Chapter 6

Representations of quivers

Quivers and their representations arise often.

6.1 Basics

Definition 1.1 A quiver is a directed graph with a finite number of vertices and arrows. It may have multiple arrows between two vertices, and loops. ◊

We denote vertices by Roman letters, and arrows by Greek letters.

A path from vertex u to vertex v is a sequence of arrows starting at u and ending at v such that the end of each arrow is the start of the next one.

If α is an arrow from a to b and β an arrow from b to c, we write $\beta\alpha$ for the path from a to c that is "first traverse α, then traverse β". Composition of paths is defined by concatenation in the obvious way.

If we adjoin a path that we call zero, and denote by 0, the paths form a monoid provided we adopt the rule that $\alpha\beta = 0$ if the endpoint of the path β is different from the starting point of the path α.

At each vertex v we introduce the trivial path ε_v, the effect of which is that if α is a path ending at v, and β is a path beginning at v, then $\beta\varepsilon_v\alpha = \beta\alpha$.

A quiver determines a category in which the objects are the vertices, and the morphisms from u to v are the paths beginning at u and ending at v. Composition of morphisms is given by concatenation of paths. The identity morphism on an object v is the trivial path ε_v. The category determined by a quiver Q will also be denoted by the letter Q.

Definition 1.2 Let k be a field, and Q a quiver. A representation of Q over k is a functor $F : Q \to \mathsf{Mod}k$. A map $\tau : F \to G$ between two representations is a natural transformation of the functors. We write $\mathsf{Mod}Q$ for the category of representations.

Thus a representation of a quiver assigns to each vertex v a vector space M_v, and to each arrow $\alpha : u \to v$ a linear map $M_u \to M_v$. This data determines the representation. A map $\tau : M \to N$ between two representations consists of

linear maps $\tau_v : M_v \to N_v$ for each vertex v, such that the obvious diagrams commute; that is, $\tau_v \circ M(\alpha) = N(\alpha) \circ \tau_u$ whenever α is an arrow from u to v.

Example 1.3 A representation of the quiver

can be denoted by drawing a picture that is the same shape as the quiver with each vertex replaced by the corresponding vector space, and each arrow labelled with the corresponding linear map. For example, there are representations

$$\begin{aligned}
P(n) &= k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} \cdots \qquad \cdots \xleftarrow{1} k \xleftarrow{1} k \\
P(n-1) &= k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} \cdots \qquad \cdots \xleftarrow{1} k \longleftarrow 0 \\
&\vdots \\
P(2) &= k \xleftarrow{1} k \longleftarrow 0 \longleftarrow \cdots \qquad \cdots \longleftarrow 0 \longleftarrow 0 \\
P(1) &= k \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots \qquad \cdots \longleftarrow 0 \longleftarrow 0
\end{aligned}$$

in which all the maps $k \to k$ are the identity map, multiplication by 1. There are non-zero maps

$$P(1) \to P(2) \to \cdots \to P(n-1) \to P(n)$$

between these representations. We also have representations

$$S(1) = k0\cdots00, \quad S(2) = 0k0\cdots00, \quad \ldots, S(n) = 00\cdots0k.$$

We don't need to draw the arrows because the linear maps are zero. Each of these representations has no proper sub-representations; they are *irreducible* representations.

We shall shortly see that the representations of a quiver form an abelian catgeory. It will then make sense to say that we have short exact sequences

$$0 \to P(i-1) \to P(i) \to S(i) \to 0 \tag{1-1}$$

for all i. Actually, a map $M \to N$ between two representations is injective (resp., surjective) if and only if, for every vertex v, the maps $M_v \to N_v$ are injective (resp., surjective). We will see that (1-1) provides a projective resolution of $S(i)$ (section 6.3). $\Diamond$

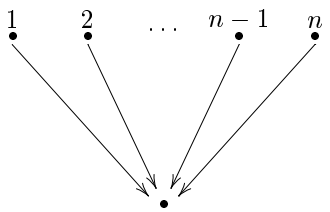
One can speak of representations of Q in any category C, such a representation being a functor $F : Q \to \mathsf{C}$. For example, one might take C to be vector bundles on a space X. Representations of quivers are ubiquitous. This is because the notion of a quiver is so basic—it consists of vertices and arrows: the vertices are simply place-holders for objects, and the arrows are place-holders

for morphisms. Whenever one has objects and morphisms between them one is dealing with a representation of a quiver.

Remark. Classifying the representations of a quiver can be interpreted as a classification problem in linear algebra. For example, the classification of vector spaces endowed with a single linear map is the same thing as the classification of representations of the quiver consisting of one vertex and one arrow (2-3), and Theorem 2.8 says that this is equivalent to the classification of modules over $k[x]$.

The classification of pairs of linear maps on a vector space (up to simultaneous conjugation) is equivalent to the representation theory of the quiver with one vertex and two arrows (2-4). By Theorem 2.8 this is equivalent to the classification of modules over the free algebra on two variables. This is a famously intractable problem and is a benchmark against which other algebraic classification problems are measured. A classification problem that includes the problem of classifying all such pairs of linear maps is said to be *wild.*

The n-subspace problem for $n \geq 5$ is wild: the problem is to classify all n-tuples of subspaces of a given vector space. This is almost the same problem as that of classifying the representations of the quiver



(1-2)

If M is a representation of this quiver in which one of the arrows is not injective, then $M = M' \oplus K$ where K is the kernel of that non-injective arrow, and M' is a representation of the subquiver for the $(n-1)$-subspace problem. Thus, excluding the $n+1$ simple modules, the indecomposable representations are in bijection with the classification of n-tuples of subspaces of a vector space. If $n \leq 3$ there are only finitely many such indecomposables up to isomorphism. If $n = 4$ there are a finite number of one-parameter families of indecomposables [**?**]; the problem is said to be *tame.* For $n \geq 5$ the problem is wild.

6.2 The path algebra

Definition 2.1 Let k be a field, and Q a quiver. The path algebra kQ is the k-vector space with basis given by all the paths in Q, and multiplication defined by concatenation of paths. ◊

In some sense kQ is a linearization of the category Q; the morphisms in the category form a basis for kQ and the multiplication in kQ is induced by the composition of morphisms.

The identity element in kQ is the sum of all the trivial paths ε_v. The elements ε_v are mutually orthogonal idempotents. They are also primitive: this is because each ε_v has degree zero in the grading defined below.

