Chapter 5

Artinian rings

The importance of the descending chain condition, or artinian condition as it is now called, was brought to the fore by Emil Artin in his 1927 paper *Zur Theorie der hyperkomplexen Zahlen* in which he extended Wedderburn's 1908 result (in *Hypercomplex Numbers*) to prove what we now call the Artin-Wedderburn Theorem: a semisimple artinian ring is isomorphic to a direct sum of matrix algebras over division rings. Wedderburn had worked with a finite dimensional algebra over a field, and the descending chain condition was introduced as a substitute for this.

In 1888 Hilbert proved his famous Basis Theorem: the ring of invariants $k[x_1, \ldots, x_n]^G$ for the action of a finite group on a polynomial ring is finitely generated. A key step is his result that every ideal in $k[x_1, \ldots, x_n]$ is finitely generated; equivalently every ascending chain of ideals in $k[x_1, \ldots, x_n]$ is eventually stationary. The abstract approach Hilbert took had such an impact that it virtually killed off the constructive approach to invariant theory that had taken center stage prior to his 1888 paper. The leading proponent of the pre-Hilbert approach to invariant theory was Paul Gordan. In 1907 Emmy Noether received her Ph.D. under Gordan at Erlangen. Under his influence, she gave a list of 331 covariants in her thesis!

Noether was greatly influenced by Hilbert's approach, and in 1915 she joined Hilbert and Klein in Göttingen. In 1921 she published her fundamental paper *Idealtheorie in Ringbereichen* which takes a whole-heartedly abstract approach. Emil Artin received his Ph.D. in 1921 from Leipzig, and then spent the year 1922-23 at Göttingen. In 1924 van der Waerden spent a year at Göttingen where he took a course from Emmy Noether on Hypercomplex Numbers. Noether gave a course on the same subject in 1926/27. Much of the material in that course appeared in her 1929 paper *Hyperkomplexe Größe und Darstellungstheorie*. In 1930 the first volume of van der Waerden's book *Moderne Algebra* appeared. Van der Waerden's book, which had been influenced by the work and lectures of Noether and Artin, codified the abstract approach to algebra. Seventy-five years later, this book is still a great place to learn modern algebra. One can see the radical change in perspective by comparing it to Weber's *Lehrbuch der Algebra* published in 1898. Despite the success of the abstract approach to

algebra, it is a chastening experience to see in Weber's book the rich earth that gave birth to modern algebra, and to realize that almost all this wonderful material has been expunged from today's graduate algebra courses.

5.1 The Jacobson radical

Definition 1.1 The Jacobson radical of a ring, denoted $J(R)$, consists of the elements that annihilate all simple right R-modules.

Lemma 1.2 *$J(R)$ is a two-sided ideal.*

Proof. If x and y annihilate all simple right R-modules so does $x \pm y$. If x annihilates all simple right R-modules, so does every element of RxR. Hence the result. □

Theorem 1.5 will show that $J(R)$ is also the set of elements that annihilate all simple *left* R-modules.

Lemma 1.3 *$J(R)$ is the intersection of all the maximal right ideals of R.*

Proof. Let I denote the intersection of all the maximal right ideals. Let M be a simple right R-module. If m is a non-zero element of M, then $\mathrm{Ann}(m)$ is a maximal right ideal, so contains I. Thus $mI = 0$. Because m and M are arbitrary it follows that I annihilates all simple modules, so is contained in $J(R)$.

Conversely, if L is a maximal right ideal, then R/L is simple, so is annihilated by $J(R)$. Hence $J(R) \subset L$, and it follows that $J(R) \subset I$. Hence $J(R) = I$. □

Proposition 1.4 *The following conditions on an element x in a ring R are equivalent:*

1. *$x \in J(R)$;*
2. *$1 - axb$ is a unit for all $a, b \in R$.*

Proof. (2) $\Rightarrow$ (1) Let x be an element satisfying condition (2). Let M be a simple right R-module. If $Mx \neq 0$, then $mx \neq 0$ for some $m \in M$, whence $mxR = M$. It follows that $m = mxb$ for some $b \in R$. Thus $m(1 - xb) = 0$. But $1 - xb$ is a unit, so $m = 0$. This is a contradiction, so we conclude that $Mx = 0$, whence $x \in J(R)$.

