 5.5.2 in combination with the bijection (5.5.1). Thus F is an equivalence.

(2) The functor $\operatorname{coh} \mathbb{P}^1_k \to \operatorname{mod} k(y)$ is exact and its kernel is the category $\operatorname{coh}_0 \mathbb{P}^1_k$ of torsion sheaves. Thus there is an induced functor $\frac{\operatorname{coh} \mathbb{P}^1_k}{\operatorname{coh}_0 \mathbb{P}^1_k} \to \operatorname{mod} k(y)$ which is faithful. The structure sheaf is the unique indecomposable object, and the argument given in the proof of Proposition 5.1.1 shows that the functor is an equivalence. □

5.6. **Support.** Given a point $\mathfrak{p} \in \operatorname{Proj} k[x_0, x_1]$ of $\mathbb{P}^1_k$, the *associated local ring* $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$ is by definition $(k[x_0,x_1]_{\mathfrak{p}})_0$. Denote by $k(\mathfrak{p})$ the *residue field* at $\mathfrak{p}$ which is by definition the residue field of the local ring $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$.

Note that dehomogenization induces isomorphisms

$$\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}} \cong \begin{cases} k[y]_{\mathfrak{p}'} & \text{if } \mathfrak{p} \neq (x_0), \\ k[y^{-1}]_{\mathfrak{p}''} & \text{if } \mathfrak{p} \neq (x_1), \end{cases}$$

where

$$\mathfrak{p}' = \{P(1,y) \in k[y] \mid P \in \mathfrak{p}\} \quad \text{and} \quad \mathfrak{p}'' = \{P(y^{-1},1) \in k[y^{-1}] \mid P \in \mathfrak{p}\}.$$

The *stalk* of a sheaf $\mathcal{F} = (M', M'', \mu)$ at $\mathfrak{p}$ is the $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$-module

$$\mathcal{F}_{\mathfrak{p}} = \begin{cases} M'_{\mathfrak{p}'} & \text{if } \mathfrak{p} \neq (x_0), \\ M''_{\mathfrak{p}''} & \text{if } \mathfrak{p} \neq (x_1), \end{cases}$$

where $M'_{\mathfrak{p}'}$ and $M''_{\mathfrak{p}''}$ are identified via μ, if $\mathfrak{p} \neq (x_i)$ for $i = 0, 1$. The *support* of $\mathcal{F}$ is by definition

$$\operatorname{supp} \mathcal{F} = \{\mathfrak{p} \in \operatorname{Proj} k[x_0,x_1] \mid \mathcal{F}_{\mathfrak{p}} \neq 0\}.$$

Note that a torsion sheaf $\mathcal{F}$ admits a unique decomposition

$$\mathcal{F} = \bigoplus_{0 \neq \mathfrak{p} \in \operatorname{Proj} k[x_0,x_1]} \mathcal{F}_{\{\mathfrak{p}\}}$$

such that each $\mathcal{F}_{\{\mathfrak{p}\}}$ is a sheaf supported at $\mathfrak{p}$. This follows directly from properties of torsion modules over a polynomial ring in one variable.

The functor sending a sheaf $\mathcal{F}$ to the family $(\mathcal{F}_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Proj} k[x_0,x_1]}$ provides an equivalence

$$\operatorname{coh}_0 \mathbb{P}^1_k \xrightarrow{\sim} \coprod_{0\neq\mathfrak{p}\in\operatorname{Proj} k[x_0,x_1]} \operatorname{mod}_0 \mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}. \tag{5.6.1}$$

Note that this yields a quasi-inverse for the equivalence from Proposition 5.5.4 if the functor is composed with the family of equivalences $\operatorname{mod}_0 \mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}} \xrightarrow{\sim} \operatorname{mod}^{\mathbb{Z}} k[x_0,x_1]_\mathfrak{p}$ from Lemma 5.5.2.

Let $\mathfrak{p}$ be a closed point and choose a homogeneous irreducible polynomial P of degree d that generates $\mathfrak{p}$. The bijection (5.4.1) identifies for each integer $r>0$ the polynomial P^r with a morphism $\phi\colon \mathcal{O}(-rd)\to\mathcal{O}(0)=\mathcal{O}$. Denote by $\mathcal{O}_{\mathfrak{p},r}$ the cokernel of this morphism. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{O}(-rd) \xrightarrow{\phi} \mathcal{O} \longrightarrow \mathcal{O}_{\mathfrak{p},r} \longrightarrow 0. \tag{5.6.2}$$

Proposition 5.6.1. *Let $\mathfrak{p}$ be a closed point of $\mathbb{P}^1_k$ and $r>0$ an integer. An indecomposable sheaf in $\operatorname{coh}\mathbb{P}^1_k$ has support $\{\mathfrak{p}\}$ and length r if and only if it is isomorphic to $\mathcal{O}_{\mathfrak{p},r}$.*

Proof. We compute the support and the length of $\mathcal{O}_{\mathfrak{p},r}$. We may assume that $\mathfrak{p}\neq(x_0)$. The morphism ϕ is given by multiplication with $P^r(1,y)$ and $P^r(y^{-1},1)$ respectively. Thus for each point $\mathfrak{q}\neq\mathfrak{p}$, the stalk morphism $\phi_\mathfrak{q}$ is an isomorphism, and therefore $(\mathcal{O}_{\mathfrak{p},r})_\mathfrak{q}=0$. On the other hand, the $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$-module $(\mathcal{O}_{\mathfrak{p},r})_\mathfrak{p}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}/\mathfrak{m}^r$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$. Note that this module is indecomposable. Thus $\mathcal{O}_{\mathfrak{p},r}$ has length r and is indecomposable.

The equivalence (5.6.1) shows that an indecomposable torsion sheaf is uniquely determined by its support and its length. ☐

Remark 5.6.2. Let $S_\mathfrak{p}=\mathcal{O}_{\mathfrak{p},1}$ be the simple sheaf concentrated at $\mathfrak{p}$. The sequence (5.6.2) induces an isomorphism $\operatorname{End}(S_\mathfrak{p})\xrightarrow{\sim}\operatorname{Hom}(\mathcal{O},S_\mathfrak{p})$. Moreover, we have an isomorphism $\operatorname{End}(S_\mathfrak{p})\cong k(\mathfrak{p})$ of algebras.

5.7. **Automorphisms.** Let $\operatorname{PGL}(2,k)$ denote the *projective linear group*, that is, the group of invertible 2×2 matrices over k modulo the subgroup of matrices of the form $\left[\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right]$. Any element $\sigma=\left[\begin{smallmatrix}\sigma_{00} & \sigma_{01}\\ \sigma_{10} & \sigma_{11}\end{smallmatrix}\right]$ in $\operatorname{PGL}(2,k)$ induces an automorphism $k[x_0,x_1]\xrightarrow{\sim}k[x_0,x_1]$ by sending x_i to $\sigma_{i0}x_0+\sigma_{i1}x_1$ $(i=0,1)$. This yields a map $\operatorname{PGL}(2,k)\to\operatorname{Aut}\mathbb{P}^1_k$ into the automorphism group of the projective line. Recall that a rational point $[\lambda_0:\lambda_1]$ is identified with the prime ideal $(\lambda_1x_0-\lambda_0x_1)$. Then the automorphism corresponding to σ sends a rational point $[\lambda_0:\lambda_1]$ to $[\sigma_{00}\lambda_0+\sigma_{01}\lambda_1:\sigma_{10}\lambda_0+\sigma_{11}\lambda_1]$.

Proposition 5.7.1. *The map $\operatorname{PGL}(2,k)\to\operatorname{Aut}\mathbb{P}^1_k$ is an isomorphism.*

Proof. It is clear that the map is injective. We provide an inverse map as follows. Let $\phi\colon\mathbb{P}^1_k\xrightarrow{\sim}\mathbb{P}^1_k$ be an automorphism. This morphism sends rational points to rational points. In particular for $i=0,1$ the inverse ϕ^{-1} sends the prime ideal (x_i) to an ideal of the form (P_i) for some homogenous irreducible polynomial $P_i=\sigma_{i0}x_0+\sigma_{i1}x_1$ in $k[x_0,x_1]$. Let $U_i=\mathbb{P}^1_k\smallsetminus\{(x_i)\}$ and denote by U_i' its image under ϕ. Then ϕ induces isomorphisms of affine lines

$$\phi_0\colon\ \operatorname{Spec}k[P_1/P_0]=U_0'\xrightarrow{\sim}U_0=\operatorname{Spec}k[x_1/x_0]$$

and

$$\phi_1\colon\ \operatorname{Spec}k[P_0/P_1]=U_1'\xrightarrow{\sim}U_1=\operatorname{Spec}k[x_0/x_1]$$

which preserve the origins. Thus there are non-zero scalars a, b such that

$$\phi_0^*(x_1/x_0) = a(P_1/P_0) \quad \text{and} \quad \phi_1^*(x_0/x_1) = b(P_0/P_1).$$

Here, ϕ_i^* denotes the induced morphism between the rings of regular functions ($i = 0, 1$). The morphisms ϕ_0 and ϕ_1 agree on $U_0 \cap U_1$, and therefore $b = a^{-1}$. It follows that ϕ is given by the linear transformation $\left[\begin{smallmatrix} \sigma_{00} & \sigma_{01} \\ a\sigma_{10} & a\sigma_{11} \end{smallmatrix}\right]$ in $\mathrm{PGL}(2, k)$. ☐

5.8. **Tilting objects.** The category $\operatorname{coh}\mathbb{P}^1_k$ admits a tilting object which is actually unique up to a shift and up to multiplicities of its indecomposable direct summands.

Proposition 5.8.1. *An object T in $\operatorname{coh}\mathbb{P}^1_k$ is a tilting object if and only if*

$$\operatorname{add} T = \operatorname{add}(\mathcal{O}(n) \oplus \mathcal{O}(n+1)) \quad \textit{for some} \quad n \in \mathbb{Z}. \tag{5.8.1}$$

For each $n \in \mathbb{Z}$, the endomorphism algebra of the tilting object $\mathcal{O}(n) \oplus \mathcal{O}(n+1)$ is isomorphic to the Kronecker algebra Λ (i.e. the path algebra of the quiver $\cdot \rightrightarrows \cdot$), and this yields a derived equivalence $\mathbf{D}^b(\operatorname{coh}\mathbb{P}^1_k) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$.

Proof. Consider $T = \mathcal{O} \oplus \mathcal{O}(1)$. We apply the bijection (5.4.1) and Serre duality. This yields $\operatorname{Ext}^1(T, T) = 0$. Let $\mathcal{F}$ be an indecomposable sheaf. If $\mathcal{F}$ is torsion, then $\operatorname{Hom}(\mathcal{O}, \mathcal{F}) \neq 0$; see §5.6. If $\mathcal{F}$ is locally free, say $\mathcal{F} \cong \mathcal{O}(n)$, then $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(n)) \neq 0$ if $n \geq 0$, and $\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(n)) \neq 0$ if $n < 0$. Thus $T = \mathcal{O} \oplus \mathcal{O}(1)$ is a tilting object, and its endomorphism algebra equals the Kronecker algebra. From this it follows that any object T in $\operatorname{coh}\mathbb{P}^1_k$ satisfying (5.8.1) is a tilting object.

Now let T be any tilting object in $\operatorname{coh}\mathbb{P}^1_k$. Then T is locally free since any non-zero torsion sheaf $\mathcal{F}$ has $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \neq 0$. Another application of the bijection (5.4.1) and Serre duality yields the condition (5.8.1).

The derived equivalence is a consequence of Theorem 3.2.5. ☐

Corollary 5.8.2. *The Grothendieck group of $\operatorname{coh}\mathbb{P}^1_k$ is free of rank two and the corresponding Euler form is non-degenerate.* ☐

6. Coherent sheaves on weighted projective lines

Following work of Lenzing [17], we describe the abelian categories that arise as categories of coherent sheaves on weighted projective lines. We provide two different approaches: a list of axioms and a description in terms of expansions of abelian categories.

The axioms basically say that these abelian categories are hereditary and noetherian, admit a tilting object, and have no non-zero projective objects. We collect some direct consequences of these axioms. In particular, we investigate the quotient category modulo the Serre subcategory of finite length objects; it is a length category with a unique simple object.

An abelian category satisfying these axioms has a Grothendieck group that is free of finite rank. We show that the rank is minimal if and only if the category is equivalent to the category of coherent sheaves on the projective line.

The axioms are invariant under forming expansions. Moreover, an expansion increases the rank of the Grothendieck group by one. Thus the formation of expansions reflects the insertion of weights for specific points of the projective line. These observations provide the basis for the final description of categories of coherent sheaves on weighted projective lines.

Throughout this section we fix an arbitrary field k.

6.1. **Weighted projective lines.** A *weighted projective line* over a field k is by definition a triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ is a (possibly empty) collection of distinct closed points of the projective line $\mathbb{P}^1_k$, and $\mathbf{p} = (p_1, \dots, p_n)$ is a *weight sequence*, that is, a sequence of positive integers. In this work we make the additional assumption that the closed points of $\boldsymbol{\lambda}$ are supposed to be *rational.* This assumption simplifies our exposition. In fact, there is no substantial difference between this case and the case where the field k is algebraically closed.