A grading on the path algebra. The path algebra can be given the structure of a $\mathbb{Z}$-graded algebra by defining the degree of a path to be its length. The trivial paths ε_v are given degree zero. Thus kQ is $\mathbb{N}$-graded, and its degree zero component is the product of fields $\oplus_{v\in Q} k\varepsilon_v$. The degree one component of kQ has basis the arrows of Q. It is a bimodule over $(kQ)_0$.

Example 2.2 Let Q be the quiver with n vertices, and no arrows. Then kQ is isomorphic to a product of n copies of the base field. We can also view kQ as the ring of k-valued functions on the set $\{1, 2, \ldots, n\}$. ◊

Example 2.3 The path algebra of

(2-3)

is isomorphic to the polynomial ring on one variable, $k[x]$, where x^n denotes the path that traverses the loop n times. The path algebra of

(2-4)

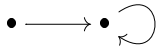
is the free algebra on two variables. The path algebra of the quiver with one vertex and n arrows is isomorphic the free algebra on n variables. ◊

Example 2.4 The path algebra of the quiver

(2-5)

with n vertices is the ring of upper triangular $n \times n$ matrices. The isomorphism is implemented by writing e_{ij} for the path beginning at the jth vertex and ending at the ith vertex. The degree i component of the upper triangular matrix algebra is the ith upper diagonal. ◊

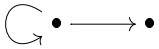
Example 2.5 The path algebra of the quiver



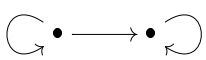
is isomorphic to

$$\begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix}.$$

This ring is right, but not left, noetherian. The path algebra of the opposite quiver



is left, but not right, noetherian. The path algebra of



is neither right nor left noetherian. ◊

The proof of the next result is obvious.

Lemma 2.6 *The path algebra kQ is finite dimensional if and only if the quiver has no cycles (i.e., no non-trivial path begins and ends at the same vertex).*

The next result is not essential for our purposes.

Proposition 2.7 *Let k be a field, and Q a quiver. Then kQ is isomorphic to the tensor algebra of $(kQ)_1$ viewed as a $(kQ)_0$-bimodule.*

Theorem 2.8 *Let k be a field, and Q a quiver. The category of k-representations of Q is equivalent to the category of left kQ-modules.*

Proof. If F is a representation of Q, we define $M := \oplus_{v \in Q}$. The direct sum of all these is denoted by M. If γ is a path in Q starting at the vertex v, we define the action of $\gamma \in kQ$ on $m \in M$ by

$$\gamma.m = F(\gamma)(m_v),$$

where m_v is the component of m belonging to $F(v)$. It is easily verified that by extending this action linearly to kQ, one makes M a kQ-module.

Conversely, if M is a kQ-module, then $M = \oplus \varepsilon_v M$, where the sum is over all vertices v of Q. Let $\iota_v : \varepsilon_v M \to M$ be the inclusion and $\pi_v : M \to \varepsilon_v M$ be the projection. We associate to M a functor $F : Q \to \mathsf{Mod}k$ as follows. For each vertex v, $F(v) := \varepsilon_v M$. If γ is a path from u to v, we define

$$F(\gamma) = \pi_v \gamma \iota_u : M_u \to M_v.$$

It is easily verified that F is a functor, and the rule $M \mapsto F$ is inverse to the rule $F \mapsto M$ described in the previous paragraph. □

We use all the standard language of modules to talk about representations of quivers. For example, we have irreducible representations, or simple modules, direct sums, submodules, etc.

Notation. Let R and S be rings, and let ${}_R M_S$ and ${}_S N_R$ be bimodules. Suppose there are bimodule maps $f : M \otimes_S N \to R$ and $g : N \otimes_R M \to S$. Then we form a ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} := \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \;\middle|\; r \in R,\, s \in S,\, m \in M,\, n \in N \right\}. \tag{2-6}$$

The multiplication is defined using the maps f and g in the obvious way.

Example 2.9 We call

$$\overset{2}{\bullet} \leftleftarrows \overset{1}{\bullet} \tag{2-7}$$

the Kronecker quiver. Its path algebra is isomorphic to

$$R := \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}. \tag{2-8}$$

We call this the Kronecker algebra. Let $A = k[x, y]$ be the commutative polynomial ring with its standard grading. Then R is isomorphic to the subring

$$\begin{pmatrix} A_0 & A_1 \\ 0 & A_0 \end{pmatrix}$$

of $M_2(A)$. The column-module $P_n := (A_n, A_{n-1})^{\mathsf{trans}}$ becomes a left R-module via left multiplication. Viewed as a representation of the quiver, P_n has the vector space A_{n-1} at the vertex labelled 1, and A_n at the vertex labelled 2, and the two arrows are multiplication by x and y.

The modules P_0 and P_1 are indecomposable projectives. It is an easy exercise to show that every P_n is indecomposable, and that $\mathrm{Hom}_R(P_n, P_{n+m}) \cong A_m$, with $a \in A_m$ acting by *right* multiplication on P_n, $(b, c).a = (ab, ac)$. A classification of all indecomposable R-modules can be found in [**?**, Theorem 8.5.7] and in [**?**, VIII.7].

The following representations indexed by $\lambda \in \mathbb{P}^1$ are indecomposable provided $\lambda \notin \{0, \infty\}$. Let $\lambda = (\alpha, \beta)$. There is a 2-dimensional representation

$$k \underset{\alpha}{\overset{\beta}{\leftleftarrows}} k \tag{2-9}$$

where α and β are the linear maps "multiply by α and β" respectively. It is a good exercise to check that the isomorphism class of the representation depends only on $(\alpha, \beta) \in \mathbb{P}^1$. $\Diamond$

Example 2.10 Let Q be the Kronecker quiver. There is a functor

$$F = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{F}, -) : \mathsf{Qcoh}\mathbb{P}^1 \to \mathsf{Rep}Q$$

where

$$\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

To see this, first observe that $\mathrm{End}_{\mathbb{P}^1} \mathcal{F} \cong kQ$, and that F is naturally a functor to the category of right $\mathrm{End}_{\mathbb{P}^1} \mathcal{F}$-modules, the action being given by composition of homomorphisms

$$\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{F}, \mathcal{G}) \times \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{F}, \mathcal{G});$$

second, the categories of left and right kQ-modules are equivalent. Under the isomorphism $\mathrm{End}_{\mathbb{P}^1}\mathcal{F} \cong kQ$, the homomorphisms $x, y : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ provide a basis for the nilpotent radical of kQ, the subspace

$$\begin{pmatrix} 0 & k^2 \\ 0 & 0 \end{pmatrix}$$

in the above matrix description of kQ.