(1) $\Rightarrow$ (2) Let $x \in J(R)$. Then $axb \in J(R)$ also. If $(1 - axb)R \neq R$, then $(1 - axb)R$ is contained in a maximal right ideal; by Lemma 1.3, axb is in that right ideal, from which it follows that 1 is also in it. This is absurd, so we conclude that $(1 - axb)R = R$. Therefore $(1 - axb)u = 1$ for some $u \in R$. Thus $1 - axb$ has a right inverse. Now $u = 1 + axbu$, and $axbu \in J(R)$, so $1 + axbu$ has a right inverse, say v. Since u has both a left and a right inverse, those inverses are equal. It follows that u is a unit, and so is $u^{-1} = 1 - axb$. □

Condition (2) in Proposition 1.4 is left/right symmetric, so we immediately deduce that $J(R)$ can equally well be defined in terms of simple *left* R-modules; that is,

$$J(R) := \{x \in R \mid xM = 0 \text{ for all simple left modules } M\}.$$

Theorem 1.5 *The Jacobson radical is the set of elements that annihilate all simple left R-modules.*

Proposition 1.6 *The following conditions on an element x in a ring R are equivalent:*

1. $x \in J(R)$;
2. $1 - xb$ *is a unit for all* $b \in R$;
3. $1 - ax$ *is a unit for all* $a \in R$.

Proof. (2) $\Rightarrow$ (1) This was already proved in the proof of Proposition 1.4.

(3) $\Rightarrow$ (1) Using the fact that the left and right Jacobson radicals are the same, the argument showing that (2) $\Rightarrow$ (1) can be mimicked to prove this.

Both (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are special cases of the implication (1) $\Rightarrow$ (2) in Proposition 1.4. □

5.2 Artinian rings are Noetherian

It is not trivial to prove that a left artinian ring is left noetherian. Neither Emil Artin nor Emmy Noether appear to have known this when they were doing their seminal work on rings with chain conditions in the 1920s.

Kaplansky, in his book *Fields and Rings* (1969), says this about the theory of artinian rings: "After Artin introduced rings with descending chain condition there was a peaceful decade of development. The subject was thought of as distinct from (and in some sense dual to) the earlier theory of rings with ascending chain condition. It came as quite a surprise when Charles Hopkins proved [...] that the descending chain condition implies the ascending chain condition." This result of Hopkins, which we prove in Theorem 2.3 below, appeared in 1939 in volume 40 of the *Annals of Mathematics*.

In an artinian ring, the Jacobson radical has the following nice characterization. Recall that a semisimple ring may be defined as one over which every module is a sum, and hence a direct sum, of simple modules.

Lemma 2.1 *If R is right artinian, then the Jacobson radical is the smallest two-sided ideal I such that R/I is semisimple as a right R-module.*

Proof. Let I be such an ideal. Since R/I is a direct sum of simple right modules, it is annihilated by $J(R)$, whence $J(R) \subset I$.

By Lemma 1.3, $J(R)$ is an intersection of maximal right ideals. Since R has the descending chain condition on right ideals, this is a finite intersection. But, an induction argument shows that if $L_1, \ldots, L_n$ are distinct maximal right ideals, then

$$R/L_1 \cap \ldots \cap L_n \cong R/L_1 \oplus \ldots R/L_n.$$

Therefore $R/J(R)$ is semisimple. Hence $J(R)$ is the smallest ideal such that the quotient is semisimple as a right R-module. □

Proposition 2.2 *If R is right artinian, then $J(R)^n = 0$ for large n.*

Proof. Write $J = J(R)$. Because R is right artinian, the chain $J \supset J^2 \supset \ldots$ stabilizes. Suppose that $J^n = J^{n+1} = \ldots$. Suppose that $J^n \neq 0$. Choose a right ideal I that is minimal subject to the condition that $IJ^n \neq 0$. Choose $a \in I$ such that $aJ^n \neq 0$. Since $aJ^nJ^n = aJ^{2n} = aJ^n \neq 0$, the minimality of I implies that $aJ^n = I$. Therefore there is some $x \in J^n$ such that $ax = a$. Hence $a(1-x) = 0$. But $1 - x$ is a unit, so $a = 0$. From this contradiction we deduce that $J^n = 0$. □

Theorem 2.3 (Hopkins 1939) *If R is right artinian, then it is right noetherian.*