We refer to the introduction for the definition of the category $\operatorname{coh} \mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$.

Let us remark that since the field k is not necessarily algebraically closed, the $\mathbf{L}(\mathbf{p})$-graded algebra $S(\mathbf{p}, \boldsymbol{\lambda})$, even up to isomorphism, might depend on the choice of the homogeneous coordinates $\lambda_{i0}, \lambda_{i1}$ for each $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$. However, up to equivalence the associated category $\operatorname{coh} \mathbb{X}$ of coherent sheaves is independent of this choice; see Corollary 7.4.4.

6.2. **Hereditary noetherian categories with tilting object.** Let $\mathcal{A}$ be a k-linear abelian category. We consider the following set of axioms:

(H1) The category $\mathcal{A}$ is skeletally small, connected, and Ext-finite.
(H2) The category $\mathcal{A}$ is noetherian, that is, each object of $\mathcal{A}$ is noetherian.
(H3) The category $\mathcal{A}$ is hereditary and has no non-zero projective object.
(H4) The category $\mathcal{A}$ has a tilting object.
(H5) The Euler form associated to $\mathcal{A}$ is non-degenerate and has discriminant ± 1.

Let us collect the basic properties of a category satisfying (H1)–(H4) so that we can use them from now on freely without any further reference.

Recall that $\mathcal{A}_0$ denotes the full subcategory consisting of all finite length objects in $\mathcal{A}$ and that $\mathcal{A}_+$ is the full subcategory consisting of all objects A in $\mathcal{A}$ satisfying $\operatorname{Hom}_{\mathcal{A}}(A_0, A) = 0$ for all A_0 in $\mathcal{A}_0$.

One should think of objects in $\mathcal{A}_0$ as *torsion objects*, whereas the objects in $\mathcal{A}_+$ are *torsion-free* or *vector bundles.*

Proposition 6.2.1. *A k-linear abelian category $\mathcal{A}$ satisfying* (H1)–(H4) *has the following properties:*

(1) *The category $\mathcal{A}$ admits a Serre functor $\tau \colon \mathcal{A} \to \mathcal{A}$.*
(2) *The Grothendieck group $K_0(\mathcal{A})$ is free of finite rank and the Euler form associated to $\mathcal{A}$ is non-degenerate.*
(3) *Every object in $\mathcal{A}$ is a direct sum of an object in $\mathcal{A}_0$ and an object in $\mathcal{A}_+$.*
(4) *The category of finite length objects admits a decomposition $\mathcal{A}_0 = \coprod_{x \in \mathbf{X}} \mathcal{A}_x$ into connected uniserial categories.*

Proof. (1) follows from Proposition 3.4.5, (2) from Proposition 3.5.2, (3) from Proposition 1.8.1, and (4) from Proposition 1.8.2. □

The noetherianness of $\mathcal{A}$ implies that $\mathcal{A}_0$ is non-trivial. On the other hand, $\mathcal{A} \neq \mathcal{A}_0$ because a length category with Serre duality and a Grothendieck group of finite rank has a degenerate Euler form; see Example 3.5.3.

Denote by Σ_0 a set of representatives of the isomorphism classes of simple objects of $\mathcal{A}$. We identify the set Σ_0/τ of τ-orbits with the index set $\mathbf{X}$ of the decomposition $\mathcal{A}_0 = \coprod_{x \in \mathbf{X}} \mathcal{A}_x$. For each $x \in \mathbf{X}$, let $p(x)$ denote the number of isomorphism classes of simple objects of $\mathcal{A}_x$.

Lemma 6.2.2. *Each $p(x)$ is finite. More precisely, $\sum_{x \in \mathbf{X}} (p(x) - 1)$ is bounded by the rank of $K_0(\mathcal{A})$.*

Proof. First observe that for two simple objects $S, T \in \mathcal{A}$, we have $\operatorname{Ext}^1_{\mathcal{A}}(S,T) \neq 0$ if and only if $T \cong \tau S$; see Theorem 1.7.1 and Proposition 1.8.1. Now choose for each $x \in \mathbf{X}$ a simple object $S_x \in \mathcal{A}_x \cap \Sigma_0$ and let $\Sigma_0' = \Sigma_0 \setminus \{S_x \mid x \in \mathbf{X}\}$. The set Σ_0' admits a linear ordering such that $S > T$ implies $\langle [S], [T] \rangle = 0$. It follows that the corresponding classes $[S]$, $S \in \Sigma_0'$, are linearly independent in $K_0(\mathcal{A})$; see Lemma 6.2.3 below. Thus $\operatorname{card} \Sigma_0' = \sum_{x \in \mathbf{X}} (p(x) - 1)$ is bounded by the rank of $K_0(\mathcal{A})$. □

Lemma 6.2.3. *Let G be an abelian group and $X \subseteq G$ a subset. Suppose there is a non-degenerate bilinear form ϕ on G and a linear ordering on X such that $\phi(x,x) \neq 0$ for all $x \in X$ and $\phi(x,y) = 0$ for all $x > y$ in X. Then X is linearly independent.*

Proof. Straightforward. □

Lemma 6.2.4. *There exists a linear map $\rho\colon K_0(\mathcal{A}) \to \mathbb{Z}$ such that*

(1) $\rho([A]) \geq 0$ *for all* $A \in \mathcal{A}$;
(2) $\rho([A]) = 0$ *if and only if* $A \in \mathcal{A}_0$;
(3) $\rho([\tau A]) = \rho([A])$ *for all* $A \in \mathcal{A}$.

Proof. For each $x \in \mathbf{X}$ choose a simple object $S_x \in \mathcal{A}_x$ and let $w_x = \sum_{i=1}^{p(x)} [\tau^i S_x]$. Choose elements $x_1, \ldots, x_r$ from $\mathbf{X}$ such that the subgroup of $K_0(\mathcal{A})$ generated by $w_{x_1}, \ldots, w_{x_r}$ contains each w_x and set $w = w_{x_1} + \cdots + w_{x_r}$.

Serre duality implies that $\langle [A], w \rangle = 0$ for all A in $\mathcal{A}_0$. If A is a non-zero object in $\mathcal{A}_+$, then A has a simple quotient, say S_x, by noetherianness. Using that $\operatorname{Ext}^1_{\mathcal{A}}(A, A_0) = 0$ for all A_0 in $\mathcal{A}_0$ by Serre duality, this yields $\langle [A], w_x \rangle > 0$ and therefore $\langle [A], w \rangle > 0$. Thus the map $\rho = \langle -, w \rangle$ has the desired properties. □

Proposition 6.2.5. *The abelian category $\mathcal{A}/\mathcal{A}_0$ is a length category.*

Proof. Let A be an object in $\mathcal{A}$. We prove by induction on $\rho([A])$ that A has finite length in $\mathcal{A}/\mathcal{A}_0$. If $\rho([A]) = 0$, then $A = 0$ in $\mathcal{A}/\mathcal{A}_0$. Now suppose $\rho([A]) > 0$. The category $\mathcal{A}/\mathcal{A}_0$ is noetherian and therefore each non-zero object has a simple quotient. Thus there exists a subobject $A' \subseteq A$ such that A/A' is simple in $\mathcal{A}/\mathcal{A}_0$. Then $\rho([A']) < \rho([A])$ and therefore A' has finite length in $\mathcal{A}/\mathcal{A}_0$. It follows that A has finite length in $\mathcal{A}/\mathcal{A}_0$. □

For each object A in $\mathcal{A}$, we denote by $\operatorname{rank} A$ the length of A in $\mathcal{A}/\mathcal{A}_0$ and call it the *rank*. This function extends to a linear map $K_0(\mathcal{A}) \to \mathbb{Z}$. This linear map is surjective and satisfies the conditions in Lemma 6.2.4. Indeed, such a map is unique by Proposition 6.3.7.

6.3. **Line bundles.** Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H4). An indecomposable object in $\mathcal{A}$ of rank one is called *line bundle*. Thus line bundles are precisely the objects in $\mathcal{A}_+$ of rank one; they form the building blocks of the category $\mathcal{A}_+$. Let us collect their basic properties.

Proposition 6.3.1. *Every object A in $\mathcal{A}_+$ admits a filtration $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ of length $n = \operatorname{rank} A$ such that each factor A_i/A_{i-1} is a line bundle.*

Proof. We proceed by induction on n. The case $n \leq 1$ is clear. If $n > 1$, choose a monomorphism $U \to A$ in $\mathcal{A}/\mathcal{A}_0$ with simple cokernel. This morphism is represented by a morphism $\phi\colon U' \to A/A'$ in $\mathcal{A}$ such that $U' \subseteq U$ and $A' \subseteq A$ are subobjects with U/U' and A' in $\mathcal{A}_0$; see Lemma 1.2.2. It follows that $A' = 0$ since $A \in \mathcal{A}_+$. Passing from U' to the image of ϕ, we may assume that $\operatorname{Ker} \phi = 0$. The cokernel $C = \operatorname{Coker}\phi$ is simple in $\mathcal{A}/\mathcal{A}_0$. Thus there is a decomposition $C = C_0 \oplus C_1$ with C_0 of finite length and C_1 a line bundle. Forming the pullback of the exact sequence $0 \to U' \to A \to C \to 0$ along the inclusion $C_0 \to C$ yields a monomorphism $A_{n-1} \to A$ with cokernel C_1. Clearly, A_{n-1} belongs to $\mathcal{A}_+$ and has rank $n-1$. □

Lemma 6.3.2. *Let A be a non-zero object in $\mathcal{A}_+$ and suppose that $\operatorname{Ext}^1_{\mathcal{A}}(S, A) \neq 0$ for some simple object S.*

(1) *There exists a chain of monomorphisms $A = A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \cdots$ in $\mathcal{A}$ such that $A_i \in \mathcal{A}_+$ and $A_i/\operatorname{Im}\phi_i \cong \tau^{-i+1}S$ for all $i > 0$.*

(2) *For each $n \geq 0$, there exists an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that $B \in \mathcal{A}_+$ and $[C] = \sum_{i=0}^{n}[\tau^{-i}S]$.*

Proof. (1) Choose a non-split exact sequence $0 \to A \xrightarrow{\phi_1} A_1 \to S \to 0$. For each simple object T, the induced morphism $\operatorname{Hom}_{\mathcal{A}}(T, A) \to \operatorname{Hom}_{\mathcal{A}}(T, A_1)$ is an isomorphism. Thus A_1 belongs to $\mathcal{A}_+$. Serre duality implies that $\operatorname{Ext}^1_{\mathcal{A}}(\tau^{-1}S, A_1) \neq 0$. Thus we can iterate the construction and obtain a sequence of morphisms $\phi_i \colon A_{i-1} \to A_i$

(2) Apply (1) by taking the composite $\phi_{n+1} \ldots \phi_1$ for the morphism $A \to B$. □

Lemma 6.3.3. *Let L, L' be line bundles and $0 \to L \to L' \to S \to 0$ an exact sequence such that S is simple. Then for each $x \in \mathbf{X}$, $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L', \mathcal{A}_x) = 0$.*

Proof. Note that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L, S_x) \neq 0$ for some simple object $S_x \in \mathcal{A}_x$. The assumptions imply $\operatorname{Hom}_{\mathcal{A}}(L', S) \neq 0$ and $\operatorname{Hom}_{\mathcal{A}}(L, \tau S) \neq 0$. If T is a simple object not lying in the τ-orbit of S, then $L \to L'$ induces an isomorphism $\operatorname{Hom}_{\mathcal{A}}(L', T) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(L, T)$. □

Lemma 6.3.4. *Let L, L' be line bundles. Then the following are equivalent:*

(1) *For each $x \in \mathbf{X}$, $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L', \mathcal{A}_x) = 0$.*

(2) *There exists $x \in \mathbf{X}$ such that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$ and $\operatorname{Hom}_{\mathcal{A}}(L', \mathcal{A}_x) \neq 0$.*

(3) *There exists $n \in \mathbb{Z}$ such that $L' \cong \tau^n L$ in $\mathcal{A}/\mathcal{A}_0$.*

Proof. (1) ⇒ (2): Observe that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L, S) \neq 0$ for some simple object $S \in \mathcal{A}_x$. Thus noetherianness of L implies that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$ for some $x \in \mathbf{X}$.

(2) ⇒ (3): Choose a simple object S and a non-zero morphism $\phi \colon L \to S$. Modulo some power of τ, there is a non-zero morphism $\phi' \colon L' \to S$. Forming the pullback of ϕ and ϕ', one obtains the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \phi'} & & \\
0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{\phi} & S & \longrightarrow & 0
\end{array}
$$

If the top row splits, then $\operatorname{Hom}_{\mathcal{A}}(L', L) \neq 0$ and therefore $L' \cong L$ in $\mathcal{A}/\mathcal{A}_0$. Otherwise, $\operatorname{Hom}_{\mathcal{A}}(K, \tau L') \cong D\operatorname{Ext}^1_{\mathcal{A}}(L', K) \neq 0$. Thus $L \cong \tau L'$ in $\mathcal{A}/\mathcal{A}_0$.