It is interesting to see how the representations of Q in the previous example arise by evaluating F. First, $F(\mathcal{O}(n))$ gives the module P_n for $n \geq 0$, and if $p = \lambda = (\alpha, \beta)$, then F applied to the skyscraper sheaf $\mathcal{O}_p$ gives the representation (2-9) where the two linear maps are induced by multiplication by x and y.

The functor F is not an equivalence of categories because, for example, $F(\mathcal{O}_{\mathbb{P}^1}(-n)) = 0$ for all $n > 0$, but F extends to an equivalence of derived categories

$$\mathbf{R}F : \mathsf{D}^{\mathrm{b}}(\mathsf{coh}\mathbb{P}^1) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}kQ).$$

$\Diamond$

The representations of the Kronecker quiver that are parametrized by $\mathbb{P}^1$ can be characterized in the following way: they are indecomposable, meaning that they are not a direct sum of two non-zero submodules, and the component of the representation at each vertex has dimension one.

If M is a representation of a quiver Q we define its dimension vector or simply its dimension to be the tuple $(\dim M_v)_{v \in Q}$ indexed by the vertices of Q. If we fix a dimension vector, $\underline{d} = (d_v)$, the set of all representations having dimension $\underline{d}$ forms a variety; explicitly, if we write $s(\alpha)$ and $t(\alpha)$ for the start and end of an arrow α, this variety is

$$\prod_{\text{arrows } \alpha} \mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}).$$

The group $\mathrm{GL}(\underline{d}) = \prod_{v \in V} \mathrm{GL}(d_v)$ acts on this variety, the orbits being the isomorphism classes of representations. Using the tools of geometric invariant theory, one can construct various "quotient" varieties that parametrize the isomorphism classes of various types of representations. Often these are interesting varieties. The point is that many interesting varieties can be realized as the parameter spaces of representations of quivers. The previous example is a simple illustration of this showing that $\mathbb{P}^1$ arises as the parameter space for the isomorphism classes of indecomposable representations of the Kronecker quiver having dimension vector $(1, 1)$.

Lemma 2.11 *A path algebra kQ is prime if and only if for every pair of distinct vertices u and v there is a path from u to v.*

Proof. Suppose that there is no path from u to v. Then $\varepsilon_v \alpha \varepsilon_u = 0$ for all paths α. Hence $\varepsilon_v R \varepsilon_u = 0$, so R is not prime.

Conversely, suppose that there is a path between any two vertices. Let x and y be non-zero elements of R, each written as a linear combination of paths.

Pick a path α occurring in the expression for x, and a path γ appearing in the expression for y. Let u denote the vertex where γ ends, and let v denote the vertex where α begins. Let β be a path from u to v. Then $\alpha\beta\gamma \neq 0$, and it follows from this that $x\beta y \neq 0$. Thus $xRy \neq 0$, showing that R is prime. □

Our definition of a quiver required there to be only a finite number of vertices and arrows. However, much of the theory extends to quivers having an infinite number of vertices and/or arrows.

If Q has infinitely many vertices, then the path algebra does not have an identity. A representation of Q gives a module over the path algebra in the usual way. If we define $\mathsf{Mod}kQ$ to consist of those right kQ-modules M such that $M = \sum M\varepsilon_v$, where v runs over all the vertices of Q, then each kQ-module gives a representation of Q. Thus one obtains a version of Theorem 2.8 for infinite quivers.

As a simple example consider the quiver consisting of vertices indexed by $\mathbb{Z}$ and no arrows. Then the representations are just vector spaces with a specified decomposition $V = \oplus_n V_n$ into subspaces. It is a tautology that the category of representations of this quiver is equivalent to the category of graded modules over the base field.

Example 2.12 The path algebra of the doubly infinite quiver

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \tag{2-10}$$

is isomorphic to the ring of doubly infinite lower triangular matrices having only finitely many non-zero entries. The category of representations of the quiver is equivalent to the category $\mathsf{GrMod}k[x]$ of graded modules over the polynomial ring in one variable x having degree one. If the vertices are labelled by the integers $\ldots, n-1, n, n+1, \ldots$ beginning at the left, then a representation $M = \oplus M_n$ of the quiver can be viewed as a graded $k[x]$-module with M_n being the degree n component, and x acting on M_n via the arrow from n to $n+1$. Conversely each graded $k[x]$-module gives a representation by placing the degree n component of the module at the vertex labelled n.

Let T_n denote the ring of $n \times n$ upper triangular matrices, and view it as the path algebra of the quiver in Example 1.3. There is a fully faithful embedding $\mathsf{Mod}T_n \to \mathsf{GrMod}k[x]$ given by sending a T_n-module to the representation of the quiver (2-10) that is the representation of T_n inserted at the vertices $1, \ldots, n$ and zeroes elsewhere. ◊

6.3 Simples and projectives

Example 3.1 Fix a vertex v in a quiver Q. We write $S(v)$ for the representation of Q that has a copy of k at the vertex v and zeroes elsewhere; that is,

$$S(v)_u := \begin{cases} k & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

All arrows are zero of course. This is an irreducible representation of Q or, equivalently, a simple kQ-module. $\Diamond$

Proposition 3.2 *Let Q be a quiver without cycles. Then*

$$\{\textit{simple left } kQ\textit{-modules}\} = \{S(v) \mid v \textit{ is a vertex}\}.$$

Proof. The k-linear span of the non-trivial paths is a two-sided ideal of kQ, say I. The subalgebra

$$\bigoplus_{v \in Q} k\varepsilon_v$$

is a complement to I in kQ, so kQ/I is isomorphic to a product of fields. The nth power of I is spanned by the paths of length $\geq n$. Since the quiver has no cycles, there is a bound to the length of the paths, so I is nilpotent. It follows that I is the nilpotent radical of kQ. So the simple kQ-modules are exactly the simple modules over $\oplus_{v \in Q} k\varepsilon_v$, namely the $S(v)$'s. $\square$

This result fails if Q has cycles. For example, the polynomial ring $k[x]$, which is the path algebra of the quiver with a single vertex and a single arrow, has infinitely many simple modules.

Each $S(v)$ can be given the structure of a graded kQ-module, and, up to shifting degree, these form a complete set of simple modules in $\mathsf{GrMod}kQ$. This follows from the fact that $(kQ)_0 = \oplus k\varepsilon_v$.

Example 3.3 Fix a vertex v. We define the representation $P(v)$ by

$$P(v) := \bigoplus_{u \in Q} P(v)_u \qquad \text{where}$$

$$P(v)_u := k\text{-linear span of the paths beginning at } v \text{ and ending at } u.$$

The paths beginning at v provide a basis for $P(v)$, so $P(v) = (kQ)\varepsilon_v$. Because the $\{\varepsilon_v \mid v \text{ is a vertex}\}$ is a complete set of orthogonal idempotents,

$$kQ = \bigoplus_{v \in Q} (kQ)\varepsilon_v = \bigoplus_{v \in Q} P(v).$$

In particular, $P(v)$ is a projective left kQ-module.