Proof. Write $J = J(R)$. The Artin-Wedderburn theorem implies that R/J has finite length because it is semisimple. Since all the factors J^i/J^{i+1} are right R/J-modules, they are semisimple. But they are also right artinian, so they have finite length as right R-modules. It therefore follows from consideration of the series $R \supset J \supset J^2 \supset \ldots \supset J^n = 0$ that R must also have finite length as a right R-module. This implies R is right noetherian. □

Example 2.4 The ring

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

is right artinian, but not left noetherian. ◊

Our proof of Hopkin's Theorem differs slightly from his original proof in that we use the notion of the Jacobson radical whereas he used the notion of the nilpotent radical. Jacobson defined his radical in 1945. Nathan Jacobson received his Ph.D. in 1934 from Princeton. He heard Noether lecture at the Institute of Advanced Studies at Princeton in 1934-35, but by the time he joined Bryn Mawr in 1935, where Emmy Noether had been hired in 1933, Noether had died at the age of 53.

Definition 2.5 A right ideal is nil if every element in it is nilpotent.

Lemma 2.6 *The Jacobson radical contains every nil right ideal.*

Proof. Let I be a nil right ideal, and M a simple right R-module. If $MI \neq 0$, then $mI \neq 0$ for some $m \in M$. But mI is a right module, so $mI = M$. Therefore $m = mx$ for some $x \in I$. It follows that $m = mx^n$ for all n. But x is nilpotent, so $m = 0$. This is absurd, so we conclude that $MI = 0$. Since I annihilates every simple right R-module it is contained in the Jacobson radical. □

Theorem 2.7 *In a right artinian ring the Jacobson radical is the largest nilpotent ideal.*

Proof. The proof of Proposition 2.2 showed that the Jacobson radical is nilpotent. Combined with Lemma 2.6 this gives the result. □

Although the Jacobson radical and the nilpotent radical coincide in a right artinian ring, the definitions of these two radicals are quite different. The definition of the Jacobson radical in Definition 1.1 emphasizes the module-theoretic approach, whereas the definition of the nilpotent radical places the ring at center stage.

As we mentioned, Hopkins used the language of the nilpotent radical which was defined as the sum of all the nil right ideals. A key step in Hopkin's proof that right artinian implies right noetherian was the result that in a right artinian ring every nil right ideal is nilpotent. This follows from Lemma 2.6 and the fact that the Jacobson radical is nilpotent, but the argument we used to prove the latter is essentially the argument Hopkin's used. Try it yourself, that is, replace $J(R)$ by a nil right ideal in the proof of Proposition 2.2, and show that every nil right ideal is nilpotent.

Proposition 2.8 *Let J be the Jacobson radical of a left artinian ring R. Then*

1. *$J = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the prime (=maximal) ideals of R;*
2. *R/J is semisimple;*
3. *$J^n = 0$ for $n \gg 0$.*

Proof. (1) By Theorem 3.1.8, the prime ideals of a left artinian ring are its maximal ideals and these are the annihilators of the simple R-modules.

(2) Because $\mathfrak{p}_i + \mathfrak{p}_j = R$ whenever $i \neq j$, we have

$$R/J = R/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n \cong R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n.$$

But each $R/\mathfrak{p}_i$ is a simple artinian ring, so isomorphic to $M_{n_i}(D_i)$ for some integer n_i and division ring D_i. Thus R/J is semisimple.

(3) Because R is left artinian, there is a finite chain of left ideals $R = I_0 \supset I_1 \supset \cdots \supset I_d = 0$ such that each I_j/I_{j+1} is a simple module. Because J annihilates every simple module, $JI_j \subset I_{j+1}$. It follows that $J^d = J^d R = 0$. □

EXERCISES

2.1 Let M be the $\mathbb{Z}$-module $\{ap^{-n} \mid a, n \in \mathbb{Z}\}$. Show that the $\mathbb{Z}$-module M is not finitely generated, but every proper submodule of it is cyclic.

5.3 Fitting's Lemma

If ϕ is an endomorphism of a finite dimensional vector space V, there is a basis for V with respect to which the matrix representing V is upper triangular. This may be rephrased as saying that there are ϕ-stable subspaces U and W such that $V = U \oplus W$ and $\phi|_U$ is an automorphism and $\phi|_W$ is nilpotent. Fitting's Lemma (Proposition 3.2) extends this result to finite length modules.

