Chapter 1

Quick Homological Algebra

As its title suggests, this chapter provides an introduction to some aspects of homological algebra. The emphasis is on Ext groups between modules over a ring, and the relation of these to various geometric ideas.

1.1 Ext groups

Let M and N be left modules over a ring R.

For each integer $n \geq 0$ there is an abelian group

$$\operatorname{Ext}_R^n(M,N).$$

If k is a commutative ring and R a k-algebra then each $\operatorname{Ext}_R^n(M,N)$ is a k-module.

We have

$$\operatorname{Ext}_R^0(M,N) = \operatorname{Hom}_R(M,N).$$

It is not unreasonable to think of the elements of each $\operatorname{Ext}_R^n(M,N)$ as "generalized homomorphisms from M to N".

As support for this point of view, the natural map

$$\operatorname{Hom}_R(M,N) \times \operatorname{Hom}_R(L,M) \to \operatorname{Hom}_R(L,N)$$

given by composition, $(f,g) \mapsto f \circ g$, generalizes to a map

$$\operatorname{Ext}_R^i(M,N) \times \operatorname{Ext}_R^j(L,M) \to \operatorname{Ext}_R^{i+j}(L,N)$$

for all i and j. Thus, for example, each $\operatorname{Ext}_R^*(M,M) = \oplus_i \operatorname{Ext}_R^i(M,M)$ is a graded ring.

Further support for thinking of $\operatorname{Ext}_R^i(M,N)$ as consisting of "generalized homomorphisms from M to N" is that if $\operatorname{Ext}_R^i(M,N) = 0$ for all i, then "M and N have nothing to do with each other" and conversely.

For example, if R is commutative and the supports of M and N are disjoint, then $\operatorname{Ext}_R^i(M,N) = 0$ for all i; in particular, if M and N are non-isomorphic simple R-modules, then $\operatorname{Ext}_R^i(M,N) = 0$ for all i. A ring R that is not commutative typically has non-isomorphic simple modules M and N such that $\mathrm{Ext}^i_R(M,N) \neq 0$. A huge part of non-commutative algebra concerns just this question—for which simples are these Ext-groups non-zero, and what are the Ext-groups in that case. For example, the representation theory of a finite group G over a field k whose characteristic divides $|G|$ is really the study of such Ext-groups.

In some sense the essential difference between the commutative and non-commutative worlds is this behavior of Ext-groups. However, to formalize, and make more precise, my vague remark that elements of $\mathrm{Ext}^i_R(M,N)$ are like generalized homomorphisms one introduces the derived category of $\mathsf{Mod}R$, and it then turns out that there can be commutative and non-commutative rings and varieties having equivalent derived catgories. One should interpret this as saying that from the appropriate standpoint, sometimes the commutative and non-commutative worlds are simply different views of the same object. The connection between the representations of the path algebra of the Kronecker quiver and sheaves on $\mathbb{P}^1$ illustrates this.

As a final example illustrating these points, we note that if u and v are vertices in a quiver Q, and $S(v)$ and $S(u)$ the simple modules at those vertices, then $\dim_k \mathrm{Ext}^1_{kQ}(S(u), S(v))$ is equal to the number of arrows $u \to v$.

1.1.1 Long exact sequences

Associated to a module M is a left exact functor $\mathrm{Hom}_R(M,-)$. That is, if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence, there is an exact sequence

$$0 \to \mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2) \to \mathrm{Hom}_R(M,N_3).$$

The last of these maps is not surjective in general. The higher Ext-groups fit into an exact sequence extending this: there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M,N_1) &\to \mathrm{Hom}_R(M,N_2) \to \mathrm{Hom}_R(M,N_3) \to \\ &\mathrm{Ext}^1_R(M,N_1) \to \mathrm{Ext}^1_R(M,N_2) \to \mathrm{Ext}^1_R(M,N_3) \to \\ &\mathrm{Ext}^2_R(M,N_1) \to \mathrm{Ext}^2_R(M,N_2) \to \mathrm{Ext}^2_R(M,N_3) \to \\ &\mathrm{Ext}^3_R(M,N_1) \to \cdots \end{aligned}$$

Associated to a module N is a left exact functor $\mathrm{Hom}_R(-,N)$. That is, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, there is an exact sequence

$$0 \to \mathrm{Hom}_R(M_3,N) \to \mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N)$$