Moreover, $P(v)$ is a graded left ideal of kQ, its degree zero component is $k\varepsilon_v$, and a complement to this is given by

$$P(v)_{\geq 1} = \bigoplus_{\text{arrows } \alpha : v \to u} P(u)\alpha.$$

Thus

$$P(v)/P(v)_{\geq 1} \cong S(v).$$

$\Diamond$

Lemma 3.4 *Let $P(v)$ be the indecomposable projective and $S(v)$ the simple module corresponding to the vertex v. Then*

$$\mathrm{Hom}_Q(P(v), S(u)) = \begin{cases} k & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Proof. A homomorphism $f : P(v) \to S(u)$ is completely determined by $f(\varepsilon_v)$. Because $\dim_k S(u) = 1$ it follows that $\dim_k \mathrm{Hom}_Q(P(v), S(u)) \leq 1$. We observed in Example 3.3 that there is a non-zero map $P(v) \to S(v)$; this proves the lemma when $u = v$. If $u \neq v$, then $\varepsilon_v S(u) = 0$, so $f(\varepsilon_v) = f(\varepsilon_v^2) = \varepsilon_v f(\varepsilon_v) = 0$ so $f = 0$; this proves the lemma when $u \neq v$. □

Example 3.5 View kQ as a left module over itself. This gives a representation of Q for which the vector space concentrated at v is the *right* ideal $P'(v) := \varepsilon_v(kQ)$; a basis for $P'(v)$ consists of all the paths that end at v; left multiplication by $\alpha : u \to v$ corresponds to the linear map $P'_u \to P'(v)$ that sends a path $\beta \in P'(u)$ to the path $\alpha\beta \in P'(v)$. ◊

Proposition 3.6 *Let Q be a quiver without cycles. Then the indecomposable projective kQ-modules are the modules $P(v)$, $v \in Q$, in Example 3.3.*

Proof. Recall that $1 = \sum_{v \in Q} \varepsilon_v$ expresses 1 as a sum of primitive orthogonal idempotents. By the general theory of modules over artinian rings the indecomposable projective kQ-modules are exactly the modules $(kQ)\varepsilon_v$. □

Theorem 3.7 *Let k be a field and Q a quiver. Then*

1. $\mathrm{gldim}\, kQ \leq 1$;
2. $\mathrm{gldim}\, kQ = 0$ *if and only if Q has no arrows.*

Proof. (2) If Q has no arrows, then kQ is a product of copies of k, so has global dimension zero.

(1) Now suppose that Q has some arrows.

Fix a vertex v. Define

$$\Theta := \{\text{the arrows beginning at } v\},$$

and for each $\alpha \in \Theta$, define

$$t(\alpha) := \text{the vertex at which } \alpha \text{ ends.}$$

Let $S(v)$ denote the simple module at v. Then $\varepsilon_u S(v)$ is zero if $u \neq v$, and equals $S(v)$ if $u = v$. Hence there is a surjective map $\varphi : P(v) \to S(v)$, the kernel of which is spanned by the non-trivial paths beginning at v. If γ is a non-trivial path beginning at v, then $\gamma = \beta\alpha$ for a unique $\alpha \in \Theta$, and a unique path β beginning at $t(\alpha)$. Therefore $\ker \varphi$ is the direct sum of the left ideals

$P(t(\alpha))\alpha$ where α runs through Θ. Left multiplication by $\alpha \in \Theta$ is an injective right kQ-module map $P(t(\alpha)) \to P(v)$ so $P(t(\alpha))\alpha \cong P(t(\alpha))$. Hence there is an exact sequence

$$0 \to \bigoplus_{\alpha \in \Theta} P(t(\alpha)) \to P(v) \to S(v) \to 0. \tag{3-11}$$

Therefore $\operatorname{pdim} S(v) \leq 1$. Since Q has some arrow, $S(v) \neq P(v)$ for some v, whence $\operatorname{pdim} S(v) \geq 1$.

If kQ is finite dimensional we are done since $\{S(v)\}$ is a complete set of simple modules, and every finite dimensional module has a composition series.

If kQ is infinite dimensional we note that (3-11) is a projective resolution in the graded category. Thus each graded simple module has projective dimension at most one in $\mathsf{GrMod}(kQ)$. Since kQ is positively graded, it follows from ??? that $\operatorname{gldim} kQ \leq 1$. □

An algebra of global dimension one is said to be hereditary. D. Benson [?, page 96] gives a nice application of the fact that $\mathsf{Mod}kQ$ has global dimension one to inverse limits.

Example 3.8 The path algebra of the quiver

is the ring of $n \times n$ upper triangular matrices. The idempotent ε_i corresponding to the vertex i is the matrix unit e_{ii}. The irreducible and indecomposable projective representations of this quiver were given in Example 1.3.

The indecomposable projectives are the columns

$$P(1) = \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad P(2) = \begin{pmatrix} 0 & k & \cdots & 0 \\ 0 & k & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \cdots,$$

$$P_{n-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ k & k & \cdots & k & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, P(n) = \begin{pmatrix} 0 & 0 & \cdots & k \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & k \\ 0 & 0 & \cdots & k \end{pmatrix}$$

of the matrix algebra.

The simple module $S(i) = 0 \cdots 0k0 \cdots 0$ corresponds to the module of column vectors having zeroes in all except possibly the ith position, on which the matrix ring acts by left multiplication.

Right multiplication by $e_{i,i-1}$ gives the first map in the projective resolution $0 \to P(i-1) \to P(i) \to S(i) \to 0$. ◊

The arrows in a quiver control the homological behavior of the simple modules.