(3) ⇒ (1): Suppose that $L' \cong \tau^n L$ in $\mathcal{A}/\mathcal{A}_0$. Then there is a subobject $L'' \subseteq L'$ with L'/L'' of finite length and there is a monomorphism $L'' \to \tau^n L$ with cokernel of finite length. For each $x \in \mathbf{X}$, it follows then by iterating Lemma 6.3.3 that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L', \mathcal{A}_x) \neq 0$. □

Proposition 6.3.5 (Lenzing)**.** *Let L be a line bundle and $x \in \mathbf{X}$. Then we have $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_x) \neq 0$.*

Proof. Let $\mathbf{X}_1$ be the set of all $y \in \mathbf{X}$ such that $\operatorname{Hom}_{\mathcal{A}}(L, \mathcal{A}_y) \neq 0$, and let $\mathbf{X}_2 = \mathbf{X} \smallsetminus \mathbf{X}_1$. We show that $\mathbf{X}_1 = \mathbf{X}$. Let $\mathcal{L}_1$ be the class of line bundles L' such that $L' \cong \tau^n L$ in $\mathcal{A}/\mathcal{A}_0$ for some $n \in \mathbb{Z}$, and let $\mathcal{L}_2$ be the class of all remaining line bundles. Denote by $\mathcal{A}_i$ $(i = 1, 2)$ the full subcategory consisting of objects $A \in \mathcal{A}$ having a filtration with factors in $\mathcal{L}_i$ or $\bigcup_{x \in \mathbf{X}_i} \mathcal{A}_x$. There are no non-zero morphisms between objects from different $\mathcal{A}_i$'s, and therefore also no extensions by Serre duality. This follows from Lemma 6.3.4 and the fact that each non-zero morphism between line bundles in $\mathcal{A}$ induces an isomorphism in $\mathcal{A}/\mathcal{A}_0$. On the other hand, each indecomposable object belongs to one of the $\mathcal{A}_i$'s. This is clear for objects of finite length. An object A from $\mathcal{A}_+$ has a finite filtration $0 = A_0 \subseteq \cdots \subseteq A_r = A$ such that each factor A_i/A_{i-1} is a line bundle; see Proposition 6.3.1. The factors belong to a single $\mathcal{L}_i$ since there are no non-split extensions between different $\mathcal{L}_i$'s. Thus $\mathcal{A} = \mathcal{A}_1 \amalg \mathcal{A}_2$, but this implies $\mathcal{A}_2 = 0$ since $\mathcal{A}$ is connected. □

Lemma 6.3.6. *Let L be a line bundle and $0 \neq A \in \mathcal{A}_+$. Then there are monomorphisms $L \to A'$ and $A \to A'$ such that A' belongs to $\mathcal{A}_+$ and the cokernel of $A \to A'$ belongs to $\mathcal{A}_0$. Moreover, each non-zero morphism $L \to A$ is a monomorphism.*

Proof. The object A has a simple quotient since it is noetherian. Thus $\operatorname{Ext}^1_{\mathcal{A}}(S, A) \neq 0$ for some simple object S by Serre duality. Depending on the value of $\langle [L], [A] \rangle$ and using that L admits a non-zero morphism to the τ-orbit of S by Proposition 6.3.5, we choose in Lemma 6.3.2 the number n sufficiently big so that there exists an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $B \in \mathcal{A}_+$, $[C] = \sum_{i=0}^{n} [\tau^{-i} S]$, and

$$\langle [L], [B] \rangle = \langle [L], [A] \rangle + \langle [L], [C] \rangle > 0.$$

Now set $A' = B$.

If $\phi \colon L \to A$ is a non-zero morphism, then $\operatorname{rank} \operatorname{Ker} \phi = 0$. Thus $\operatorname{Ker} \phi = 0$ since L belongs to $\mathcal{A}_+$. □

The discussion of line bundles yields further properties of $\mathcal{A}/\mathcal{A}_0$ and $K_0(\mathcal{A})$.

Proposition 6.3.7. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H4). *Then the following holds:*

(1) *The abelian category $\mathcal{A}/\mathcal{A}_0$ has, up to isomorphism, a unique simple object.*
(2) *Each non-zero object A in $\mathcal{A}$ satisfies $[A] \neq 0$ in $K_0(\mathcal{A})$.*
(3) *Let L be a line bundle in $\mathcal{A}$. Then $K_0(\mathcal{A}) = \mathbb{Z}[L] \oplus K_0'(\mathcal{A})$, where $K_0'(\mathcal{A})$ denotes the image of the canonical map $K_0(\mathcal{A}_0) \to K_0(\mathcal{A})$.*

Proof. (1) Let L, L' be line bundles in $\mathcal{A}$. There are monomorphisms $L \to L''$ and $L' \to L''$ with both cokernels in $\mathcal{A}_0$, by Lemma 6.3.6. Thus $L \cong L'' \cong L'$ in $\mathcal{A}/\mathcal{A}_0$.

(2) Let A be a non-zero object. If A is not of finite length, then $\operatorname{rank}[A] \neq 0$ and therefore $[A] \neq 0$. Now suppose that A is of finite length. It follows from Proposition 6.3.5 that $\operatorname{Hom}_{\mathcal{A}}(L, A) \neq 0$ for some line bundle L. On the other hand, $\operatorname{Ext}^1_{\mathcal{A}}(L, A) = 0$ by Serre duality. Thus $\langle [L], [A] \rangle \neq 0$, and it follows that $[A] \neq 0$.

(3) We have $\mathbb{Z}[L] \cap K_0'(\mathcal{A}) = 0$ since $\operatorname{rank} L > 0$ and $\operatorname{rank} x = 0$ for all $x \in K_0'(\mathcal{A})$. We show by induction on the rank that each class $[A]$ belongs to $\mathbb{Z}[L] + K_0'(\mathcal{A})$. This is clear if $\operatorname{rank} A = 0$. If $\operatorname{rank} A > 0$, then there is an exact sequence $0 \to L \to A' \to A'' \to 0$ such that $A' \cong A$ in $\mathcal{A}/\mathcal{A}_0$; see Lemma 6.3.6. It follows that $\operatorname{rank} A'' = \operatorname{rank} A' - 1$, and therefore $[A'] = [L] + [A'']$ belongs to $\mathbb{Z}[L] + K_0'(\mathcal{A})$. Finally observe that $[A] - [A']$ belongs to $K_0'(\mathcal{A})$. □

An immediate consequence is the fact that the rank of $K_0(\mathcal{A})$ is at least two.

6.4. **Exceptional objects.** Let $\mathcal{A}$ be a k-linear abelian category that is Hom-finite and hereditary. An object A is called *exceptional* if $\operatorname{Ext}^1_{\mathcal{A}}(A, A) = 0$ and $\operatorname{End}_{\mathcal{A}}(A)$ is a division ring.

Lemma 6.4.1 (Happel-Ringel)**.** *Let A, B be indecomposable objects in $\mathcal{A}$. If $\operatorname{Ext}^1_{\mathcal{A}}(B, A) = 0$, then each non-zero morphism $A \to B$ is a monomorphism or an epimorphism.*

Proof. Let $\phi\colon A \to B$ be a non-zero morphism and $A \xrightarrow{\phi'} \operatorname{Im}\phi \xrightarrow{\phi''} B$ its canonical factorization. We obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B/\operatorname{Im}\phi & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Im}\phi & \xrightarrow{\phi''} & B & \longrightarrow & B/\operatorname{Im}\phi & \longrightarrow & 0 \end{array}$$

since $\operatorname{Ext}^1_{\mathcal{A}}(B/\operatorname{Im}\phi, -)$ is right exact. The induced exact sequence $0 \to A \to \operatorname{Im}\phi \oplus E \to B \to 0$ splits. Thus ϕ' or ϕ'' is a split monomorphism since $\operatorname{End}_{\mathcal{A}}(A)$ is local. In the first case ϕ is a monomorphism, and in the second case ϕ is an epimorphism. □

Let us collect some immediate consequences.

Proposition 6.4.2. *Let $\mathcal{A}$ be a k-linear abelian category that is Hom-finite and hereditary. Then the following holds:*

(1) *An indecomposable object A satisfying $\operatorname{Ext}^1_{\mathcal{A}}(A, A) = 0$ is exceptional.*
(2) *Let A, B be non-isomorphic exceptional objects. Suppose that $\operatorname{Ext}^1_{\mathcal{A}}(A, B) = 0$ and $\operatorname{Ext}^1_{\mathcal{A}}(B, A) = 0$. Then $\operatorname{Hom}_{\mathcal{A}}(A, B) = 0$ or $\operatorname{Hom}_{\mathcal{A}}(B, A) = 0$.*
(3) *Assume further that $\mathcal{A}$ is Ext-finite. Let A, B be exceptional objects and $[A] = [B]$ in $K_0(\mathcal{A})$. Then $A \cong B$.*

Proof. We apply Lemma 6.4.1 and use the fact that for each indecomposable object, an endomorphism is either nilpotent or invertible. In particular, an endomorphism that is a monomorphism or an epimorphism is invertible. From this, (1) and (2) are clear.

(3) Observe that $\operatorname{Hom}_{\mathcal{A}}(A, B) \neq 0$ since $\langle [A], [B] \rangle > 0$. Let $\phi\colon A \to B$ be a non-zero morphism and $B' = \operatorname{Im}\phi$. Applying the right exact functor $\operatorname{Ext}^1_{\mathcal{A}}(-, B)$ to the inclusion $B' \to B$ shows that $\operatorname{Ext}^1_{\mathcal{A}}(B', B) = 0$. Thus $\langle [B'], [A] \rangle = \langle [B'], [B] \rangle > 0$. Composing a non-zero morphism $B' \to A$ with the epimorphism $A \to B'$ induced by ϕ yields an isomorphism since $\operatorname{End}_{\mathcal{A}}(A)$ is a division ring. Thus ϕ is a monomorphism. The dual argument shows that ϕ is an epimorphism. Thus $A \cong B$. □

6.5. **Expansions of abelian categories.** We consider expansions of abelian categories that satisfy the axioms (H1)–(H4).

Lemma 6.5.1. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H4). *For a full subcategory $\mathcal{B}$ of $\mathcal{A}$, the following are equivalent:*

(1) *The inclusion $\mathcal{B} \to \mathcal{A}$ is a non-split expansion of abelian categories.*
(2) *There exists a simple object S such that $\tau S \not\cong S$ and $S^\perp = \mathcal{B}$.*

Proof. Apply Lemma 4.1.2. If $\mathcal{B} \to \mathcal{A}$ is an expansion, then $\mathcal{B} = S^\perp$ for some localizable object S, and Serre duality implies $\tau S \not\cong S$. Conversely, if S is simple and $\tau S \not\cong S$, then S is a (co)localizable object with $S^\perp = {}^\perp\tau S$. Thus the inclusion $S^\perp \to \mathcal{A}$ is an expansion, and this is non-split since $\mathcal{A}$ is connected. □

Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of k-linear abelian categories satisfying (H1)–(H4). Then i restricts to an expansion $\mathcal{B}_0 \to \mathcal{A}_0$ by Proposition 4.3.3. Let $\mathcal{A}_0 = \coprod_{x\in \mathbf{X}_\mathcal{A}} \mathcal{A}_x$ and $\mathcal{B}_0 = \coprod_{x\in \mathbf{X}_\mathcal{B}} \mathcal{B}_x$ be the decompositions into connected uniserial categories; see Proposition 1.8.2.

Proposition 6.5.2. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of k-linear abelian categories satisfying* (H1)–(H4)*. There exists a bijection $\phi\colon \mathbf{X}_\mathcal{B} \to \mathbf{X}_\mathcal{A}$ and $x_0 \in \mathbf{X}_\mathcal{B}$ such that i restricts to an expansion $\mathcal{B}_{x_0} \to \mathcal{A}_{\phi(x_0)}$ and to equivalences $\mathcal{B}_x \xrightarrow{\sim} \mathcal{A}_{\phi(x)}$ for all $x \neq x_0$ in $\mathbf{X}_\mathcal{B}$.*

Proof. Apply Lemma 4.5.1. □

Next we investigate the existence of tilting objects for expansions of abelian categories.

Lemma 6.5.3. *Let $i\colon \mathcal{B} \to \mathcal{A}$ be a non-split expansion of k-linear abelian categories that are Ext-finite.*

(1) *If $\mathcal{B}$ admits a tilting object, then $\mathcal{A}$ admits a tilting object.*
(2) *Suppose in addition that $\mathcal{A}$ is hereditary and has no non-zero projective object. If $\mathcal{A}$ admits a tilting object, then $\mathcal{B}$ admits a tilting object.*

Proof. (1) Let T be a tilting object in $\mathcal{B}$. Choose an exact sequence $0 \to S \to T' \to iT \to 0$ in $\mathcal{A}$ with S in $\operatorname{add} S_\lambda$ such that the induced map $\operatorname{Hom}_\mathcal{A}(S, S_\lambda) \to \operatorname{Ext}^1_\mathcal{A}(iT, S_\lambda)$ is an epimorphism. We claim that $U = T' \oplus S_\lambda$ is a tilting object for $\mathcal{A}$.