The last of these maps is not surjective in general. The higher Ext-groups fit into an exact sequence extending this: there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M_3,N) &\to \mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N) \\ &\mathrm{Ext}^1_R(M_3,N) \to \mathrm{Ext}^1_R(M_2,N) \to \mathrm{Ext}^1_R(M_1,N) \to \\ &\mathrm{Ext}^2_R(M_3,N) \to \mathrm{Ext}^2_R(M_2,N) \to \mathrm{Ext}^2_R(M_1,N) \to \\ &\mathrm{Ext}^3_R(M_3,M) \to \cdots \end{aligned}$$

1.2 How to compute Ext groups

We compute $\mathrm{Ext}^i_R(M,N)$ by taking a projective resolution of M, applying the functor $\mathrm{Hom}_R(-,N)$, then taking homology.

First, a projective R-module P is a module satisfying any one of the following three equivalent properties:

1. $\mathrm{Hom}_R(P,-)$ is an exact functor; that is, whenever $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact, so is $0 \to \mathrm{Hom}_R(P,N_1) \to \mathrm{Hom}_R(P,N_2) \to \mathrm{Hom}_R(P,N_3) \to 0$;
2. there is a module Q such that $P \oplus Q$ is a free R-module;
3. every short exact sequence of the form $0 \to L \to M \to P \to 0$ splits; i.e., there is a map $P \to M$ such that the composition $P \to M \to P$ is the identity id_P;
4. if $\beta : M \to N$ is surjective and $\alpha : P \to N$ is any map, there is a map $\gamma : P \to M$ such that $\alpha = \beta\gamma$.

Notice that a free module is projective. You should check that a direct sum of projective modules is again projective and that a diret summand of a projective module is again projective.

A projective resolution of a module M is an exact sequence of the form

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in which each P_i is projective.

To see that such a resolution exists, simply observe that every module is a quotient of a free module. This is sometimes expressed by saying that $\mathsf{Mod}R$ has enough projectives.

Given such a projective resolution, we may apply the functor $\mathrm{Hom}_R(-,N)$ to it (deleting the first term) to obtain a sequence

$$0 \to \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N) \to \mathrm{Hom}_R(P_2,N) \to \cdots .$$

This sequence is not generally exact, but it is a complex, meaning that the composition of any two adjacent maps is zero. We may therefore take homology, meaning *kernel modulo image* at each place, and these homology groups are the Ext groups: that is,

$$\mathrm{Ext}^i_R(M,N) = \frac{\ker\big(\mathrm{Hom}_R(P_i,N) \to \mathrm{Hom}_R(P_{i+1},N)\big)}{\mathrm{im}\big(\mathrm{Hom}_R(P_{i-1},N) \to \mathrm{Hom}_R(P_i,N)\big)}.$$

It is easy to check that this really does give $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$. The elements of $\mathrm{Ext}^i_R(M,N)$ are equivalence classes of certain homomorphisms $P_i \to N$; if one thinks of P_i as some kind of "approximation" to M, then the elements of $\mathrm{Ext}^i_R(M,N)$ are some kind of approximation to maps from M to N. This is vague! but I say it to emphasize again that elements of $\mathrm{Ext}^i_R(M,N)$

should be thought of as some kind of generalized homomorphisms from M to N.

A module M has many projective resolutions, and one must check that the computation of $\mathrm{Ext}^i_R(M,N)$ does not depend on the choice of projective resolution. We will not do this, but you can find it in dozens of books.

In fact, we have not *defined* the Ext groups; rather, we gave a method for computing them.

The projective dimension of M is the smallest integer n such that M has a projective resolution of length n, i.e., a projective resolution of the form

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

We write $\operatorname{pdim} M = n$ in this case. If $\operatorname{pdim} M = n$, then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > n$ and all n. The converse is also true. Thus

$$\operatorname{pdim} M = \max\{n \mid \mathrm{Ext}^n(M,N) \neq 0 \text{ for some } N\}.$$

Proposition 2.1 *The following conditions on a module M are equivalent:*

1. $\operatorname{pdim} M = n < \infty$;
2. *n is the smallest integer such that in every projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$ the kernel of the map $P_n \to P_{n-1}$ is projective.*
3. $\mathrm{Ext}^{n+1}_R(M,N) = 0$ *for all N, and* $\mathrm{Ext}^n_R(M,N) \neq 0$ *for some N;*
4. $\mathrm{Ext}^i_R(M,N) = 0$ *for all N and all $i > n$, and* $\mathrm{Ext}^n_R(M,N) \neq 0$ *for some N.*

In particular, P is projective if and only if $\mathrm{Ext}^1_R(P,N) = 0$ for all N, if and only if $\mathrm{Ext}^i_R(P,N) = 0$ for all N and all $i \geq 1$.