Lemma 3.1 *Let ϕ be an endomorphism of an R-module M. Then ϕ is an automorphism if either*

1. *M is artinian and ϕ is injective;*
2. *M is noetherian and ϕ is surjective.*

Proof. (1) By hypothesis, the map $M \to \phi(M)$ is injective. It is also surjective, so an isomorphism. An artinian module cannot be isomorphic to a proper submodule of itself (because that would lead to a proper infinite chain of descending submodules), so we conclude that $\phi(M) = M$. Hence the result.

(2) Somewhat similar. □

Proposition 3.2 *[Fitting's Lemma, date?] Let M be a finite length R-module and $\phi \in \operatorname{End}_R M$. There is a decomposition $M = L \oplus N$ such that*

1. *$\phi(L) \subset L$ and $\phi(N) \subset N$, and*
2. *$\phi|_L$ is an automorphism, and*
3. *$\phi|_N$ is nilpotent.*

Proof. Since M is both artinian and noetherian there is an integer d such that, for all integers $n \geq d$, $\phi^n(M) = \phi^d(M)$ and $\ker \phi^n = \ker \phi^d$. Set $L = \phi^d(M)$ and $N = \ker \phi^d$.

Then $\phi(L) = \phi^{d+1}(M) = L$ so, by Lemma 3.1, $\phi|_L$ is an automorphism.

Now $\phi(N) = \phi(\ker \phi^{d+1}) \subset \ker \phi^d = N$ so $\phi^{d+1}(N) = 0$, so $\phi|_N$ is nilpotent.

On $L \cap N$, ϕ is both injective and nilpotent, so $L \cap N = 0$. Finally,

$$\begin{aligned} M &= (\phi^d)^{-1}(\phi^d(M)) = (\phi^d)^{-1}(\phi^{2d}(M)) = (\phi^d)^{-1}(\phi^d(\phi^d(M))) \\ &= \phi^d(M) + \ker \phi^d = L + N \end{aligned}$$

so $M = L \oplus N$ as claimed. □

5.4 Indecomposable modules

The structure of modules over a semisimple ring is elementary: there are a finite number of simple modules up to isomorphism, every module is isomorphic to a direct sum of those simple modules, and modules M and N are isomorphic if and only if each simple module occurs with the same multiplicity in M and N. This result is a model that we might try to emulate for other rings.

For example, we might ask if there is a small (finite?) list of special modules such that every module is isomorphic to a direct sum of copies of those on the list, and that two modules are isomorphic if and only if the ways in which they are expressed as direct sums of the special modules are the same. This is too optimistic, but parts of this are possible, and for some rings and modules this desirable result can be achieved.

A module is indecomposable if it is not a direct sum of non-zero submodules.

Our first result is that indecomposable modules are the basic building blocks of (almost) all modules.

Proposition 4.1 *Let M be a module that is either noetherian or artinian. Then M can be written as a finite direct sum of indecomposable modules.*

Proof. The zero module is indecomposable, so suppose that $M \neq 0$. If M is artinian, then it has a submodule N that is minimal subject to the condition that N is a direct summand of M. If M is noetherian, it has a submodule L that is *maximal* subject to this condition, and if N is then chosen so that $M = L \oplus N$, then N is minimal subject to this condition.

It is clear that N is indecomposable. We may write $M = N \oplus M_1$. Since M_1 is a quotient of M it inherits the chain condition from M, so we may apply the same argument with M_1 in place of M to get $M_1 = N_1 \oplus M_2$ with N_2 indecomposable. Continuing in this way the chain condition ensures that the process comes to a halt in a finite number of steps, thus giving the result. □

We now address the question of uniqueness: if a module can be written as a direct sum of indecomposables in two ways are the two decompositions essentially the same?

First we need a way of recognizing that a module is indecomposable. For a finite length module Theorem 4.3 provides such a criterion.

A ring R is local if $R/J(R)$ is a division ring.

Notice this is not the same as saying that R has a unique maximal two-sided ideal.