The formula $\operatorname{Ext}^n_\mathcal{A}(iT, A) \cong \operatorname{Ext}^n_\mathcal{B}(T, i_\rho A)$ for all $A \in \mathcal{A}$ and $n \geq 0$ implies $\operatorname{proj.dim} iT \leq 1$. Therefore $\operatorname{proj.dim} U \leq 1$. The construction of T' implies $\operatorname{Ext}^1_\mathcal{A}(T', S_\lambda) = 0$, and $\operatorname{Ext}^1_\mathcal{A}(S_\lambda, T') = 0$ is clear since $\operatorname{Ext}^1_\mathcal{A}(S_\lambda, S) = 0$ and $\operatorname{Ext}^1_\mathcal{A}(S_\lambda, iT) = 0$. We have

$$\operatorname{Ext}^1_\mathcal{A}(T', T') \cong \operatorname{Ext}^1_\mathcal{A}(T', iT) \cong \operatorname{Ext}^1_\mathcal{B}(i_\lambda T', T) \cong \operatorname{Ext}^1_\mathcal{B}(T, T) = 0,$$

and therefore $\operatorname{Ext}^1_\mathcal{A}(U, U) = 0$. Finally, assume that $\operatorname{Ext}^n_\mathcal{A}(U, A) = 0$ for some A in $\mathcal{A}$ $(n = 0, 1)$. The condition $\operatorname{Ext}^n_\mathcal{A}(S_\lambda, A) = 0$ implies that A belongs to the image of i, say $A = iB$, and that $\operatorname{Ext}^n_\mathcal{A}(iT, A) = 0$. Then $0 = \operatorname{Ext}^n_\mathcal{A}(iT, iB) \cong \operatorname{Ext}^n_\mathcal{B}(T, B)$ implies $B = 0$. Thus U is a tilting object.

(2) Let T be a tilting object in $\mathcal{A}$. We intend to show that $i_\lambda T$ is a tilting object for $\mathcal{B}$. The formula $\operatorname{Ext}^n_\mathcal{B}(i_\lambda T, B) \cong \operatorname{Ext}^n_\mathcal{A}(T, iB)$ for all $B \in \mathcal{B}$ and $n \geq 0$ shows that $\operatorname{proj.dim} i_\lambda T \leq 1$ and that $\operatorname{Ext}^n_\mathcal{B}(i_\lambda T, B) = 0$ $(n = 0, 1)$ implies $B = 0$. It remains to show that $\operatorname{Ext}^1_\mathcal{B}(i_\lambda T, i_\lambda T) = 0$. In fact, it is equivalent to show that $\operatorname{Ext}^1_\mathcal{A}(T, ii_\lambda T) = 0$.

We proceed by cases. First assume that $\operatorname{Ext}^1_\mathcal{A}(S_\lambda, T) = 0$. Thus the adjunction morphism $\eta_T\colon T \to ii_\lambda T$ is an epimorphism, and therefore $\operatorname{Ext}^1_\mathcal{A}(T, ii_\lambda T) = 0$ since $\operatorname{Ext}^1_\mathcal{A}(T, -)$ is right exact.

Next assume that $\operatorname{Ext}^1_\mathcal{A}(T, S_\lambda) = 0$. Then $\operatorname{Ext}^1_\mathcal{A}(T, ii_\lambda T) = 0$ follows since $\operatorname{Ker} \eta_T$ and $\operatorname{Coker} \eta_T$ belong to $\operatorname{add} S_\lambda$.

Finally, assume that $\operatorname{Ext}^1_\mathcal{A}(S_\lambda, T) \neq 0$. We apply the Auslander-Reiten formula from Proposition 4.4.1 and have $\operatorname{Hom}_\mathcal{A}(T, S_\rho) \neq 0$. Thus there is an epimorphism $T \to S_\rho$, and this implies $\operatorname{Ext}^1_\mathcal{A}(T, S_\rho) = 0$. The category $\mathcal{A}$ admits a Serre functor $\tau\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ by Proposition 3.4.5, and the Auslander-Reiten formula implies $\tau S_\lambda = S_\rho$. Thus $\operatorname{Ext}^1_\mathcal{A}(\tau^{-1}T, S_\lambda) = 0$. The object $U = \tau^{-1}T$ is a tilting object for $\mathcal{A}$, and the above argument shows that $i_\lambda U$ is a tilting object for $\mathcal{B}$. □

The following result says that the concept of an expansion of abelian categories is compatible with the list of axioms (H1)–(H5).

Theorem 6.5.4. *Let k be a field and $\mathcal{B} \to \mathcal{A}$ a non-split expansion of k-linear abelian categories with associated division ring k. Then $\mathcal{A}$ satisfies* (H1)–(H5) *if and only if $\mathcal{B}$ satisfies* (H1)–(H5)*. In that case the rank of $K_0(\mathcal{B})$ is one less than that of $K_0(\mathcal{A})$.*

Proof. We provide the references for each axiom. The final assertion about the rank of $K_0(\mathcal{B})$ follows from Lemma 3.5.7.
(H1) Lemmas 1.3.1, 4.5.1, and 4.6.2.
(H2) Lemma 1.3.3.
(H3) Lemma 4.6.1.
(H4) Lemma 6.5.3.
(H5) Proposition 3.5.2 and Lemma 3.5.7. □

6.6. **An equivalence via tilting.** We give a criterion so that an equivalence of derived categories $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$ restricts to an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$.

Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H4) and consider its bounded derived category $\mathbf{D}^b(\mathcal{A})$. Recall that

$$\mathbf{D}^b(\mathcal{A}) = \bigsqcup_{n\in\mathbb{Z}} \mathcal{A}[n]$$

with non-zero morphisms $\mathcal{A}[i] \to \mathcal{A}[j]$ only if $j - i \in \{0, 1\}$ since $\mathcal{A}$ is hereditary; see Corollary 2.2.2.

The isomorphism $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathbf{D}^b(\mathcal{A}))$ yields a rank function $K_0(\mathbf{D}^b(\mathcal{A})) \to \mathbb{Z}$. Note that for each complex X concentrated in degree n, we have $(-1)^n{\cdot}\operatorname{rank}[X] \geq 0$.

Let T be an indecomposable object in $\mathcal{A}_+$ and view it as a complex concentrated in degree zero. Define $\mathcal{L}(T)$ to be the class of indecomposable objects $L \in \mathbf{D}^b(\mathcal{A})$ of rank one such that $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L', L) \neq 0$ for some indecomposable object $L' \in \mathbf{D}^b(\mathcal{A})$ of rank one satisfying $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L', T) \neq 0$.

Lemma 6.6.1. *Let T be an indecomposable object in $\mathcal{A}_+$. Then the objects in $\mathcal{L}(T)$ are precisely those that are isomorphic to a line bundle in $\mathcal{A}$, viewed as a complex concentrated in degree zero.*

Proof. Let L' be a complex in $\mathbf{D}^b(\mathcal{A})$ that is concentrated in one degree, say n. If $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L', T) \neq 0$, then $n = 0$ or $n = 1$. If the rank of L' is positive, then $n = 0$. The same argument shows that every indecomposable object L in $\mathbf{D}^b(\mathcal{A})$ of rank one and satisfying $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L', L) \neq 0$ is isomorphic to a line bundle, viewed as a complex concentrated in degree zero.

Conversely, let L be a line bundle. Then there exists a non-zero morphism $L \to T'$ for some object T' that admits an exact sequence $0 \to T \to T' \to C \to 0$ such that C has finite length; see Lemma 6.3.6. The pullback of $T \to T'$ and $L \to T'$ yields a line bundle L' with non-zero morphisms to T and L. Thus L belongs to $\mathcal{L}(T)$. □

Lemma 6.6.2. *Let T be an indecomposable object in $\mathcal{A}_+$. Then the following are equivalent for an indecomposable object X in $\mathbf{D}^b(\mathcal{A})$:*

(1) *$H^iX = 0$ for all $i \neq 0$.*
(2) *$\operatorname{rank}[X] \geq 0$ and $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L, X) \neq 0$ for some $L \in \mathcal{L}(T)$.*

Proof. We apply Lemma 6.6.1. Because X is indecomposable, there exists an integer n such that $H^iX = 0$ for all $i \neq n$.

(1) ⇒ (2): If $n = 0$, then $\operatorname{rank}[X] \geq 0$ and $\operatorname{Hom}_{\mathcal{A}}(L, H^0X) \neq 0$ for some line bundle L in $\mathcal{A}$. Thus $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L, X) \neq 0$ for some $L \in \mathcal{L}(T)$.

(2) ⇒ (1): If $\operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(L,X) \neq 0$ for some $L \in \mathcal{L}(T)$, then $n=0$ or $n=1$. It follows that $n=0$ if X has positive rank. If the rank of X is zero, then $H^n X$ has finite length and therefore $\operatorname{Ext}^1_{\mathcal{A}}(H^0L, H^nX)=0$. Thus $n=0$. □

Proposition 6.6.3. *Let $\mathcal{A},\mathcal{A}'$ be abelian categories satisfying* (H1)–(H4) *with tilting objects $T\in\mathcal{A}$ and $T'\in\mathcal{A}'$. Suppose that $\operatorname{End}_{\mathcal{A}}(T)\cong \operatorname{End}_{\mathcal{A}'}(T')$ and that the induced equivalence $\operatorname{add}T \xrightarrow{\sim} \operatorname{add}T'$ preserves the rank. Then $\mathcal{A}$ and $\mathcal{A}'$ are equivalent categories.*

Proof. We identify $\Lambda = \operatorname{End}_{\mathcal{A}}(T) = \operatorname{End}_{\mathcal{A}'}(T')$ and obtain equivalences

$$\mathbf{D}^b(\mathcal{A}) \xrightarrow{\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T,-)} \mathbf{D}^b(\operatorname{mod}\Lambda) \xleftarrow{\mathbf{R}\operatorname{Hom}(T',-)} \mathbf{D}^b(\mathcal{A}').$$

This yields an equivalence $F\colon \mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$ taking T to T'. The functor F preserves the rank since the indecomposable direct summands of T form a basis of $K_0(\mathbf{D}^b(\mathcal{A}))$. It follows from Lemma 6.6.2 that F identifies $\mathcal{A}$ with $\mathcal{A}'$. □

6.7. **The homogeneous case.** Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H4). Call $\mathcal{A}$ *homogeneous* if $\tau A\cong A$ for each object A of finite length, or equivalently, $\tau S\cong S$ for each simple object S. This property together with the conditions (H1)–(H5) characterizes the category $\operatorname{coh}\mathbb{P}^1_k$. The following characterization of the property of $\mathcal{A}$ to be homogeneous will be useful.

Proposition 6.7.1. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H4). *Then the following are equivalent:*

1. *If S is a simple object in $\mathcal{A}$, then $\tau S\cong S$.*
2. *If A,B are finite length objects in $\mathcal{A}$, then $\langle [A],[B]\rangle = 0$.*
3. *The rank of $K_0(\mathcal{A})$ equals two.*
4. *If A has infinite length and S is a simple object in $\mathcal{A}$, then $\operatorname{Hom}_{\mathcal{A}}(A,S)\neq 0$.*

Proof. (1) ⇒ (2): It suffices to show that $\langle [S],[T]\rangle = 0$ for each pair of simple objects S,T. The equality $\langle [S],[T]\rangle = 0$ is an immediate consequence of Serre duality if $\tau T\cong T$.

(2) ⇒ (3): Let $K_0'(\mathcal{A})$ be the image of the canonical map $K_0(\mathcal{A}_0)\to K_0(\mathcal{A})$. We have $K_0(\mathcal{A})\cong \mathbb{Z}\oplus K_0'(\mathcal{A})$ by Proposition 6.3.7. Now apply Lemma 6.7.2 below.

(3) ⇒ (1): Suppose there exists a simple object S such that $\tau S\ncong S$. Then $\mathcal{B}=S^{\perp}$ satisfies (H1)–(H4) and $K_0(\mathcal{A})\cong \mathbb{Z}[S]\oplus K_0(\mathcal{B})$, by Theorem 6.5.4 and Lemma 6.5.1. If $\mathcal{B}$ is homogeneous, then $K_0(\mathcal{A})\cong\mathbb{Z}^3$ by the first part of the proof. Otherwise, we proceed as before and reduce to the homogeneous case. In any case, the rank of $K_0(\mathcal{A})$ is at least 3.

(1) ⇒ (4): This follows from Proposition 6.3.5 since each infinite length object A admits a subobject A' such that A/A' is a line bundle.