Definition 2.2 The (left) global homological dimension of R is

$$\operatorname{gldim} R := \max\{\operatorname{pdim} M \mid M \in \mathsf{Mod}R\}.$$

◊

1.2.1 Injectives

Although we have restricted attention to modules over rings, one can do homological algebra in many other abelian categories. Of particular importance is the category $\mathsf{Qcoh}X$ of quasi-coherent sheaves on a scheme X. There is a big difference though—our Ext-groups were “defined” in terms of projective resolutions, but if X is an irreducible projective variety of dimension ≥ 1, the only projective in $\mathsf{Qcoh}X$ is the zero sheaf!

Fortunately, one may also define the Ext-groups in terms of *injective* modules rather than projective modules and the answer is the same, and $\mathsf{Qcoh}X$ has enough injectives, meaning that every $\mathcal{G} \in \mathsf{Qcoh}X$ has an *injective* resolution, that being an exact sequence

$$0 \to \mathcal{G} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

in which every $\mathcal{I}_j$ is injective. One then defines

$$\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G}) = \frac{\ker\bigl(\mathrm{Hom}_R(\mathcal{F}, \mathcal{I}_i) \to \mathrm{Hom}_R(\mathcal{F}, \mathcal{I}_{i+1})\bigr)}{\mathrm{im}\bigl(\mathrm{Hom}_R(\mathcal{F}, \mathcal{I}_{i-1}) \to \mathrm{Hom}_R(\mathcal{F}, \mathcal{I}_i)\bigr)}.$$

Notice that in this case one takes an *injective* resolution of the *second* variable in $\mathrm{Ext}^i(-,-)$ rather than a *projective* resolution of the *first* variable.

As intimated, for modules over a ring $\mathrm{Ext}^i_R(M, N)$ can also be computed by taking an injective resolution of the second variable.

An object I in an abelian category C is injective if $\mathrm{Hom}_{\mathsf{C}}(-, I)$ is an exact functor, i.e., sends short exact sequences to short exact sequences.

1.2.2 Projectives and vector bundles

Projective modules are the algebraists vector bundle. This is more than an analogy. Let X be a compact Hausdorff space and $C(X)$ the ring of continuous $\mathbb{C}$-valued functions. Then the functor sending a complex vector bundle E on X to its sections $\Gamma(X, E)$, made into a $C(X)$-module in the obvious way, is an equivalence between the category of such vector bundles and the category of finitely generated projective $C(X)$-modules.

This immediately suggests that the machinery of topological K-theory can by transferred to modules over rings. This is indeed what one does and it is the first step leading into the important subject of algebraic K-theory.

1.3 Some results

Proposition 3.1 *A ring has global dimension zero if and only if it is semisimple artinian.*

Proof. ($\Rightarrow$) Suppose that $\mathrm{gldim}\, R = 0$. Let M be any module and L a submodule of it. Then the exact sequence $0 \to L \to M \to M/L \to 0$ splits, so N is a direct summand of M. But this characterizes semisimple rings—every submodule has a complement.

($\Leftarrow$) Suppose that R is a semisimple ring; i.e., every R-module is isomorphic to a direct sum of simples. Let M be a simple left R-module. Then $M \cong R/I$ for some maximal left ideal I, and hence $R \cong M \oplus I$. Thus M is projective, and we conclude that every left R-module is isomorphic to a direct sum of projectives, and is therefore projective. R-module □

In particular, if G is a finite group and k a field whose characteristic does not divide the order of G, the group algebra kG has global dimension zero.

Proposition 3.2 $\operatorname{gldim} R = \max\{\operatorname{pdim} R/I \mid I \textit{ is a left ideal}\}$.