Proposition 3.9 *If u and v are vertices, then $\dim_k \operatorname{Ext}^1_{kQ}(S(v), S(u))$ is the number of arrows $v \to u$.*

Proof. We use the notation in the proof of Theorem 3.7. It follows from the projective resolution (3-11) that $\operatorname{Ext}^1_{kQ}(S(v), S(u))$ is the cokernel of the natural map

$$\Phi : \operatorname{Hom}_Q(P(v), S(u)) \to \operatorname{Hom}_Q(\bigoplus_{\alpha\in\Theta} P(t(\alpha)), S(u)).$$

Let β be a path beginning at $t(\alpha)$; the map $P(t(\alpha)) \to P(v)$ in (3-11) is $\beta \mapsto \beta\alpha$; hence

$$\Phi(f)(\beta) = f(\beta\alpha) = f(\beta\alpha\varepsilon_v) = \beta\alpha f(\varepsilon_v) = 0.$$

Since $\Phi = 0$,

$$\begin{aligned}\dim_k \operatorname{Ext}^1_Q(S(v), S(u)) &= \dim_k \operatorname{Hom}_Q(\oplus_{\alpha\in\Theta} P(t(\alpha)), S(u)) \\ &= \sum_{\alpha\in\Theta} \dim_k \operatorname{Hom}_Q(S(t(\alpha)), S(u)),\end{aligned}$$

where the second equality follows from Lemma 3.4. This number is the cardinality of $\{\alpha \in \Theta \mid t(\alpha) = u\}$, which is what we needed to prove. □

One can be more explicit about the way in which the arrows $u \to v$ give non-split extensions.

Lemma 3.10 *Let Q be a quiver and u and v distinct vertices. There is an arrow $u \to v$ if and only if there is a non-split extension of $S(u)$ by $S(v)$.*

Proof. ($\Rightarrow$) Suppose that $u \neq v$. Let M be the representation with k at both u and v, zeroes elsewhere, and assign to the arrow α the identity map $k = k_u \to k = k_v$. It is clear that $S(v) = (0 \to k = k_v)$ is a sub-representation of M, and that the quotient is isomorphic to $S(u)$. Hence M is an extension of $S(u)$ by $S(v)$. Because the map $k \to k$ is non-zero, $S(v)$ is the only proper sub-representation of M. Thus, by Lemma 4.6.2, the extension does not split.

Suppose that $u = v$. Let M be the representation with k^2 at v, zeroes elsewhere, and assign to the arrow α any rank one map $T : k^2 \to k^2$. The image of T is a submodule of M and it is isomorphic to $S(v)$; this is the only proper submodule of M so, by Lemma 4.6.2, the extension does not split.

($\Leftarrow$) Suppose that $0 \to S(v) \to M \to S(u) \to 0$ is a non-split extension. Then $\dim_k M = 2$. We consider $S(v)$ as a submodule of M.

Suppose that $u \neq v$. The representation of Q associated to M has k at the vertex labelled u and k at the vertex labelled v. If there were no arrow from u to v, then $S(u)$ would be a sub-representation of M, so by Lemma 4.6.2, M would split. This is a contradiction, so we conclude that there must be an arrow from u to v.

Suppose that $u = v$. Then the representation M has k^2 at the vertex labelled v and zeroes elsewhere. If there were no arrow $v \to v$, then every 1-dimensional subspace of M would be a submodule, so M would split by Lemma 4.6.2. Again, we conclude that there is an arrow from u to v. □

Propositions 3.2 and 3.9 give a recipe for recovering the quiver from a finite dimensional path algebra. One takes a vertex for each simple module, and from the vertex u to the vertex v one draws $\dim_k \operatorname{Ext}^1(S(u), S(v))$ arrows.

The next result, which we do not need, illustrates the fundamental nature of path algebras. Recall that an algebra is hereditary if it has global dimension ≤ 1.

Theorem 3.11 *Suppose that k is a perfect field. Then, up to Morita equivalence, the path algebras kQ give all the finite dimensional hereditary algebras.*

6.4 Some infinite-dimensional path algebras

We now examine a family of infinite dimensional path algebras consisting of non-maximal hereditary orders.

Proposition 4.1 *Let Q be the following quiver with n vertices and n arrows:*

$$Q \;=\; \begin{array}{ccccccc} \bullet & \longleftarrow & \bullet & \longleftarrow & \cdots & \longleftarrow & \bullet \\ n & & n-1 & & & & 2 \qquad 1 \end{array} \qquad \text{(4-12)}$$

Write x_i for the arrow that ends at the vertex labelled i.

1. *The path algebra is isomorphic to the ring*

$$S = \begin{pmatrix} R & xR & \cdots & xR & xR \\ R & R & \cdots & xR & xR \\ \vdots & & & & \vdots \\ R & R & \cdots & R & xR \\ R & R & \cdots & R & R \end{pmatrix}, \qquad \text{(4-13)}$$

   where $R = k[x]$.

2. *Under this isomorphism, $x = \sum_{i=1}^{n} x_i x_{i-1} \ldots x_{i+1}$.*

3. *The element $u = x_1 + \ldots + x_n$ is normal, meaning that $uS = Su$. If we identify kQ with S, then*

$$u = \begin{pmatrix} 0 & 0 & \ldots & 0 & x \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

4. $u^n = x$.

5. *If k is algebraically closed, then the simple Q-modules are as follows:*

   (a) *There are n one-dimensional simple modules V_i, the representations defined by placing k at the vertex labelled i and zero at all others. Under the isomorphism $V_i e_{ii} \neq 0$, where $e_i i$ is the primitive diagonal idempotent in the ii-position.*

   (b) *There is a family of n-dimensional simples S_λ parametrized by $\lambda \in \mathbb{A}^1 \backslash \{0\}$ obtained by placing k at each vertex, and making each arrow the identity map, except the arrow from 1 to n which is multiplication by λ.*

6. *If $\lambda \neq 0$, then $S/(x - \lambda) \cong M_n(k)$.*

7. *The element u annihilates all the one-dimensional simple S-modules.*

8. *The auto-equivalence $- \otimes_S Su$ of $\mathsf{Mod}S$ sends the one-dimensional simple V_i to V_{i-1} and fixes each n-dimensional simple.*

Proof. To show that x is central it suffices to show that it commutes with the idempotents ε_i (the trivial paths) and with each x_i. This is a routine calculation.

A non-trivial path begins at some vertex i, goes t times around the loop, but not $t+1$ times, and ends at the vertex j. This path is

$$(x_{i-1}x_{i-2}\dots x_i)^t x_{i-1}x_{i-2}\dots x_j,$$

which equals $x^t x_{i-1}x_{i-2}\dots x_j$. Thus, $k[x]$ is a central subalgebra of kQ, and a basis for kQ over $k[x]$ is given by the elements $x_{i-1}x_{i-2}\dots x_j$ together with the idempotents ε_i.

The claimed isomorphism is obtained as follows. If $i = j$, we send ε_i to e_{ii}. If $i > j$, we send $x_{i-1}x_{i-2}\dots x_j$ to e_{ij}. Otherwise, we send $x_{i-1}x_{i-2}\dots x_j$ to xe_{ij}. For example, $x_1 \mapsto e_{21}$, $x_2 \mapsto e_{32}$, and $x_n \mapsto xe_{1n}$.