(4) ⇒ (1): Suppose there exists a simple object S such that $\tau S\ncong S$. Then $\mathcal{B}=S^{\perp}={}^{\perp}\tau S$ yields an expansion $\mathcal{B}\to\mathcal{A}$ by Lemma 6.5.1, and this induces an equivalence $\mathcal{B}/\mathcal{B}_0\to\mathcal{A}/\mathcal{A}_0$ by Proposition 4.3.3. Thus any infinite length object in $\mathcal{A}$ yields one in $\mathcal{B}$, say A, satisfying $\operatorname{Hom}_{\mathcal{A}}(A,\tau S)=0$ by construction. □

Lemma 6.7.2. *Let G be a free abelian group of finite rank with a non-degenerate bilinear form ϕ. Suppose there is a subgroup $0\neq H\subseteq G$ such that $G/H\cong\mathbb{Z}$ and $\phi(x,y)=0$ for all $x,y\in H$. Then $G\cong\mathbb{Z}^2$.*

Proof. The assumption on H implies that for each pair of non-zero elements $x,y\in H$, there are non-zero integers α_x,α_y with $\alpha_x x=\alpha_y y$. This implies $H\cong\mathbb{Z}$ since H is free. □

Lemma 6.7.3. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H4) *and suppose that $\mathcal{A}$ is homogeneous. Then the following holds:*

(1) *Each line bundle L is exceptional.*
(2) *There exists a simple object S such that in $K_0(\mathcal{A})$ the class of each simple object S' is of the form $[S'] = n \cdot [S]$ for some $n > 0$.*

Proof. (1) Set $e_1 = [L]$ and choose a second basis vector e_2 of $K_0(\mathcal{A}) \cong \mathbb{Z}^2$ lying in the image of the map $K_0(\mathcal{A}_0) \to K_0(\mathcal{A})$; see Proposition 6.3.7. For $a = \alpha_1 e_1 + \alpha_2 e_2$ in $K_0(\mathcal{A})$, we have $\langle a, a\rangle = \alpha_1^2 \langle e_1, e_1\rangle$ since $\langle e_1, e_2\rangle = -\langle e_2, e_1\rangle$ and $\langle e_2, e_2\rangle = 0$. From the existence of a tilting object T in $\mathcal{A}$, it follows that $\langle e_1, e_1\rangle > 0$ since $\langle [T], [T]\rangle > 0$. Now observe that $\langle e_1, e_1\rangle = \dim_k \operatorname{Hom}_{\mathcal{A}}(L, L) - \dim_k \operatorname{Ext}^1_{\mathcal{A}}(L, L)$ is divisible by $\dim_k \operatorname{End}_{\mathcal{A}}(L)$ since the endomorphism ring of any line bundle is a division ring by Lemma 6.3.6. Thus $\operatorname{Ext}^1_{\mathcal{A}}(L, L) = 0$.

(2) Choose a simple object S such that $d = \langle [L], [S]\rangle$ is minimal. Given any simple object S', there are integers $q, r \geq 0$ with $\langle [L], [S']\rangle = q \cdot d + r$ and $r < d$. Applying Proposition 6.3.5 and Lemma 6.3.2, we obtain extensions $0 \to L \to E \to C \to 0$ and $0 \to L \to E' \to C' \to 0$ such that E, E' are line bundles, $[C] = q \cdot [S]$, and $[C'] = [S']$. Hence

$$\langle [E], [E']\rangle = \langle [L], [L]\rangle + \langle [L], [S']\rangle - q \cdot \langle [L], [S]\rangle > 0.$$

This gives an exact sequence $0 \to E \to E' \to F \to 0$, where F is an object of finite length and $\langle [L], [F]\rangle = r < d$. The minimality of $\langle [L], [S]\rangle$ implies $F = 0$, and therefore $[S'] = q \cdot [S]$. □

The class $[S]$ in Lemma 6.7.3 yields a generator for the image of the map $K_0(\mathcal{A}_0) \to K_0(\mathcal{A})$. Thus for any finite length object A in $\mathcal{A}$, there exists some $n \geq 0$ with $[A] = n \cdot [S]$. We call this number the *degree* of A and observe that it is independent of the choice of S.

Next we describe tilting objects for an abelian category that satisfies (H1)–(H5) and is homogeneous.

Proposition 6.7.4. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H5) *and suppose that $\mathcal{A}$ is homogeneous. Let L be a line bundle and S a simple object of degree one. Then*

$$\operatorname{Hom}_{\mathcal{A}}(L, S) = k, \quad \operatorname{Ext}^1_{\mathcal{A}}(S, L) = k, \quad \operatorname{End}_{\mathcal{A}}(L) = k, \quad \textit{and} \quad \operatorname{End}_{\mathcal{A}}(S) = k.$$

Let $0 \to L \to L' \to S \to 0$ be a non-split extension. Then $L \oplus L'$ is a tilting object and its endomorphism algebra is isomorphic to the Kronecker algebra (i.e. the path algebra of the quiver $\cdot \rightrightarrows \cdot$). Moreover, the simple objects of degree one are precisely the objects that arise as the cokernel of a non-zero morphism $L \to L'$.

Proof. It follows from Proposition 6.3.7 and Lemma 6.7.3 that $[L]$ and $[S]$ form a basis of $K_0(\mathcal{A})$. The corresponding matrix $\left[\begin{smallmatrix} \langle [L],[L]\rangle & \langle [L],[S]\rangle \\ \langle [S],[L]\rangle & \langle [S],[S]\rangle \end{smallmatrix}\right]$ has determinant ± 1. Thus $1 = \langle [L], [S]\rangle = -\langle [S], [L]\rangle$. This implies $\operatorname{Hom}_{\mathcal{A}}(L, S) = k$ and $\operatorname{Ext}^1_{\mathcal{A}}(S, L) = k$. The space $\operatorname{Hom}_{\mathcal{A}}(L, S)$ is a module over $\operatorname{End}_{\mathcal{A}}(L)$ and over $\operatorname{End}_{\mathcal{A}}(S)$. It follows that $\operatorname{End}_{\mathcal{A}}(L) = k$ and $\operatorname{End}_{\mathcal{A}}(S) = k$.

Next we show that $T = L \oplus L'$ is a tilting object. An application of $\operatorname{Hom}_{\mathcal{A}}(L, -)$ to $0 \to L \to L' \to S \to 0$ yields $\operatorname{Ext}^1_{\mathcal{A}}(L, L') = 0$, while application of $\operatorname{Hom}_{\mathcal{A}}(-, L)$ implies $\operatorname{Ext}^1_{\mathcal{A}}(L', L) = 0$. Thus $\operatorname{Ext}^1_{\mathcal{A}}(T, T) = 0$. For any non-zero object A in $\mathcal{A}$, we have $\langle [L], [A]\rangle \neq 0$ or $\langle [L'], [A]\rangle \neq 0$ since $[L]$ and $[L']$ form a basis of $K_0(\mathcal{A})$ and $[A] \neq 0$. Thus T is a tilting object.

A simple computation shows that $\dim_k \operatorname{Hom}_{\mathcal{A}}(L, L') = 2$, while $\operatorname{Hom}_{\mathcal{A}}(L', L) = 0$. Thus $\operatorname{End}_{\mathcal{A}}(T)$ is isomorphic to the Kronecker algebra.

Let $\phi\colon L \to L'$ be a non-zero morphism. This is a monomorphism since L is a line bundle. The cokernel $C = \operatorname{Coker}\phi$ is of finite length since L and L' have the same rank. The degree of C is one since $[C] = [L'] - [L] = [S]$. In particular, C is simple.

Now let S' be a simple object of degree one. Choose a non-split extension $0 \to L \to E \to S' \to 0$. Then E is a line bundle and therefore exceptional by Lemma 6.7.3. We have $[E] = [L']$ in $K_0(\mathcal{A})$ since $[S] = [S']$, and it follows from Proposition 6.4.2 that $E \cong L'$. Thus S' arises as the cokernel of a morphism $L \to L'$. □

The next theorem provides an axiomatic description of the category $\operatorname{coh}\mathbb{P}^1_k$.

Theorem 6.7.5 (Lenzing)**.** *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H5) *and suppose that $\mathcal{A}$ is homogeneous. Then $\mathcal{A}$ is equivalent to* $\operatorname{coh}\mathbb{P}^1_k$.

Proof. The categories $\operatorname{coh}\mathbb{P}^1_k$ and $\mathcal{A}$ admit each a tilting object such that its endomorphism algebra is isomorphic to the Kronecker algebra, see Propositions 5.8.1 and 6.7.4. Note that in both cases the indecomposable direct summands of a tilting object have rank one. Now apply Proposition 6.6.3. □

6.8. **Coherent sheaves on weighted projective lines.** The following theorem characterizes the abelian categories that arise as categories of coherent sheaves on weighted projective lines in the sense of Geigle and Lenzing [10].

Theorem 6.8.1 (Lenzing)**.** *Let k be a field and $\mathcal{A}$ a k-linear abelian category. Then the following are equivalent:*

1. *The category $\mathcal{A}$ satisfies* (H1)–(H5).
2. *There is a finite sequence $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of full subcategories such that $\mathcal{A}^0$ is equivalent to $\operatorname{coh}\mathbb{P}^1_k$ and $\mathcal{A}^{i+1}$ is a non-split expansion of $\mathcal{A}^i$ with associated division ring k for $0 \le i \le r-1$.*
3. *The category $\mathcal{A}$ is equivalent to $\operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$.*

Proof. (1) ⇒ (2): Suppose that $\mathcal{A}$ satisfies (H1)–(H5). The rank of $K_0(\mathcal{A})$ is finite, say n. So one constructs a filtration $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of length $r = n-2$ by reducing the rank of the Grothendieck group as follows. If $\mathcal{A}$ is homogeneous, then $\mathcal{A}$ is equivalent to $\operatorname{coh}\mathbb{P}^1_k$ by Theorem 6.7.5. Otherwise, there is a simple object S such that $S \not\cong \tau S$ by Proposition 6.7.1. Then put $\mathcal{A}^{r-1} = S^{\perp}$ and the inclusion $\mathcal{A}^{r-1} \to \mathcal{A}$ is a non-split expansion by Lemma 6.5.1. Moreover, $\mathcal{A}^{r-1}$ satisfies (H1)–(H5) by Theorem 6.5.4, and the associated division ring is k by Lemma 3.5.7. Note that the rank of $K_0(\mathcal{A}^{r-1})$ is one less than that of $K_0(\mathcal{A})$. So one proceeds and constructs a sequence of subcategories $\mathcal{A}^i$. The process stops after r steps when $\mathcal{A}^0$ is homogeneous.

(2) ⇒ (1): Suppose that $\mathcal{A}$ admits a filtration $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ such that $\mathcal{A}^0$ is equivalent to $\operatorname{coh}\mathbb{P}^1_k$ and $\mathcal{A}^{i+1}$ is a non-split expansion of $\mathcal{A}^i$ with associated division ring k for $0 \le i \le r-1$. The discussion in §5 shows that $\operatorname{coh}\mathbb{P}^1_k$ satisfies (H1)–(H5). An iterated application of Theorem 6.5.4 yields that $\mathcal{A}$ satisfies (H1)–(H5).

(2) ⇒ (3): Suppose again that $\mathcal{A}$ admits a sequence $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of expansions such that $\mathcal{A}^0$ is equivalent to $\operatorname{coh}\mathbb{P}^1_k$. This yields a fully faithful exact functor $\operatorname{coh}\mathbb{P}^1_k \to \mathcal{A}$, and it follows from Proposition 6.5.2 that this functor identifies the index set of the decomposition (5.6.1)

$$\operatorname{coh}_0\mathbb{P}^1_k \xrightarrow{\sim} \coprod_{0\neq\mathfrak{p}\in\operatorname{Proj}k[x_0,x_1]} \operatorname{mod}_0\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$$

into connected components with the index set of the decomposition $\mathcal{A}_0 = \coprod_{x\in\mathbf{X}} \mathcal{A}_x$. Thus there is a canonical bijection between the set of closed points of $\mathbb{P}^1_k$ and the set $\mathbf{X}$. Moreover, if $x \in \mathbf{X}$ is a point with $p(x) > 1$, then the corresponding closed point $\mathfrak{p}$ of $\mathbb{P}^1_k$ is rational since the residue field of the corresponding local ring $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}}$ equals k. This follows from the fact that in the filtration $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ the associated division ring of each expansion equals k.

Let $\boldsymbol{\lambda}$ be the finite collection of points $\{x \in \mathbf{X} \mid p(x) > 1\}$, viewed as points of $\mathbb{P}^1_k$, and denote by $\mathbf{p}$ the corresponding sequence of positive integers $p(x)$. Then there exists a tilting object T such that $\operatorname{End}_{\mathcal{A}}(T) \cong \operatorname{Sq}(\mathbf{p},\boldsymbol{\lambda})$; see Proposition 6.9.1 below. On the other hand, let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ be the weighted projective line that is determined by the parameters $\boldsymbol{\lambda}$ and $\mathbf{p}$. The category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ admits the following tilting object

$$\mathcal{O} \oplus \mathcal{O}(\vec{c}) \oplus (S_1^{[1]} \oplus \cdots \oplus S_1^{[p_1-1]}) \oplus \cdots \oplus (S_n^{[1]} \oplus \cdots \oplus S_n^{[p_n-1]})$$

with endomorphism algebra $\operatorname{Sq}(\mathbf{p},\boldsymbol{\lambda})$, where the notation is taken from the introduction with $S_i = S_{i1}$; see [20, Example 4.4]. This yields a derived equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{coh}\mathbb{X})$ which restricts to an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{coh}\mathbb{X}$ by Proposition 6.6.3.