Theorem 3.3 *Let R be a left noetherian ring. If $\operatorname{gldim} R < \infty$, then*

$$\operatorname{gldim} R = \max\{\operatorname{pdim} M \mid M \textit{ is simple}\}.$$

Proof. [?, Cor. 7.1.14]. □

Proposition 3.4 *If I is a two-sided ideal in a ring R, then*

$$\operatorname{Ext}^1_R(R/I, R/I) \cong \operatorname{Hom}_R(I/I^2, R/I).$$

Proof. If we apply $\operatorname{Hom}_R(-, R/I)$ to the exact sequence $0 \to I \to R \to R/I \to 0$, and use the fact that $\operatorname{Ext}^1_R(R, -) = 0$ because R is projective, the long exact sequence for Ext gives an exact sequence

$$\operatorname{Hom}_R(R, R/I) \xrightarrow{\ \phi\ } \operatorname{Hom}_R(I, R/I) \to \operatorname{Ext}^1_R(R/I, R/I) \to 0.$$

The map ϕ is zero because any homomorphism $R \to R/I$ vanishes on I. Hence $\operatorname{Ext}^1_R(R/I, R/I) \cong \operatorname{Hom}_R(I, R/I)$. But a homomorphism $I \to R/I$ vanishes on I^2, so the natural map $\operatorname{Hom}_R(I/I^2, R/I) \to \operatorname{Hom}_R(I, R/I)$ is an isomorphism. These two isomorphisms give the result. □

Consider Proposition 3.4. Since I/I^2 is naturally an R/I-module, we have

$$\operatorname{Ext}^1_R(R/I, R/I) \cong \operatorname{Hom}_{R/I}(I/I^2, R/I) = (I/I^2)^*,$$

the dual of I/I^2 as an R/I-module.

Relation to the tangent space. Now consider the case where $I = \mathfrak{m}$ is a maximal ideal of a commutative ring R and $R/\mathfrak{m} = \mathcal{O}_p$ is the simple module corresponding to a closed point $p \in \operatorname{Spec} R$; then

$$\operatorname{Ext}^1_R(\mathcal{O}_p, \mathcal{O}_p) \cong (\mathfrak{m}/\mathfrak{m}^2)^* \cong T_pX,$$

the tangent space to the scheme $X = \operatorname{Spec} R$ at p. Thus, a point p on an irreducible variety X is smooth if and only if $\dim \operatorname{Ext}^1_R(\mathcal{O}_p, \mathcal{O}_p) = \dim X$.

Now suppose that p is a point on an irreducible projective algebraic variety X over an algebraicaly closed field; then it is still true that

$$\operatorname{Ext}^1_R(\mathcal{O}_p, \mathcal{O}_p) \cong T_p^*X.$$

To put this in a broader context recall that if X is a smooth irreducible algebraic variety, and $Y \subset X$ a smooth subvariety cut out by the sheaf $\mathcal{I}$ of ideals in $\mathcal{O}_X$, then we define $\mathcal{I}/\mathcal{I}^2$ to be the conormal sheaf of Y in X, and $\mathcal{N}_{Y/X} := \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ to be the normal sheaf of Y in X.

1.3.1 Graded rings

A k-algebra A is connected graded if $A = A_0 \oplus A_1 \oplus \cdots$, $A_iA_j \subset A_{i+j}$ for all i and j, and $A_0 = k$. We say that elements in A_i are homogeneous of degree i.

The polynomial ring $k[x_1, \ldots, x_n]$ can be made into a connected graded k-algebra by setting $\deg x_i = 1$ for all i. The next result tells us that the global dimension of $k[x_1, \ldots, x_n]$ is the largest i such that $\mathrm{Ext}^i_R(k,k) \neq 0$, where k is the module $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$. We show in section 1.5 that this is n.

Theorem 3.5 *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded k-algebra such that $\dim_k A_0 < \infty$. Then*

$$\begin{aligned}\operatorname{gldim} A &= \max\{\operatorname{pdim}_A M \mid M \textit{ is a simple } A_0\textit{-module}\}.\\ &= \max\{n \mid \mathrm{Ext}^n_A(M,N) \neq 0 \textit{ for some simple } A_0\textit{-modules } M \textit{ and } N\}.\end{aligned}$$

In the previous theorem an A_0-module is made into an A-module via the ring homomorphism $A \to A_0$ sending $A_{\geq 1}$ to zero.