Lemma 4.2 *Let U be the group of units in a ring R. The following are equivalent:*

1. *R is local;*
2. *$R \backslash U \subset J(R)$;*
3. *$R \backslash U$ is closed under $+$.*

Proof. (1) $\Rightarrow$ (2) The condition in (2) is equivalent to $R\backslash J(R) \subset U$, and this is what we will prove. If $x \notin J(R)$, then (1) implies there is an element $y \in R$ such that $1 - xy \in J(R)$ and $1 - yx \in J(R)$. By Proposition 1.4, $xy = 1 - (1 - xy)$ is a unit and $yx = 1 - (1 - yx)$ is also a unit. Thus x has both a right and left inverse, so is a unit.

(2) $\Rightarrow$ (3) Since $J(R)$ is not equal to R, $U \cap J(R) = \phi$, so condition (2) is equivalent to $R\backslash U = J(R)$; thus (3) is a consequence of the fact that $J(R)$ is closed under $+$.

(3) $\Rightarrow$ (1) Suppose $x \notin J(R)$; we must show that the image of x in $R/J(R)$ is a unit. By Proposition 1.6, there are elements $a, b \in R$ such that $1 - ax$ and $1 - xb$ are not units. Thus ax is a unit, else the hypothesis (3) would imply that $1 = ax + (1 - ax)$ is a non-unit. Similarly, xb is a unit. Thus x has both a left and right inverse, and so is a unit in R. It follows that the image of x in $R/J(R)$ is a unit too. □

Theorem 4.3 *Let M be a finite length R-module. Then M is indecomposable if and only if* $\mathrm{End}_R M$ *is local.*

Proof. ($\Rightarrow$) Let $\phi : M \to M$. Because M is indecomposable, Fitting's Lemma implies that ϕ is either nilpotent or and automorphism. Hence the non-units in $\mathrm{End}_R M$ consist of the nilpotent endomorphisms; by Lemma 4.2 it therefore suffices to show that a sum of two nilpotent endomorphisms is nilpotent.

Let ϕ and ψ be nilpotent endomorphisms and suppose to the contrary that their sum, say η, is a unit. Replacing ϕ and ψ by $\phi\eta^{-1}$ and $\psi\eta^{-1}$ we may in fact assume that $\psi + \phi = \mathrm{id}_M$. Thus $1 - \phi$ is not a unit; but this is absurd because if $\phi^n = 0$, $\mathrm{id}_M = (1 - \phi)(1 + \phi + \cdots + \phi^{n-1})$. We conclude that $\phi + \psi$ is not a unit, and hence nilpotent, as required.

($\Leftarrow$) Suppose that $M = L \oplus N$. Let $\lambda : M \to L$ and $\eta : M \to N$ be the projections. Then $\lambda + \eta = \mathrm{id}_M$. Since $\mathrm{End}_R M$ is local, either λ or η is a unit by part (3) of Lemma 4.2. But $\lambda^2 = \lambda$ and $\eta^2 = \eta$, so whichever of these is a unit must, in fact, be the identity. Hence either L or N equals M. □

Our proof of the second implication did not use the finite length hypothesis so we conclude that if the endomorphism ring is local the module is indecomposable.

Theorem 4.4 (Krull-Remak-Schmidt) *Let A be a left artinian ring and M a finitely generated left A-module. Then M is a direct sum of indecomposable modules in an essentially unique way.*

Proof. After Proposition 4.1, M is a finite direct sum of indecomposables so it remains to prove uniqueness. This is pretty tedious! Look in, say, Pierce's book or Lam's. □

This result is sometimes called the Krull-Schmidt Theorem, but both Remak and Azumaya also contributed to it.

5.5 Projectives

Lemma 5.1 *Write J for the Jacobson radical. Let P and Q be right R-modules. If P is projective, then the natural map*

$$\Phi : \mathrm{Hom}_R(P, Q) \to \mathrm{Hom}_R(P/PJ, Q/QJ)$$

is surjective.

Proof. Notice first that Φ exists because every R-module homomorphism $P \to Q$ sends PJ to QJ. Now let $\alpha : P/PJ \to Q/QJ$ be a module homomorphism. Because P is projective, there is a map $\beta : P \to Q$ making the following diagram commute:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ & & P/PJ & & \\ & & \downarrow{\scriptstyle \alpha} & & \\ Q & \longrightarrow & Q/QJ & \longrightarrow & 0 \end{array}$$

In other words, $\Phi(\beta) = \alpha$. □

Lemma 5.2 *Let R be a right artinian ring, and J its Jacobson radical. Let P be a projective right R-module, $E = \mathrm{Hom}_R(P, P)$ its endomorphism ring, and $J(E)$ the Jacobson radical of E. Then the natural map*

$$\Psi : E/J(E) \to \mathrm{End}_R\, P/PJ$$

is an isomorphism of rings.