(3) $\Rightarrow$ (1): See [10]. □

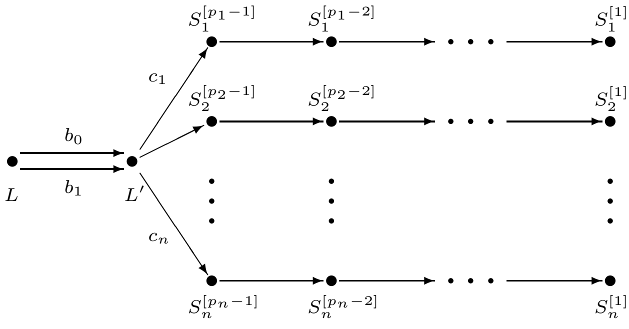
Remark 6.8.2. Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H5).

(1) The reduction to the homogeneous case in the proof of Theorem 6.8.1 shows that the rank of the Grothendieck group of $\mathcal{A}$ is $2 + \sum_{x\in\mathbf{X}}(p(x) - 1)$.

(2) Let $x \in \mathbf{X}$ and $p(x) > 1$. Then $\mathcal{A}_x$ is equivalent to the category of finite dimensional nilpotent representations of a quiver of extended Dynkin type $\tilde{\mathbb{A}}_{p(x)-1}$ with cyclic orientation; see Example 1.8.3.

6.9. **A tilting object.** Let k be a field and $\mathcal{A}$ a k-linear abelian category satisfying (H1)–(H5). We construct a tilting object and compute its endomorphism algebra, which is a *squid algebra* in the sense of Brenner and Butler [5].

Given a collection $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of distinct rational points $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ of $\mathbb{P}^1_k$, and a sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of positive integers, we define $\operatorname{Sq}(\mathbf{p},\boldsymbol{\lambda})$ to be the finite dimensional associative algebra given by the quiver



modulo the relations

$$c_i(\lambda_{i0}b_1 - \lambda_{i1}b_0) = 0 \quad (i = 1, \ldots, n).$$

Proposition 6.9.1 (Lenzing-Meltzer)**.** *A k-linear abelian category satisfying* (H1)–(H5) *admits a tilting object with endomorphism algebra isomorphic to* $\operatorname{Sq}(\mathbf{p},\boldsymbol{\lambda})$ *for some pair* $\mathbf{p},\boldsymbol{\lambda}$.

Proof. Fix a k-linear abelian category $\mathcal{A}$ satisfying (H1)–(H5). We apply Theorem 6.8.1(2) and follow its proof. Thus there exists a sequence $\mathcal{B} = \mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ of expansions such that $\mathcal{B}$ is equivalent to $\operatorname{coh}\mathbb{P}^1_k$. Let $(x_1, \ldots, x_n)$ be the collection of distinct points $x \in \mathbf{X}$ with $p(x) > 1$ and set $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i = p(x_i)$ for each i. Choose line bundles L and L' in $\mathcal{B}$ forming a tilting object $L \oplus L'$ for $\mathcal{B}$, and choose a basis b_0, b_1 of $\operatorname{Hom}_{\mathcal{A}}(L, L')$; see Proposition 6.7.4. The inclusion $F\colon \mathcal{B} \to \mathcal{A}$ restricts to a family of inclusions $\mathcal{B}_{x_i} \to \mathcal{A}_{x_i}$; see Proposition 6.5.2. Note that each inclusion $\mathcal{B}_{x_i} \to \mathcal{A}_{x_i}$ is a composite of p_i non-split expansions. Thus there are simple objects $S'_i \in \mathcal{B}_{x_i}$ and $S_i \in \mathcal{A}_{x_i}$ such that $FS'_i = S_i^{[p_i]}$. Here, $S_i^{[p_i]}$ denotes the uniserial object with top S_i and length p_i, and we use that an expansion sends a specific simple object to an object of length two; see Lemma 4.3.1. In particular, $S_i^{[j]}$ belongs to ${}^{\perp}\mathcal{B}$ for $1 \leq j < p_i$. Note that $\operatorname{End}_{\mathcal{A}}(S'_i) = k$ since the division ring of each expansion is k. In $\operatorname{coh}\mathbb{P}^1_k$, a simple object with trivial endomorphism ring has degree one. Thus each simple object S'_i fits into an exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda_{i0}b_1 - \lambda_{i1}b_0} L' \longrightarrow S'_i \longrightarrow 0 \tag{6.9.1}$$

by Proposition 6.7.4, and this yields a collection $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of rational points $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ in $\mathbb{P}^1_k$. Moreover, there are canonical morphisms $c_i\colon L' \twoheadrightarrow S_i^{[p_i]} \twoheadrightarrow S_i^{[p_i-1]}$ in $\mathcal{A}$ satisfying the relations $c_i(\lambda_{i0}b_1 - \lambda_{i1}b_0) = 0$. It is straightforward to verify that the object

$$T = L \oplus L' \oplus (S_1^{[1]} \oplus \cdots \oplus S_1^{[p_1-1]}) \oplus \cdots \oplus (S_n^{[1]} \oplus \cdots \oplus S_n^{[p_n-1]})$$

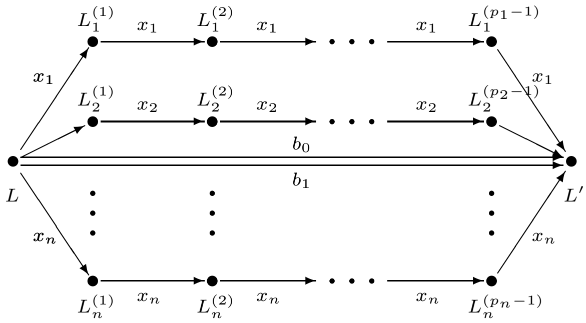
is a tilting object for $\mathcal{A}$, using the criterion of Lemma 3.5.5. Indeed, $\operatorname{Ext}^1_{\mathcal{A}}(T, T) = 0$, the indecomposable direct summands of T yield a basis of $K_0(\mathcal{A})$, and $[A] \neq 0$ for each object $A \neq 0$ by Proposition 6.3.7. Finally, one checks that $\operatorname{End}_{\mathcal{A}}(T)$ is isomorphic to $\operatorname{Sq}(\mathbf{p}, \boldsymbol{\lambda})$. Here, one uses that each epimorphism $S_i^{[p_i]} \to S_i^{[j]}$ induces an isomorphism $\operatorname{Hom}_{\mathcal{A}}(L \oplus L', S_i^{[p_i]}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(L \oplus L', S_i^{[j]})$. □

7. Canonical algebras

The canonical algebras in the sense of Ringel [22, 23] provide a link between weighted projective lines and the representation theory of finite dimensional algebras. In fact, Geigle and Lenzing constructed in [10] for each weighted projective line $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ a tilting object in $\operatorname{coh}\mathbb{X}$ such that its endomorphism algebra is isomorphic to the canonical algebra with the same parameters; see Example 7.4.3. It turns out that this tilting object is somehow canonical. From this it follows that the parameters $\boldsymbol{\lambda}$ and $\mathbf{p}$ can be reconstructed from the category $\operatorname{coh}\mathbb{X}$. To be more precise, we consider an abelian category $\mathcal{A}$ that is equivalent to $\operatorname{coh}\mathbb{X}$ for some weighted projective line $\mathbb{X}$. For each line bundle L in $\mathcal{A}$ one constructs a canonical tilting object T_L such that its endomorphism algebra is a canonical algebra. Then one shows that for each pair of line bundles L, L' there is a sequence of tubular mutations in the sense of Lenzing and Meltzer [19, 21, 18] that yields an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ taking L to L' and therefore T_L to $T_{L'}$. In particular, the endomorphism algebras of T_L and $T_{L'}$ are isomorphic and therefore an invariant of $\mathcal{A}$. Finally one observes that the parameters $\boldsymbol{\lambda}$ and $\mathbf{p}$ form an invariant of the canonical algebra $\operatorname{End}_{\mathcal{A}}(T_L)$.

Throughout this section we fix an arbitrary field k.

7.1. **Canonical algebras from weighted projective lines.** Consider a collection $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of distinct rational points $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ of $\mathbb{P}^1_k$, and a sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of positive integers. We define the *canonical algebra* $C(\mathbf{p}, \boldsymbol{\lambda})$ to be the finite dimensional associative algebra given by the quiver



modulo the relations[8]

$$x_i^{p_i} = \lambda_{i0} b_1 - \lambda_{i1} b_0 \quad (i = 1, \ldots, n).$$

Theorem 7.1.1 (Geigle-Lenzing)**.** *A k-linear abelian category satisfying* (H1)–(H5) *admits a tilting object T such that T is a direct sum of line bundles and the endomorphism algebra of T is isomorphic to $C(\mathbf{p}, \boldsymbol{\lambda})$ for some pair $\mathbf{p}, \boldsymbol{\lambda}$.*

Proof. Fix a k-linear abelian category $\mathcal{A}$ satisfying (H1)–(H5). We adapt the proof of Proposition 6.9.1 and modify the tilting object constructed in that proof as follows. Consider for each point $x_i \in \mathbf{X}$ with $p(x_i) > 1$ the exact sequence (6.9.1)

$$0 \longrightarrow L \xrightarrow{\phi_i} L' \longrightarrow S_i^{[p_i]} \longrightarrow 0$$

in $\mathcal{A}$, where $\phi_i = \lambda_{i0} b_1 - \lambda_{i1} b_0$. We form successively the pullback along the chain of monomorphisms

$$(\tau^{-1} S_i)_{[1]} \rightarrowtail (\tau^{-1} S_i)_{[2]} \rightarrowtail \cdots \rightarrowtail (\tau^{-1} S_i)_{[p_i - 1]} \rightarrowtail (\tau^{-1} S_i)_{[p_i]} = S_i^{[p_i]}$$

and obtain the following commutative diagram with exact columns.

(7.1.1)
$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & = & L & = \cdots = & L & = & L \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi_i} \\
L_i^{(1)} & \longrightarrow & L_i^{(2)} & \longrightarrow \cdots \longrightarrow & L_i^{(p_i-1)} & \longrightarrow & L' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\tau^{-1} S_i)_{[1]} & \longrightarrow & (\tau^{-1} S_i)_{[2]} & \longrightarrow \cdots \longrightarrow & (\tau^{-1} S_i)_{[p_i-1]} & \longrightarrow & (\tau^{-1} S_i)_{[p_i]} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

Note that each object $L_i^{(j)}$ is a line bundle. It is straightforward to verify that the object

$$T = L \oplus L' \oplus (L_1^{(1)} \oplus \cdots \oplus L_1^{(p_1-1)}) \oplus \cdots \oplus (L_n^{(1)} \oplus \cdots \oplus L_n^{(p_n-1)})$$

[8]Note that the relations do not generate an admissible ideal of the path algebra, except when the collection $\boldsymbol{\lambda}$ is empty. In that case $C(\mathbf{p}, \boldsymbol{\lambda})$ equals the Kronecker algebra.

is a tilting object for $\mathcal{A}$, following the line of arguments in the proof of Proposition 6.9.1. □

Corollary 7.1.2. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H5). *Then there exists for some pair $\mathbf{p}, \boldsymbol{\lambda}$ an equivalence of derived categories*

$$\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} C(\mathbf{p}, \boldsymbol{\lambda})).$$ □

7.2. **Vector bundle presentations.** Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H4). The category $\mathcal{A}_+$ consisting of the vector bundles in $\mathcal{A}$ determines together with its exact structure the category $\mathcal{A}$. In fact, every object A in $\mathcal{A}$ admits a presentation $0 \to A_1 \to A_0 \to A \to 0$ with A_0, A_1 in $\mathcal{A}_+$. Introducing appropriate morphisms between complexes in $\mathcal{A}_+$, one can make these presentations functorial.

Lemma 7.2.1. *Every object in $\mathcal{A}$ is a factor object of an object in $\mathcal{A}_+$.*

Proof. Every object of $\mathcal{A}$ decomposes into an object of $\mathcal{A}_+$ and an object of finite length. Thus it suffices to show that for an indecomposable object A of finite length there is an epimorphism $E \to A$ with E in $\mathcal{A}_+$. We use induction on the length $\ell(A)$ of A. Up to certain power of τ, the case $\ell(A) = 1$ follows from Proposition 6.3.5. Assume that $\ell(A) > 1$. Take a maximal subobject $A' \subseteq A$. By the induction hypothesis there is an epimorphism $\phi\colon E' \to A'$. Note that ϕ induces an epimorphism $\operatorname{Ext}^1_{\mathcal{A}}(A/A', E') \to \operatorname{Ext}^1_{\mathcal{A}}(A/A', A')$. In particular, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & A/A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A/A' & \longrightarrow & 0. \end{array}$$

Note that the upper row does not split since A is indecomposable. Thus the morphism $E' \to E$ induces a bijection $\operatorname{Hom}_{\mathcal{A}}(S, E') \to \operatorname{Hom}_{\mathcal{A}}(S, E)$ for each simple object S. It follows that E belongs to $\mathcal{A}_+$, and therefore A is a quotient of an object in $\mathcal{A}_+$. □

The following result says that the subcategory $\mathcal{A}_+$ determines $\mathcal{A}$. However, it is important to notice that one uses the exact structure on $\mathcal{A}_+$ that is inherited from $\mathcal{A}$. To be precise, a morphism of complexes in $\mathcal{A}_+$ is by definition a *quasi-isomorphism* if it is a quasi-isomorphism of complexes in $\mathcal{A}$.