1.3.2 Related rings

Theorem 3.6 *Let $R[t]$ denote the polynomial extension of a ring R by a commuting (=central) indeterminate t. Then*

$$\operatorname{gldim} R[t] = \operatorname{gldim} R.$$

Proof. [?] □

Theorem 3.7 (Rees) *Let x be a regular non-unit in a ring R and suppose that $Rx = xR$. Let N be an R-module on which the map $n \mapsto xn$ is injective. Then*

$$\mathrm{Ext}^n_{R/(x)}(M, N/xN) \cong \mathrm{Ext}^{n+1}_R(M,N)$$

for all R-modules M.

Corollary 3.8 *Let x be a regular non-unit in a ring R and suppose that $Rx = xR$. If $\operatorname{gldim} R/(x) < \infty$, then*

$$\operatorname{gldim} R \geq 1 + \operatorname{gldim} R/(x).$$

Proposition 3.9 *If R_S is a localization of R, then*

$$\operatorname{gldim} R_S \leq \operatorname{gldim} R.$$

1.4 First examples

Example 4.1 Let $R = k[x]/(x^2)$. Let $M = R/(x)$. Then a projective resolution of M is given by

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow M \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, M)$ to this resolution gives a complex

$$0 \longrightarrow M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots$$

in which all the maps are zero, so taking homology gives

$$\operatorname{Ext}^i_R(M, M) \cong M$$

for all i. In particular, $\operatorname{pdim} M = \infty$ and $\operatorname{gldim} R = \infty$. ◊

Example 4.2 Let $R = k[x]/(x^n)$. Let $M = R/(x)$. Then a projective resolution of M is given by

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \longrightarrow M \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, M)$ to this resolution gives a complex

$$0 \longrightarrow M \xrightarrow{x} M \xrightarrow{x^{n-1}} M \xrightarrow{x} M \xrightarrow{x^{n-1}} \cdots$$

in which all the maps are zero, so taking homology gives

$$\operatorname{Ext}^i_R(M, M) \cong M$$

for all i. In particular, $\operatorname{pdim} M = \infty$ and $\operatorname{gldim} R = \infty$. ◊

Let k be a field of characteristic $p > 0$. If n is a positive multiple of p, then the group algebra $k\mathbb{Z}_n$ is isomorphic to $k[x]/(x^n)$, so $\operatorname{gldim} k\mathbb{Z}_n = \infty$. More generally, if G is any finite group whose order is divisible by p, then $\operatorname{gldim} kG = \infty$. Thus the representation theory of finite groups is to a large extent the understanding of Ext-groups between various representations.

1.5 The polynomial ring

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates. Write $\mathfrak{m} = (x_1, \dots, x_n)$ and $k = R/\mathfrak{m}$ for the simple module at the origin. A projective resolution of k is constructed as follows.

First, let V be the vector space with basis $e_1, \dots, e_n$. The exterior algebra on V is

$$\Lambda V := T(V)/(e_i^2, e_ie_j + e_je_i \mid 1 \leq i, j \leq n)$$

It is an good exercise in linear algebra to show that ΛV is a finite-dimensional graded k-algebra with degree i component $\Lambda^m V$ having basis

$$e_{i_1} \cdots e_{i_m} = e_{i_1} \wedge \cdots \wedge e_{i_m}, \qquad 1 \leq i_1 < i_2 < \cdots i_m \leq n.$$

Thus, $\dim_k \Lambda^m V = \binom{n}{m}$.

Now, let $P_m = R \otimes_k \Lambda^m V$ be the free R-module on this basis. Define the R-module homomorphism

$$d_m : P_m \to P_{m-1},$$
$$d_m(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{j=1}^{m} (-1)^{j+1} x_{i_j} \otimes e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_m}.$$

A little linear algebra shows that $d_m \circ d_{m+1} = 0$, or $d^2 = 0$ for short. Also, the image of $d^1 : P_1 = R \otimes V \to P_0 = R$ is $\mathfrak{m}$, so we obtain a complex

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to k \to 0. \tag{5-1}$$

This is in fact a projective resolution: a little linear algebra shows that $\ker d_m = \operatorname{im} d_{m+1}$ for all m.

We call (5-1) the Koszul complex.

To compute $\operatorname{Ext}^*_R(k,k)$ we apply $\operatorname{Hom}_R(-,k)$ to the (deleted) complex; now $\operatorname{Hom}_R(R \otimes_k \Lambda^i V, k) \cong \operatorname{Hom}_k(\Lambda^i V, k)$, and we also find that all the maps in this complex are zero; hence

$$\operatorname{Ext}^i_R(k,k) \cong (\Lambda^i V)^* \cong k^{\binom{n}{i}}.$$

It follows that $\operatorname{pdim}_R R/\mathfrak{m} = n$.