It is clear that the one-dimensional simples are as claimed. It is also clear that the modules S_λ are simple. So, it remains to show that this is a complete list of the simples.

Let M be a simple module. Then M is annihilated by $x - \lambda$ for some $\lambda \in k$. Let $M = \oplus_{i=1}^n M_i$ be the decomposition with M_i the component at the vertex i. Any non-zero submodule of M must contain some non-zero homogeneous element, say $0 \neq m \in M_i$. We have

$$\lambda m = m.x = mx_{i-1}x_{i-2}\dots x_i.$$

Therefore the elements $m, mx_{i-1}, mx_{i-1}x_{i-2}, \dots$ span a submodule of M, which must, by hypothesis, equal M itself. If $\lambda \neq 0$, this is isomorphic to S_λ (just change bases in the correct way). If $\lambda = 0$, then the last non-zero element in the list $m, mx_{i-1}, mx_{i-1}x_{i-2}, \dots$ spans a one-dimensional submodule, so we conclude that $\dim_k M = 1$.

It follows from the earlier part of the proof that S is spanned by n^2 elements as a $k[x]$-module (in fact these elements form a basis), so $\dim_k S/(x-\lambda) \leq n^2$. However, when $\lambda \neq 0$, this ring has an n-dimensional simple module, so it must be isomorphic to $M_n(k)$.

To see that u is normal, we need only observe that $u\varepsilon_i = x_i = \varepsilon_{i+1}u$, and that $ux_{i-1} = x_i x_{i-1} = x_i u$. A simple calculation shows that $u^n = x$. It is clear that $V_i u = 0$ for all simples (because, for example, $\deg u > 0$).

The action of $- \otimes_S Su$ sends any module annihilated by $x - \lambda$ to another module annihilated by $x - \lambda$, so $S_\lambda \otimes_S Su \cong S_\lambda$. The one-dimensional simple V_i is annihilated by ε_j for all $j \neq i$. Since $u\varepsilon_j = \varepsilon_{j+1}u$, $V_i \otimes_S Su$ is annihilated by ε_j for all $j \neq i-1$. Therefore $V_i \otimes_S Su \cong V_{i-1}$. □

Proposition 4.2 *Let Q be the quiver in Proposition 4.1. Then* ModkQ *is equivalent to the category* GrMod$k[t]$ *where $k[t]$ is graded by the group $\mathbb{Z}_n$, with* $\deg t = 1$.

Proof. Label the elements of $\mathbb{Z}_n$ by $0, 1, \ldots, n-1$ in the obvious way. It is helpful to give the vertices of Q the same labels in such a way that there is an arrow from i to $i+1$ for each i.

This is so obvious that explanation probably obscures the issue. Each representation of Q gives a graded $k[t]$-module with the degree i component of the $k[t]$-module being the component of the representation at the vertex labelled i, and the action of t on the degree i component being given by the action of the map corresponding to the arrow from i to $i+1$. Conversely, a graded $k[t]$-module gives a representation of Q in an obvious way. Morphisms in one category obviously are morphisms in the other category. The functors between the categories are clearly mutually inverse. □

The path algebra in Proposition 4.1 is a prime ring.

EXERCISES

4.1 Suppose that we adopt the opposite convention for the concatenation of paths—that is, we write $\alpha\beta$ to mean first traverse β, then traverse α. Write $k * Q$ for the path algebra obtained in this way. Show that $k * Q \cong (kQ)^{\mathrm{op}}$.

4.2 Let Q be the quiver $1 \to 2 \to \cdots \to n$. Show that $k * Q$ is isomorphic to the ring of lower triangular matrices in such a way that e_{ij} corresponds to the path from j to i. With this convention, the representations of the path algebra correspond to the left modules over $k * Q$.

4.3 If Q is the disjoint union of quivers Q_1 and Q_2, show that $kQ \cong kQ_1 \oplus kQ_2$.

4.4 Show that the indecomposable projectives for the path algebra $k * Q$ in the previous exercise are the columns of the triangular matrix ring. More generally, show that the left ideal of $k * Q$ consisting of all paths that begin at a vertex v is an indecomposable projective.

4.5 If Q is a quiver, let Q^{op} denote the "same" quiver with the arrows reversed. Show that $kQ^{\mathrm{op}} \cong (kQ)^{\mathrm{op}}$.

4.6 Show that the simple modules $S(v)$ defined in exercise 3.1 are the only simple objects in the category of graded kQ-modules up to shifting degree.

4.7 A vertex is called a source if there are no arrows ending at it. A vertex is called a sink, if there are no arrows beginning at it.

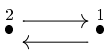
(a) Show that v is sink if and only if $S(v)$ is injective.

(b) Show that v is source if and only if $S(v)$ is projective.

4.8 Show that $\mathrm{Hom}_Q(P(u), S(v))$ is zero if $u \neq v$, and is isomorphic to k if $u = v$.

4.9 Suppose that Q has no cycles. Let B denote the linear span of all arrows in Q. Determine the $(kQ)_0$-bimodule structure of B for some of the examples in this section. Since $(kQ)_0$ is a product of fields $\prod_v k\varepsilon_v$, so is $(kQ)_0 \otimes_k (kQ)_0$. The simple $(kQ)_0$ bimodules are indexed by pairs of vertices, and we denote them by ${}_uS_v$, this being the bimodule that is *not* annihilated by ε_u on the left, and *not* annihilated by ε_v on the right.

(a) For the quiver

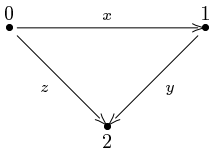


(4-14)

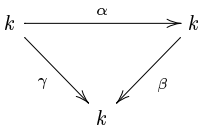
show that $B \cong {}_1S_2 \oplus {}_2S_1$, that $B^{\otimes(2n+1)} \cong B$ for all n, and that $B^{\otimes 2n} \cong {}_1S_1 \oplus {}_2S_2$ for all n.

(b) Compare this with what happens for the Kronecker quiver (2-7).

4.10 Let Q be the quiver



Let $M_{\alpha\beta\gamma}$ be the representation



where the maps are multiplication by the non-zero scalars $\alpha, \beta, \gamma \in k$. Show that $M_{\alpha\beta\gamma} \cong M_{\mu\nu\omega}$ if and only if $\alpha\beta\gamma^{-1} = \mu\nu\omega^{-1}$.

4.11 Let Q be a quiver. Let M be the matrix with rows and columns indexed by the vertices of Q, and entry m_{uv} the number of arrows from u to v. Show that the uv-entry in M^n is the number of paths of length n from u to v.