Proof. By Lemma 5.1, the natural map $\Phi : E \to \mathrm{End}_R\, P/PJ$ is surjective. It is routine to check that Φ is a ring homomorphism, so it remains to check that $\ker \Phi = J(E)$.

First we show that $J(E) \subset \ker \Phi$. Because Φ is surjective, $\Phi(J(E))$ is contained in the Jacobson radical of $\mathrm{End}_R\, P/PJ$; it therefore suffices to show that the Jacobson radical of $\mathrm{End}_R\, P/PJ$ is zero; since P/PJ is semisimple, this is a consequence of the following claim.

Claim: If M is a semisimple module, the Jacobson radical of $\mathrm{End}\, M$ is zero. Proof: Let $\alpha : M \to M$ be a non-zero homomorphism. There is a simple submodule $S \subset M$ such that $\alpha(S) \neq 0$. There is also a homomorphism $\rho : M \to M$ such that $\rho(M) = \alpha(S)$ and $\rho^2 = \rho$. It is not difficult to see that there is a map $\beta : M \to M$ such that $\rho = \alpha\beta$; roughly speaking β is the inverse of the isomorphism $\alpha|_S : S \to \alpha(S)$. It follows that $\rho(\mathrm{id}_M - \alpha\beta) = 0$, so $1 - \alpha\beta$ is not a unit in $\mathrm{End}\, M$; hence, by Proposition 1.4, α is not in the Jacobson radical of $\mathrm{End}\, M$. This proves the claim.

Conversely, suppose that $\alpha : P \to P$ is sent to zero by Φ. Then $\alpha(P) \subset PJ$, and induction gives $\alpha^n(P) \subset PJ^n$ which is zero for $n \gg 0$ by Proposition 2.2. Hence $\ker \Phi$ is a nil ideal, so is contained in $J(E)$ by Lemma 2.6.

Hence $\ker \Phi = J(E)$, as required. □

Let R be a right artinian ring, and J its Jacobson radical. Then R/RJ is semisimple and it is easy to decide if two R/J-modules are isomorphic: M and N are isomorphic if and only if each simple occurs in M with the same multiplicity as it occurs in N. The next result reduces the question of isomorphism between projective R-modules to that of isomorphism between two R/J-modules.

Proposition 5.3 *Let R be a right artinian ring, and J its Jacobson radical. Two projective right R-modules P and Q are isomorphic if and only if P/PJ and Q/QJ are isomorphic.*

Proof. ($\Rightarrow$) If P and Q are isomorphic, then certainly $P/PJ \cong Q/QJ$.

($\Leftarrow$) Let $\alpha : P/PJ \to Q/QJ$ be an isomorphism. By Lemma 5.1, α is induced by a homomorphism $\tilde{\alpha} : P \to Q$. Likewise, $\beta = \alpha^{-1}$ is induced by a homomorphism $\tilde{\beta} : Q \to P$. Because $\beta\alpha = \mathrm{id}_{P/PJ}$, $\tilde{\beta}\tilde{\alpha} - \mathrm{id}_P$ is sent to zero under the natural map $\mathrm{End}_R P \to \mathrm{End}_R(P/PJ)$; hence by Lemma 5.2, $\tilde{\beta}\tilde{\alpha} - \mathrm{id}_P$ is in the Jacobson radical of $\mathrm{End}_R P$. By Proposition 1.4, $\tilde{\beta}\tilde{\alpha}$ is therefore a unit in $\mathrm{End}_R P$. Similarly, $\tilde{\alpha}\tilde{\beta}$ is therefore a unit in $\mathrm{End}_R Q$.