Proposition 7.2.2. *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H4). *The inclusion $\mathcal{A}_+ \to \mathcal{A}$ induces an equivalence*

$$\mathbf{K}^b(\mathcal{A}_+)[\mathrm{qis}^{-1}] \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Proof. The subcategory $\mathcal{A}_+$ of $\mathcal{A}$ is closed under forming extensions and taking kernels of epimorphisms. Moreover, each object A in $\mathcal{A}$ fits into an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to A \to 0$ with each A_i in $\mathcal{A}_+$; see Lemma 7.2.1. With these properties, the assertion follows from [25, Chap. III, Prop. 2.4.3]. □

Corollary 7.2.3. *Let $F\colon \mathcal{A}_+ \xrightarrow{\sim} \mathcal{A}_+$ be an equivalence and suppose that a sequence ξ in $\mathcal{A}_+$ is exact if and only if $F\xi$ is exact. Then F extends uniquely to an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$.*

Proof. We apply Proposition 7.2.2. Thus the assumption on F implies that the functor extends to an equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$. This equivalence restricts to an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ because $\mathcal{A}$ identifies with the full subcategory consisting of complexes

$$\cdots \to 0 \to A_1 \xrightarrow{\delta} A_0 \to 0 \to \cdots$$

with A_0, A_1 in $\mathcal{A}_+$ and δ a monomorphism; see Lemma 7.2.1. □

7.3. **Line bundles and tubular mutations.** Let $\mathcal{A}$ be a k-linear abelian category satisfying (H1)–(H5) and $\mathcal{A}_0 = \coprod_{x\in\mathbf{X}} \mathcal{A}_x$ be the decomposition of $\mathcal{A}_0$ into connected uniserial components. For each $x \in \mathbf{X}$ denote by T_x the direct sum of a representative set of simple objects in $\mathcal{A}_x$ and set $\mathcal{T}_x = \operatorname{add} T_x$.

Note that $\mathcal{T}_x$ is a Hom-finite semisimple abelian category with finitely many simple objects. Thus each additive functor $\mathcal{T}_x \to \operatorname{mod} k$ is representable. This observation yields two functors $\bar{\delta}_x, \bar{\varepsilon}_x \colon \mathcal{A} \to \mathcal{T}_x$ such that for each object A in $\mathcal{A}$

$$\operatorname{Hom}_{\mathcal{A}}(A, -)|_{\mathcal{T}_x} \cong \operatorname{Hom}_{\mathcal{T}_x}(\bar{\delta}_x A, -) \quad \text{and} \quad \operatorname{Ext}^1_{\mathcal{A}}(-, A)|_{\mathcal{T}_x} \cong \operatorname{Hom}_{\mathcal{T}_x}(-, \bar{\varepsilon}_x A).$$

Fix an object A in $\mathcal{A}_+$. The identity morphism of $\bar{\delta}_x A$ corresponds to a morphism $A \to \bar{\delta}_x A$. This is an epimorphism and we complete it to an exact sequence $0 \to \delta_x A \to A \to \bar{\delta}_x A \to 0$. On the other hand, the identity morphism of $\bar{\varepsilon}_x A$ corresponds to an exact sequence $0 \to A \to \varepsilon_x A \to \bar{\varepsilon}_x A \to 0$. It is easily checked that this defines two functors $\delta_x, \varepsilon_x \colon \mathcal{A}_+ \to \mathcal{A}$.

The following lemma shows that δ_x and ε_x yield equivalences $\mathcal{A}_+ \xrightarrow{\sim} \mathcal{A}_+$.

Lemma 7.3.1. *Let $\xi \colon 0 \to A \to A' \to T \to 0$ be an exact sequence in $\mathcal{A}$ with T in $\mathcal{T}_x$ for some $x \in \mathbf{X}$. Then the following are equivalent:*

1. *ξ induces an iso $\operatorname{Hom}_{\mathcal{A}}(-, T)|_{\mathcal{T}_x} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(-, A)|_{\mathcal{T}_x}$ and A belongs to $\mathcal{A}_+$.*
2. *ξ induces an iso $\operatorname{Hom}_{\mathcal{A}}(A, -)|_{\mathcal{T}_x} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(T, -)|_{\mathcal{T}_x}$ and A belongs to $\mathcal{A}_+$.*
3. *ξ induces an iso $\operatorname{Hom}_{\mathcal{A}}(T, -)|_{\mathcal{T}_x} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(A', -)|_{\mathcal{T}_x}$ and A' belongs to $\mathcal{A}_+$.*

Proof. (1) ⇔ (2): Apply Serre duality and observe that $\tau T_x \cong T_x$.

(1) & (2) ⇒ (3): Let S be any simple object in $\mathcal{A}$ and apply $\operatorname{Hom}_{\mathcal{A}}(S, -)$ to ξ. Using (1) and the fact that A belongs to $\mathcal{A}_+$, it follows that $\operatorname{Hom}_{\mathcal{A}}(S, A') = 0$. Thus A' is in $\mathcal{A}_+$.

Now let S be any object in $\mathcal{T}_x$ and apply $\operatorname{Hom}_{\mathcal{A}}(-, S)$ to ξ. This yields the following exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(T, S) \xrightarrow{\alpha} \operatorname{Hom}_{\mathcal{A}}(A', S) \xrightarrow{\beta} \operatorname{Hom}_{\mathcal{A}}(A, S) \xrightarrow{\gamma} \operatorname{Ext}^1_{\mathcal{A}}(T, S) \to 0$$

where α is an isomorphism if and only if γ is an isomorphism. Thus (3) holds.

(3) ⇒ (2): The object A is in $\mathcal{A}_+$ since $\mathcal{A}_+$ is closed under taking subobjects. The rest follows as before by choosing S in $\mathcal{T}_x$ and applying $\operatorname{Hom}_{\mathcal{A}}(-, S)$ to ξ. □

Proposition 7.3.2. *The functors δ_x and ε_x form a pair of mutually inverse equivalences $\mathcal{A}_+ \xrightarrow{\sim} \mathcal{A}_+$. Moreover, δ_x and ε_x take exact sequences in $\mathcal{A}_+$ to exact sequences.*

Proof. The first assertion is an immediate consequence of Lemma 7.3.1. For the exactness observe that $\bar{\delta}_x$ and $\bar{\varepsilon}_x$ are exact when restricted to $\mathcal{A}_+$. The exactness of δ_x and ε_x then follows from the 3×3 lemma. □

Using Corollary 7.2.3, the functors δ_x, ε_x yield equivalences $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$. These functors are called *tubular mutations* and were introduced by Lenzing and Meltzer [19, 21, 18].

Corollary 7.3.3. *The equivalences $\delta_x, \varepsilon_x \colon \mathcal{A}_+ \xrightarrow{\sim} \mathcal{A}_+$ extend to a pair of mutually inverse equivalences $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$.* □

Next we show that for each pair of line bundles L, L' there exists a sequence of tubular mutations taking L to L'. We need the following proposition which is of independent interest.

Proposition 7.3.4. *Let L be a line bundle. For each $x \in \mathbf{X}$ there exists up to isomorphism a unique simple object S_x in $\mathcal{A}_x$ such that $\mathrm{Hom}_{\mathcal{A}}(L, S_x) \neq 0$. Moreover, any non-zero morphism $L \to S_x$ induces an isomorphism $\mathrm{End}_{\mathcal{A}}(S_x) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(L, S_x)$.*

Proof. For the purpose of this proof, call a line bundle L *Hom-simple* if the assertion of the proposition holds for L. We begin by showing that a specific line bundle is Hom-simple.

Fix a sequence of subcategories $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \cdots \subseteq \mathcal{A}^r = \mathcal{A}$ as in Theorem 6.8.1 and choose a line bundle L in $\mathcal{A}^0$. Note that the inclusion $\mathcal{A}^0 \to \mathcal{A}$ sends L to a line bundle of $\mathcal{A}$ by Proposition 4.3.3. For each $x \in \mathbf{X}$, there is at least one simple object $S_x \in \mathcal{A}_x$ with $\mathrm{Hom}_{\mathcal{A}}(L, S_x) \neq 0$ by Proposition 6.3.5. On the other hand, the intersection of $(\mathcal{A}^0)^{\perp}$ with $\mathcal{A}_x$ is a Serre subcategory having $p(x) - 1$ simple objects. In particular, $\mathrm{Hom}_{\mathcal{A}}(L, -)$ vanishes on them. Thus there is a unique simple object S_x in $\mathcal{A}_x$ with $\mathrm{Hom}_{\mathcal{A}}(L, S_x) \neq 0$.

Let $i_\rho \colon \mathcal{A} \to \mathcal{A}^0$ denote the right adjoint of the inclusion $\mathcal{A}^0 \to \mathcal{A}$. Then $i_\rho S_x$ is a simple object by Lemma 4.3.1. Choose a non-zero morphism $L \to S_x$. This yields the following commutative square, where $\eta \colon i_\rho S_x \to S_x$ denotes the adjunction morphism.

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{A}^0}(i_\rho S_x) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}^0}(L, i_\rho S_x) \\ {\scriptstyle i_\rho}\uparrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}(L,\eta)} \\ \mathrm{End}_{\mathcal{A}}(S_x) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(L, S_x) \end{array}$$

The map $\mathrm{Hom}_{\mathcal{A}}(L, \eta)$ is an isomorphism since L belongs to $\mathcal{A}^0$, and i_ρ induces an isomorphism $\mathrm{End}_{\mathcal{A}}(S_x) \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}^0}(i_\rho S_x)$ since i_ρ is a quotient functor and S_x is simple; see Lemma 1.3.6. Finally observe that $\mathcal{A}^0$ is equivalent to $\mathrm{coh}\, \mathbb{P}^1_k$. Thus the induced map $\mathrm{End}_{\mathcal{A}^0}(i_\rho S_x) \to \mathrm{Hom}_{\mathcal{A}^0}(L, i_\rho S_x)$ is an isomorphism because we may assume that L corresponds to the structure sheaf; see Remark 5.6.2. It follows that $\mathrm{End}_{\mathcal{A}}(S_x) \cong \mathrm{Hom}_{\mathcal{A}}(L, S_x)$.

Having shown the assertion for a specific line bundle, we apply Lemma 6.3.6 to verify the assertion for an arbitrary line bundle. Thus we need to show that for any pair L, L' of line bundles and each monomorphism $\phi \colon L \to L'$ with cokernel in $\mathcal{A}_0$, the object L is Hom-simple if and only if L' is Hom-simple.

Using induction on the length of the cokernel of ϕ, we may assume that the cokernel is simple. The exact sequence $0 \to L \xrightarrow{\phi} L' \to S \to 0$ induces for each simple object T an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(S, T) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{A}}(L', T) \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{A}}(L, T) \xrightarrow{\gamma} D\,\mathrm{Hom}_{\mathcal{A}}(T, \tau S) \to 0.$$

If L is Hom-simple, then a simple calculation shows that γ is an isomorphism. Thus α is an isomorphism and it follows that L' is Hom-simple. The same argument shows that L is Hom-simple if L' is Hom-simple. □

The following result is due to Kussin; see [16, Proposition 4.2.3].

Proposition 7.3.5. *Let L, L' be two line bundles in $\mathcal{A}$. Then there exists an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ taking L to L'. In particular, each line bundle is exceptional.*

Proof. We apply Lemma 6.3.6. Thus we may assume that there is an exact sequence $\xi\colon 0 \to L \to L' \to C \to 0$ with C of finite length. Using induction on the length of C, we may even assume that C is simple. Suppose that C belongs to $\mathcal{A}_x$ with $x \in \mathbf{X}$. Then ξ induces an isomorphism $\operatorname{Hom}_{\mathcal{A}}(-,C)|_{\mathcal{T}_x} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(-,L)|_{\mathcal{T}_x}$ by Proposition 7.3.4 and Serre duality. Thus $L' = \varepsilon_x L$ and the tubular mutation given by ε_x sends L to L'; see Corollary 7.3.3. □

7.4. **Weight functions.** Let k be a field and $\mathcal{A}$ a k-linear abelian category satisfying (H1)–(H5). We associate to $\mathcal{A}$ a weight function and show that it is an invariant of $\mathcal{A}$ which determines $\mathcal{A}$ up to equivalence.

In the following we identify the projective linear group $\mathrm{PGL}(2,k)$ with $\operatorname{Aut}\mathbb{P}^1_k$; see Proposition 5.7.1.