The sum of the entries in M^n is the dimension of the degree n component of kQ.

4.12 Show that the Kronecker algebra R is isomorphic to the ring of global sections of the sheaf of non-commutative algebras

$$\mathcal{R} := \begin{pmatrix} \mathcal{O} & \mathcal{O}(-1) \\ \mathcal{O}(1) & \mathcal{O} \end{pmatrix} \tag{4-15}$$

on the projective line $\mathbb{P}^1$. Fix $n \in \mathbb{Z}$. Show that the action of $\mathcal{R}$ on $\mathcal{O}(n) \oplus \mathcal{O}(n-1)$ by left matrix multiplication induces an isomorphism $\mathcal{R} \cong \mathcal{E}nd(\mathcal{O}(n) \oplus \mathcal{O}(n-1))$.

4.13 Retain the notation in Exercise 12. Write $\mathcal{P} = \mathcal{O}(0) \oplus \mathcal{O}(-1)$. Show that that there is an exact functor

$$\mathcal{H}om(\mathcal{P}, -) : \mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}\mathcal{R}$$

from quasi-coherent $\mathcal{O}_{\mathbb{P}^1}$-modules to sheaves of right $\mathcal{R}$-modules.

4.14 Retain the notation in Exercise 12. Show that the global section functor $H^0(\mathbb{P}^1, -)$ gives a left exact functor $\mathsf{Mod}\mathcal{R} \to \mathsf{Mod}R$. Let F denote the left exact functor $\mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}R$ that is the composition of the global sections functor with the functor in the previous exercise. Thus $F = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{P}, -)$.

(a) For each invertible $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}(n)$ describe $F(\mathcal{O}(n))$ as a representation of the quiver

$$\bullet \rightrightarrows \bullet \ . \tag{4-16}$$

(b) Do the same thing for each simple $\mathbb{P}^1$-module $\mathcal{O}_p$, $p \in \mathbb{P}^1$.

4.15 Retain the notation in Exercise 12. Let x be a coordinate function on $\mathbb{P}^1$, and let U be the copy of the affine line where x has no pole. Show that $\mathcal{R}(U)$, the ring of sections of $\mathcal{R}$ over U, is isomorphic to

$$\begin{pmatrix} k[x] & (x) \\ (x^{-1}) & k[x] \end{pmatrix},$$

where (x) denotes the ideal of $k[x]$ generated by x, and (x^{-1}) denotes the $k[x]$-submodule of $k(x)$ generated by x^{-1}. Show that this ring is isomorphic to $M_2(k[x])$. [Hint: conjugation by a suitable unit will give an explicit isomorphism; one can also argue by using $\mathcal{P}(U)$.]

Over the open set U' where x has no poles, $\mathcal{R}(U')$ is isomorphic to

$$\begin{pmatrix} k[x^{-1}] & (x^{-1}) \\ (x) & k[x^{-1}] \end{pmatrix},$$

which is naturally isomorphic to 2×2 matrices over $k[x^{-1}]$.

4.16 Retain the notation in Exercise 12. For each $\mathcal{R}$-module $\mathcal{F}$, $\mathcal{F}(U)$ is a module over $\mathcal{R}(U)$, so by Morita equivalence gives a module over $k[x]$. For the $\mathbb{P}^1$-modules $\mathcal{M} = \mathcal{O}(n)$ and $\mathcal{M} = \mathcal{O}_p$ describe the modules over $k[x]$ that are obtained from the $\mathcal{R}$-modules $\mathcal{H}om(\mathcal{P}, \mathcal{M})$.

4.17 Define projective cover, and give a related exercise.

4.18 Let R be a finite dimensional k-algebra. Then $\mathrm{Hom}_k(-, k)$ is a duality between left and right R-modules. Show that it interchanges projective modules and injective modules. Show that it interchanges injective envelopes and projective covers; in particular, that it sends the injective envelope of a simple module S to the projective cover of the corresponding simple module $S^* := \mathrm{Hom}_k(S, k)$. Also show that $\mathrm{Ann}S = \mathrm{Ann}S^*$.

4.19 Define representations U_i, $1 \leq i \leq n$, for the quiver (4-12) as follows. To each vertex assign a copy of k, and assign to each arrow the identity map, except for the arrow x_i ending at i. Assign the zero map to that arrow. Show that U_i is uniserial (i.e., it has a unique composition series), and that its composition factors are the one-dimensional simples $V_i, V_{i-1}, \ldots, V_{i+1}$ starting from the top.

4.20 Verify the claims in Example 2.5.

6.5 Some functorial observations

There is an obvious notion of a map from one quiver to another: a function sending vertices to vertices and arrows to arrows in a compatible way.

Definition 5.1 A morphism, or a map, $f : Q \to Q'$ between two quivers is a set map from the vertices of Q to the vertices of Q' together with a set map from the arrows of Q to the arrows of Q' such that if α is an arrow in Q from u to v, then $f(\alpha)$ is an arrow from $f(u)$ to $f(v)$.

The objects in the category of quivers are the quivers, and the morphisms are as above. We denote this category Quiver. $\Diamond$

A quiver determines a category whose objects are the vertices, and whose morphisms are the paths. A map of quivers induces a functor between the associated categories. However, a functor between the categories that sends some non-trivial path to a trivial path is not induced by a map of quivers.

Proposition 5.2 *The rule $Q \mapsto kQ$ extends to a contravariant functor from the category of quivers to k-algebras.*

Proof. Let $f : Q \to Q'$ be a map of quivers. We define $\varphi : kQ' \to kQ$ to be the linear map such that for each path α' in Q',

$$\varphi(\alpha') := \sum_{f(\alpha)=\alpha'} \alpha.$$

In particular, if $\alpha = \varepsilon_u$ is the trivial path at u, then $\varphi(\varepsilon_u) = \sum_{f(v)=u} \varepsilon_v$, from which it follows that $\varphi(1) = 1$.

To show that φ is an algebra homomorphism we must show that if α' and β' are paths in Q', then $\varphi(\alpha'\beta') = \varphi(\alpha')\varphi(\beta')$.

First suppose that $\alpha'\beta' = 0$. Then $s(\beta') \neq t(\alpha')$. Therefore $s(\beta) \neq t(\alpha)$ for any paths α and β for which $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$; hence $\alpha\beta = 0$. Thus $\varphi(\alpha')\varphi(\beta') = 0$.