The first of these conclusions implies that α has a left inverse, and the second that α has a right inverse. Thus $\tilde{\alpha} : P \to Q$ is an isomorphism. □

Lemma 5.4 *Let R be a right artinian ring, and J its Jacobson radical. Let P be a direct summand of R as a right module. Then P is indecomposable if and only if P/PJ is simple.*

Proof. By hypothesis, P is projective and has finite length. Write $E = \mathrm{End}_R P$, and let $J(E)$ be its Jacobson radical. Then P is indecomposable if and only if $\mathrm{End}_R P$ is a local ring, if and only if $E/J(E)$ is a division ring. But $E/J(E) \cong \mathrm{End}_R P/PJ$ by Lemma 5.2; because the endomorphism ring of a semisimple module is a division ring if and only if that module is simple, P is indecomposable if and only if P/PJ is simple. □

An element e in a ring R is idempotent if $e^2 = e$. A non-zero idempotent is primitive if it cannot be written as a sum of two non-zero idempotents. A set $\{e_1, \ldots, e_n\}$ of idempotents is orthogonal if $e_i e_j = 0$ when $i \neq j$.

Lemma 5.5 *Let N be a nilpotent ideal in a ring R and $\bar{e} \in \bar{R} = R/N$ an idempotent. Then*

1. *there is an idempotent $e \in R$ such that $\bar{e} = e + N$, and*
2. *if $e' \in R$ is another idempotent such that $\bar{e} = e' + N$, then there is a unit $1 + x \in R$ such that $e' = (1 + x)e(1 + x)^{-1}$.*

Proof. (1) We define inductively idempotents $e_i \in R/N^i$ as follows. First $e_1 = e$. Suppose that $i > 1$ and $e_{i-1} \in R/N^{i-1}$ is known. Let $f \in R/N^i$ be any element whose image in R/N^{i-1} is e_{i-1}. Then the image of $f^2 - f$ in R/N^{i-1} is $e_{i-1}^2 - e_{i-1}$, which is zero by the inductive hypothesis. Hence $f, f^2, f^3, \ldots$ all have the same image in R/N^{i-1}, namely e_{i-1}. So $e_i := 3f^2 - 2f^3$ has image e_{i-1} in R/N^{i-1}. Also

$$\begin{aligned} e_i^2 - e_i &= (3f^2 - 2f^3)(3f^2 - 2f^3 - 1) \\ &= -(3 - 2f)(1 + 2f)(f^2 - f)^2 \\ &= 0. \end{aligned}$$

The final equality is because $f^2 - f \in N^{i-1}/N^i$.

If $N^d = 0$, we take $e = e_d$.

(2) Left to the reader! □

Theorem 5.6 *Let R be a right artinian ring. There is a bijection*

$$\{\textit{isoclasses of indecomposable projectives}\} \longleftrightarrow \{\textit{isoclasses of simples}\}.$$

Proof. By Lemma 5.4, the rule $P \mapsto P/PJ$ gives a map from the set of isomorphism classes (isoclasses, for short) of indecomposable projectives to the set of isoclasses of simple right A-modules. This map is injective by Proposition 5.3. To show it is surjective we must show that every simple arises as P/PJ for some projective P; P is necessarily indecomposable by Lemma 5.4.

Let S be a simple right A-module. Then J annihilates S, so S is a module over $\bar{A} = A/J$, and there is an idempotent $\bar{e} \in \bar{A}$ such that $S \cong \bar{e}\bar{A}$. By Lemma 5.5, there is an idempotent $e \in A$ whose image in $\bar{A}$ is $\bar{e}$. Now $P := eA$ is a direct summand of A, so projective. Since $P/PJ = eA/eJ$ is annihilated by J it is a right $\bar{A}$-module, and is generated by the image of e, so is equal to $\bar{e}\bar{A}$. That is, $P/PJ \cong S$. □

Proposition 5.7 *Let A be a right artinian ring and J its Jacobson radical. Then*

1. *an element $e \in A$ is a primitive idempotent if and only if eA is an indecomposable projective;*
2. *if P is an indecomposable projective right A-module, then $P \cong eA$ for some primitive idempotent e;*
3. *an indecomposable projective eA has a unique maximal submodule, namely eJ;*
4. *if e and f are primitive idempotents, then $eA \cong fA$ if and only if $eA/eJ \cong fA/fJ$ if and only if $f = ueu^{-1}$ for some unit $u \in A$.*

Proof. Most of this is a restatement of previous results.

(1) Let $e \in A$. If e is an idempotent, eA is projective. Conversely, if eA is projective, then the map $A \to eA$, $a \mapsto ea$, splits, from which it follows that e is idempotent. Thus e is idempotent if and only if eA is projective.