A *weight function* $w\colon \mathbb{P}^1_k \to \mathbb{Z}$ is a map which assigns to each closed point x of $\mathbb{P}^1_k$ a positive integer $w(x)$ such that $w(x) = 1$ for almost all x. Two weight functions w, w' are *equivalent* if there exists some linear transformation $\sigma \in \mathrm{PGL}(2,k)$ such that $w'(x) = w(\sigma x)$ for every closed point x. Given a collection $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ of distinct closed points $\lambda_i \in \mathbb{P}^1_k$, and a sequence $\mathbf{p} = (p_1, \dots, p_n)$ of positive integers, there is associated a weight function $w_{\mathbf{p},\boldsymbol{\lambda}}$, where $w_{\mathbf{p},\boldsymbol{\lambda}}(\lambda_i) = p_i$ for $1 \le i \le n$ and $w_{\mathbf{p},\boldsymbol{\lambda}}(x) = 1$ otherwise.

It follows from our convention that each weight function w corresponding to a weighted projective line satisfies $w(x) = 1$ if the point x is not rational.

Theorem 7.4.1 (Lenzing)**.** *Let $\mathcal{A}$ be a k-linear abelian category satisfying* (H1)–(H5)*. For each line bundle L, there exists a canonically*[9] *defined tilting object T_L which is unique up to isomorphism. Moreover:*

1. *The object T_L determines parameters $\mathbf{p}, \boldsymbol{\lambda}$ such that $\operatorname{End}_{\mathcal{A}}(T_L) \cong C(\mathbf{p}, \boldsymbol{\lambda})$. The parameters $\mathbf{p}, \boldsymbol{\lambda}$ depend on a choice and any other choice gives parameters $\mathbf{p}', \boldsymbol{\lambda}'$ such that the weight functions $w_{\mathbf{p},\boldsymbol{\lambda}}$ and $w_{\mathbf{p}',\boldsymbol{\lambda}'}$ are equivalent.*
2. *Let M be a second line bundle. Then there exists an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ taking L to M. Thus $\operatorname{End}_{\mathcal{A}}(T_L)$ is isomorphic to $\operatorname{End}_{\mathcal{A}}(T_M)$ and the associated weight functions are equivalent.*

Proof. Let $(x_1, \dots, x_n)$ be the collection of distinct points $x \in \mathbf{X}$ with $p(x) > 1$ and set $\mathbf{p} = (p_1, \dots, p_n)$ with $p_i = p(x_i)$ for each i. We apply Proposition 7.3.4 and choose for each i a simple object $S_i = S_{x_i}$ such that $\operatorname{Hom}_{\mathcal{A}}(L, S_i) \neq 0$. Thus $\operatorname{Ext}^1_{\mathcal{A}}(\tau^{-1} S_i, L) \neq 0$ by Serre duality, and Lemma 6.3.2 yields a chain of monomorphism

$$L \xrightarrow{\psi_1} L_i^{(1)} \xrightarrow{\psi_2} L_i^{(2)} \xrightarrow{\psi_3} \cdots \xrightarrow{\psi_{p_i}} L_i^{(p_i)} \tag{7.4.1}$$

with $\operatorname{Coker}\psi_j \cong \tau^{-j} S_i$ for all j. Note that each object $L_i^{(j)}$ is a line bundle and therefore exceptional.

The class $[L_i^{(p_i)}] = [L] - \sum_{j=1}^{p(x)} [\tau^j S_i]$ does not depend on i since the inclusion $K_0(\mathcal{A}^0) \to K_0(\mathcal{A})$ sends the class of the simple object S_i' of $\mathcal{A}^0_{x_i}$ to $\sum_{j=1}^{p(x)} [\tau^j S_i]$; see Lemma 4.3.1. Here, we use that S_i' is a simple object of degree one; so its class in $K_0(\mathcal{A}^0)$ is independent of i by Lemma 6.7.3. Thus the object $L_i^{(p_i)}$ does not depend on i by Proposition 6.4.2, and we denote it by L'.

[9]This canonical choice provides another justification for the term ‘canonical algebra’.

The construction of the $L_i^{(j)}$ is parallel to the construction in the proof of Theorem 7.1.1, and we refer to the commutative diagram (7.1.1) illustrating it.

Next observe that each object $L_i^{(j)}$ depends only on L and x_i because its class has this property and the object is exceptional. Thus the object

$$T_L = L \oplus L' \oplus (L_1^{(1)} \oplus \cdots \oplus L_1^{(p_1-1)}) \oplus \cdots \oplus (L_n^{(1)} \oplus \cdots \oplus L_n^{(p_n-1)})$$

depends up to isomorphism only on L. In particular, the object equals up to equivalence the tilting object constructed in the proof of Theorem 7.1.1, because for each pair of line bundles M, N there exists an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ taking M to N, by Proposition 7.3.5. It follows that T_L is a tilting object for $\mathcal{A}$.

(1) Choosing a basis b_0, b_1 of $\operatorname{Hom}_{\mathcal{A}}(L, L')$, we obtain rational points $\lambda_i = [\lambda_{i0} : \lambda_{i1}]$ in $\mathbb{P}^1_k$ such that $\phi_i = \lambda_{i0} b_1 - \lambda_{i1} b_0$, where ϕ_i is the composite of the morphisms in (7.4.1). Note that each ϕ_i depends on the choice of the ψ_j, but it is unique up to a non-zero scalar. It follows that $\operatorname{End}_{\mathcal{A}}(T_L)$ is isomorphic to $C(\mathbf{p}, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. Any other choice of the basis b_0, b_1 gives another set of parameters $\boldsymbol{\lambda}'$ and a linear transformation $\sigma \in \mathrm{PGL}(2, k)$ such that $\lambda_i' = \sigma(\lambda_i)$ for each i. Thus the weight function $w_{\mathbf{p},\boldsymbol{\lambda}}$ is unique up to equivalence.

(2) Apply Proposition 7.3.5. ☐

Remark 7.4.2. There is an analogue of Theorem 7.4.1 with the canonical algebra $C(\mathbf{p}, \boldsymbol{\lambda})$ replaced by the squid algebra $\mathrm{Sq}(\mathbf{p}, \boldsymbol{\lambda})$.

The following example gives an explicit description of the tilting object for the category of coherent sheaves on a weighted projective line; see [10, Proposition 4.1].

Example 7.4.3. Let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ be a weighted projective line. Then $T_{\mathcal{O}} = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$ is the tilting object for $\operatorname{coh} \mathbb{X}$ that is associated to the line bundle $\mathcal{O}$. The endomorphism algebra is isomorphic to the canonical algebra $C(\mathbf{p}, \boldsymbol{\lambda})$.

It is now a consequence of Theorem 7.4.1 that a k-linear abelian category satisfying (H1)–(H5) is determined up to equivalence by its associated weight function.

Corollary 7.4.4. *For two weighted projective lines $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ and $\mathbb{X}' = (\mathbb{P}^1_k, \boldsymbol{\lambda}', \mathbf{p}')$ over a field k, the following are equivalent:*

(1) *The weight functions $w_{\mathbf{p},\boldsymbol{\lambda}}$ and $w_{\mathbf{p}',\boldsymbol{\lambda}'}$ are equivalent.*
(2) *The algebras $C(\mathbf{p}, \boldsymbol{\lambda})$ and $C(\mathbf{p}', \boldsymbol{\lambda}')$ are isomorphic.*
(3) *The categories $\operatorname{coh} \mathbb{X}$ and $\operatorname{coh} \mathbb{X}'$ are equivalent.*

Proof. (1) ⇒ (2): Suppose that $w_{\mathbf{p},\boldsymbol{\lambda}}$ and $w_{\mathbf{p}',\boldsymbol{\lambda}'}$ are equivalent via some linear transformation $\sigma = \left[\begin{smallmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{smallmatrix}\right]$ in $\mathrm{PGL}(2, k)$. Thus we may assume that the points of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are related via $\sigma(\lambda_i) = \lambda_i'$ for $1 \le i \le n$. The algebra $C(\mathbf{p}, \boldsymbol{\lambda})$ is generated by a collection of arrows b_0, b_1, x_i; analogously $C(\mathbf{p}', \boldsymbol{\lambda}')$ is generated by arrows b_0', b_1', x_i'. We obtain an isomorphism $f \colon C(\mathbf{p}, \boldsymbol{\lambda}) \xrightarrow{\sim} C(\mathbf{p}', \boldsymbol{\lambda}')$ by defining $f(b_0) = \sigma_{11} b_0' - \sigma_{01} b_1'$, $f(b_1) = \sigma_{00} b_1' - \sigma_{10} b_0'$ and $f(x_i) = x_i'$ $(1 \le i \le n)$.

(2) ⇒ (3): The category $\operatorname{coh} \mathbb{X}$ admits a tilting object T with endomorphism algebra $C(\mathbf{p}, \boldsymbol{\lambda})$, which is obtained from the tilting object

$$\mathcal{O} \oplus \mathcal{O}(\vec{c}) \oplus (S_1^{[1]} \oplus \cdots \oplus S_1^{[p_1-1]}) \oplus \cdots \oplus (S_n^{[1]} \oplus \cdots \oplus S_n^{[p_n-1]})$$

by modifying it as in the proof of Theorem 7.1.1. Analogously, $\operatorname{coh} \mathbb{X}'$ admits a tilting object T' with endomorphism algebra $C(\mathbf{p}', \boldsymbol{\lambda}')$. If both algebras are isomorphic, then Proposition 6.6.3 implies that $\operatorname{coh} \mathbb{X}$ and $\operatorname{coh} \mathbb{X}'$ are equivalent.

(3) ⇒ (1): Suppose that there is an equivalence $F \colon \operatorname{coh} \mathbb{X} \xrightarrow{\sim} \operatorname{coh} \mathbb{X}'$. We apply Theorem 7.4.1. Thus the functor F takes for any line bundle L of $\operatorname{coh} \mathbb{X}$ the canonically defined tilting object T_L to T_{FL}. The associated weight function for T_L is

equivalent to $w_{\mathbf{p},\boldsymbol{\lambda}}$, whereas for T_{FL} it is equivalent to $w_{\mathbf{p}',\boldsymbol{\lambda}'}$. It follows that $w_{\mathbf{p},\boldsymbol{\lambda}}$ and $w_{\mathbf{p}',\boldsymbol{\lambda}'}$ are equivalent. □

8. Further topics

In this section we list a few topics which have attracted interest in the past, and which are areas of present research. The lists of papers is certainly not complete and we refer to the references in the listed papers for more information.

1. The classification of indecomposable vector bundles on weighted projective lines: the trichotomy 'domestic/tubular/wild' based on the Euler characteristic.

W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, in *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*, 265–297, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

H. Lenzing and J. A. de la Peña, Wild canonical algebras, Math. Z. **224** (1997), no. 3, 403–425.

H. Lenzing, Hereditary categories, in *Handbook of tilting theory*, 105–146, Cambridge Univ. Press, Cambridge, 2007.

2. Noncommutative curves of genus zero: the study of weighted projective lines over arbitrary base fields.

H. Lenzing, Representations of finite dimensional algebras and singularity theory, in *Trends in ring theory (Miskolc, 1996)*, 71–97, Amer. Math. Soc., Providence, RI, 1998.

H. Lenzing and J. A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc. (3) **78** (1999), no. 3, 513–540.

D. Kussin, Noncommutative curves of genus zero: related to finite dimensional algebras, Mem. Amer. Math. Soc. **201** (2009), no. 942, x+128 pp.

3. Graded singularities: the study of weighted projective lines in terms of graded singularities (maximal Cohen-Macaulay modules, vector bundles, the triangulated category of singularities in the sense of Buchweitz and Orlov).

H. Kajiura, K. Saito and A. Takahashi, Matrix factorization and representations of quivers. II. Type *ADE* case, Adv. Math. **211** (2007), no. 1, 327–362.

H. Lenzing and J. A. de la Peña, Extended canonical algebras and Fuchsian singularities, arXiv:math/0611532, Math. Z., to appear.

H. Lenzing and J. A. de la Peña, Spectral analysis of finite dimensional algebras and singularities, in *Trends in representation theory of algebras and related topics*, 541–588, Eur. Math. Soc., Zürich, 2008.

A. Takahashi, Weighted projective lines associated to regular systems of weights of dual type, arXiv:0711.3907, Adv. Stud. Pure Math., to appear.

4. Kac's theorem, Hall algebras: the theorem characterizes the dimension types of indecomposable coherent sheaves over weighted projective lines in terms of loop algebras of Kac-Moody Lie algebras; the proof uses Hall algebras.

W. Crawley-Boevey, Kac's Theorem for weighted projective lines, arXiv:math/0512078, J. Eur. Math. Soc., to appear.

W. Crawley-Boevey, Quiver algebras, weighted projective lines, and the Deligne-Simpson problem, in *International Congress of Mathematicians. Vol. II*, 117–129, Eur. Math. Soc., Zürich, 2006.

O. Schiffmann, Noncommutative projective curves and quantum loop algebras, Duke Math. J. **121** (2004), no. 1, 113–168.

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