Now suppose that $\alpha'\beta' \neq 0$. Set $v' = t(\alpha') = s(\beta')$, and let $v_1, \ldots, v_n$ be the distinct vertices in Q such that $f(v_i) = v'$. Let A_i be the set of arrows in Q that end at v_i and are mapped to α'. Let B_i be the set of arrows in Q that start at v_i and are mapped to β'. Then

$$\varphi(\alpha')\varphi(\beta') = \sum_{i=1}^{n} \Big(\sum_{\alpha \in A_i} \alpha \Big) \Big(\sum_{b \in B_i} \beta \Big). \tag{5-17}$$

On the other hand,

$$\varphi(\alpha'\beta') = \sum_{f(\gamma)=\alpha'\beta'} \gamma. \tag{5-18}$$

Each such γ passes through a unique v_i. Because f sends each arrow to an arrow, there is a unique way to write $\gamma = \alpha\beta$ with $\alpha \in A_i$ and $\beta \in B_i$. It follows that the sums (5-17) and (5-18) are equal. $\square$

Corollary 5.3 *A map of quivers $f : Q \to Q'$ in the sense of Definition 5.1 induces an adjoint pair of functors $f^* : \mathsf{Mod}Q' \to \mathsf{Mod}Q$ and $f_* : \mathsf{Mod}Q \to \mathsf{Mod}Q'$ with f^* left adjoint to f_*.*

Proof. The induced algebra homomorphism $kQ' \to kQ$ gives rise to such a pair of functors where $f^* = kQ \otimes_{kQ'} -$ and f_*=view a kQ'-module as a kQ-module. □

Let $f : Q \to Q'$ be a map of quivers. There is a concrete description of f^* and f_* does not appeal to the path algebra.

A representation M of Q' is the same thing as a functor $M : Q' \to \mathsf{Mod}k$, where Q' is viewed as a category in which the morphisms are the paths. Hence each Q'-module M, viewed as a functor, determines a functor $M \circ f : Q \to \mathsf{Mod}k$. We have $f^*M = M \circ f$. Explicitly, if $v \in Q$, then $(f^*M)_v = M_{f(v)}$, and if α is an arrow from u to v in Q, then the action of α on f^*M sends $M_{f(u)}$ to $M_{f(v)}$ via the action of $f(\alpha)$.

On the other hand, if N is a Q-module, then f_*N is Q'-module with $(f_*N)_{v'} = \oplus_{f(v)=v'} N_v$, and if α' is an arrow in Q' then α' acts on f_*N as the direct sum of the actions of $f(\alpha)$ as α runs through all arrows such that $f(\alpha) = \alpha'$.

Example 5.4 We call Q' a subquiver of Q if there is a map $f : Q' \to Q$ that is injective on vertices and arrows. In this case the induced map $kQ \to kQ'$ gives an isomorphism $kQ' \cong kQ/I$, where I is the ideal generated by those ε_v for which v is not in Q' and by those arrows α which are not in Q'.

A special case of this arises when Q is a quiver with n vertices, and Q' is the subquiver consisting of the same vertices, but no arrows. The induced map $kQ \to kQ' \cong k^{\times n}$ is gotten by modding out the ideal spanned by the arrows; that is, one quotients out the homogenous ideal of kQ consisting of positive degree elements; if kQ is finite dimensional, this is the same as modding out the nilpotent radical.

As a second special case, let Q be the quiver

$$\overset{n}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \bullet \cdots \bullet \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \tag{5-19}$$

Let Q' be the subquiver of Q obtained by deleting the first vertex and arrow (or the last vertex and arrow). Then the induced map $kQ \to kQ'$ is that obtained by modding out the first row (or the last column) of the upper triangular $n \times n$ matrix ring. ◊

The next example concerns a ubiquitous infinite dimensional path algebra.

Example 5.5 (cf. Proposition 4.1 and Example **??**) Consider the unique map of quivers $h : Q \to Q'$ from

$$Q \;=\; \underset{n}{\bullet} \longrightarrow \underset{n-1}{\bullet} \longrightarrow \bullet \;\cdots\; \bullet \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{1}{\bullet} \qquad \text{(5-20)}$$

to

$$Q' \;=\; \bullet \circlearrowright \qquad \text{(5-21)}$$

Thus h sends the vertices of Q to the vertex of Q', and sends the arrows of Q to the single arrow of Q'.

The path algebra kQ' is isomorphic to a polynomial ring $k[u]$, and kQ is isomorphic to the subalgebra of $M_n(k[x])$ in (4-13). The ring homomorphism $kQ' \to kQ$ can be viewed as the inclusion $k[u] \to kQ$ that sends u to the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & x \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

This inclusion induces a map of spaces $\mathsf{Mod}kQ \to \mathbb{A}^1$.

For $0 \neq \lambda \in k$ let S_λ denote the simple kQ-module annihilated by $x - \lambda$. Then $\dim S_\lambda = n$. The action of u on $v = (v_1, \dots, v_n) \in S_\lambda$ is given by

$$(v_1, \dots, v_n).u = (v_2, \dots, v_n, \lambda v_1).$$

It follows that $h_*S_\lambda \cong k[u]/(u^n - \lambda)$. If k has n distinct n^{th} roots of unity, then h_*S_λ is a direct sum of n distinct simple $k[u]$-modules; thus h breaks the closed point corresponding to S_λ into n distinct closed points on the affine line parametrized by u. If M is a one-dimensional simple S-module, then $h_*M \cong k[u]/(u)$. On the other hand h^* sends $k[u]/(u - \lambda)$ to S_λ, and sends $k[u]/(u)$ to S/Su which is isomorphic to the direct sum of the n one-dimensional simples; to see this last isomorphism observe that the row space k^n becomes a cyclic module, with generator $(1, 1, \dots, 1)$, under the natural action of $S \subset M_n(k[x])$ with x acting as zero.

Let $g : \mathbb{A}^1 \to \mathbb{A}^1$ be the map $\alpha \mapsto \alpha^n$, and let $f : Q \to \mathbb{A}^1$ be the composition $f = g \circ h$. The image to have in mind is that $f : Q \to \mathbb{A}^1$ is a covering of the affine line by a non-commutative curve. A picture of this curve appears at (**??**) in Example **??**.

Since g is induced by the inclusions $k[u^n] \to k[u]$, and since $u^n = x$ (see Proposition 4.1), the map f is induced by the inclusion of $k[x]$ as the center of kQ. The map f sends closed points of kQ to closed points of $\mathbb{A}^1$. The fiber over 0 consists of the n points of Q corresponding to the one-dimensional simples, and the fiber over any other closed point of $\mathbb{A}^1$ consists of a single point, one corresponding to an n-dimensional simple kQ-module. $\Diamond$