Now suppose that e is an idempotent. If e is not primitive, then eA is certainly decomposable. Conversely, suppose that eA is decomposable, say $eA = P \oplus Q$ with P and Q non-zero. Let $\pi : eA \to P$ be the projection, and set $f := \pi(e)$. Clearly, $P = \pi(eA) = \pi(e)A = fA$. We have $f = ef$ since $f \in eA$; also $f = \pi(e) = \pi(e^2) = \pi(e)e = fe$. Since π is the projection, $f = \pi(f) = \pi(ef) = \pi(e)f = f^2$. The obvious calculation shows that $e - f$ is idempotent too. It follows that $e = f + (e - f)$ is not primitive.

(2) Let P be an indecomposable projective. Then P/PJ is simple; but the proof of Theorem 5.6 shows that the indecomposable projective corresponding to that simple is of the form eA. It follows that $P \cong eA$, and by (1) this e is a primitive idempotent.

(3) Let $P = eA$ be an indecomposable projective. We have already seen that P/PJ is simple, so PJ is a maximal submodule of P. If N is a maximal submodule of P, then P/N is simple, so annihilated by J, whence $N \subset PJ$.

(4) This follows from part (2) of Lemma 5.5. □

The indecomposable projective A-modules may be obtained as follows. Since A/J is semisimple artinian, we can write

$$1 = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n$$

as a sum of primitive orthogonal idempotents. There are primitive orthogonal idempotents $e_1, \ldots, e_n \in A$ such that $\bar{e}_i = e_i + J$ and $1 = e_1 + \cdots + e_n$. Then $\{e_iA \mid 1 \leq i \leq n\}$ is a complete list of the indecomposable projective A-modules up to isomorphism. A particular indecomposable projective may appear on this list several times, the number of times being determined as follows. Write

$$A/J = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$$

as a direct sum of matrix algebras over division algebras. Let $\bar{e}^i_{11}, \ldots, \bar{e}^i_{n_i n_i}$ be a complete set of primitive orthogonal idempotents in $M_{n_i}(D_i)$; let $e^i_{jj} \in A$ be a primitive idempotent such that $\bar{e}^i_{jj} = e^i_{jj} + J(A)$; then $Ae^i_{11} \cong Ae^i_{22} \cong \cdots \cong Ae^i_{n_i n_i}$ and these are the only isomorphisms among the indecomposable projectives Ae^i_{jj} over all i and j.

When A/J is a product of division rings all the n_is are equal to 1 and $\{e_iA \mid 1 \leq i \leq n\}$ is a complete list of the isoclasses of indecomposable projectives. When A/J is a product of division rings we say it is a basic ring.

For any right artinian A, there is a right artinian ring A' such that $\mathsf{Mod}A$ and $\mathsf{Mod}A'$ are equivalent categories and A' has the property in the previous paragraph. The way to obtain A' is as follows. With the above notation, set

$$e := \bar{e}^1_{11} + \bar{e}^2_{11} + \cdots + \bar{e}^n_{11}$$

and $A' = eAe$. Then $AeA = A$ and it follows from this (and the idempotence of e) that $\mathsf{Mod}A$ and $\mathsf{Mod}A'$ are equivalent. The equivalence $\mathsf{Mod}A \to \mathsf{Mod}A'$ is given by $M \mapsto Me$.

The Cartan matrix We continue to assume that A is a right artinian ring with Jacobson radical J. Let $S_1, \ldots, S_n$ be a complete set of isoclasses of the simple right A-modules (without repetitions!). Let P_i be the indecomposable projective such that $P_i/P_iJ \cong S_i$. Let

$$c_{ij} := \text{the multiplicity of } S_i \text{ as a composition factor of } P_j.$$

We call $C := (c_{ij})$ the Cartan matrix of A. This matrix is an important invariant of A. If e_j is an idempotent such that $e_jA \cong P_j$, then $c_{ij} \neq 0$ if and only if $(P_je_i =)e_jAe_i \neq 0$.

We can associate to A a directed graph with vertices the isoclasses of simple modules (equivalently, isoclasses of indecomposable projectives, or simply the indices $1, \ldots, n$). The number of arrows $S_i \to S_j$ is required to be the multiplicity of S_j as a component of the semisimple module P_iJ/P_iJ^2.