Theorem 5.1 *The global dimension of the polynomial ring $k[x_1, \ldots, x_n]$ is n.*

If k is algebraically closed, and M is any simple module, we can make a change of basis so that M is isomorphic to $R/\mathfrak{m}$; hence $\operatorname{Ext}^i_R(M,M)$ is of dimension $\binom{n}{i}$.

Example 5.2 For $A = k[x,y,z]$, the commutative polynomial ring on three variables, the Koszul complex is

$$0 \longrightarrow A(-3) \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A(-2)^3 \xrightarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}} A(-1)^3$$
$$\xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \longrightarrow k \longrightarrow 0.$$

Here we view elements of A^r as row vectors and the maps are right multiplication by the given matrices. ◊

1.5.1 Koszul complexes more generally

A careful look at the construction of the Koszul complex reveals that whenever $x_1, \ldots, x_n$ are central elements of a ring R, one can construct a complex

$$\cdots \to R \otimes_k \Lambda^m V \to \cdots \to R \otimes_k V \to R \to R/(x_1, \ldots, x_n) \to 0 \qquad \text{(5-2)}$$

using the formula for d_m given above. Although this is a complex, it need not be exact. However, it is exact if $x_1, \ldots, x_n$ is a regular sequence, meaning that x_1 is regular (i.e., a non-zero divisor) and for each $i > 1$, the image of x_i in $\bar{R} = R/(x_1, \ldots, x_{i-1})$ is regular.

In fact, one does not even need the x_is to be central; all one needs is that each x_i is normal in $\bar{R} = R/(x_1, \ldots, x_{i-1})$, meaning that $\bar{R}\bar{x}_i = \bar{x}_i\bar{R}$. We continue to call (5-2) a Koszul complex in this case.

Example 5.3 Fix $0 \neq q \in k$. Let $A = k[x, y]$ have defining relation $yx = qxy$. By the Diamond Lemma, $\{x^i y^j \mid i, j \geq 0\}$ is a basis for A. It follows that

$$0 \to A(-2) \xrightarrow{(y, -qx)} A(-1)^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

is exact, and hence a projective resolution of $k = A/(x, y)$. Applying the functor $\operatorname{Hom}_A(-, k)$ to this (deleted) resolution gives a complex $0 \to k \to k^2 \to k \to 0$ in which all the arrows are zero. Hence

$$\operatorname{Ext}^i_A(k, k) \cong \begin{cases} k & \text{if } i = 0, \\ k^2 & \text{if } i = 1, \\ k & \text{if } i = 2. \end{cases}$$

By Theorem 3.5, gldim $A = 2$. ◊

Examples abound. An important class of such examples is the the enveloping algebras of solvable Lie algebars over an algebraically closed field.

Example 5.4 Fix non-zero scalars $\alpha, \beta, \gamma \in k$ and let $A = k[x, y, z]$ be the ring subject to the relations

$$yx = \gamma xy, \; zy = \alpha yz, \; xz = \beta zx.$$

By the Diamond Lemma, $\{x^p y^q z^r \mid p, q, r \geq 0\}$ is a basis for A. It is not difficult to see that

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} \gamma x & \alpha y & \beta z \end{pmatrix}} A(-2)^3 \xrightarrow{\begin{pmatrix} 0 & z & -\alpha y \\ -\beta z & 0 & x \\ y & -\gamma x & 0 \end{pmatrix}} A(-1)^3$$

$$\xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \longrightarrow k \longrightarrow 0.$$

is a projective resolution of the left module ${}_A k = A/(x, y, z)$. ◊

1.6 Extensions between modules

Definition 6.1 An exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0 \tag{6-3}$$

is called an extension of N by L. We also say that M is an extension of N by L. The extension is split if there is a map $\gamma : N \to M$ such that $\beta \circ \gamma = \mathrm{id}_N$. Otherwise the extension is said to be non-split. ◊

The trivial extension of N by L is the sequence $0 \to L \to L \oplus N \to N \to 0$ with the obvious maps. It splits. If the extension (6-3) splits, then the image of the splitting map γ is isomorphic to N, and $M = L \oplus \gamma(N) \cong L \oplus N$.

Lemma 6.2 *Let L and N be simple modules. The extension (6-3) is non-split if and only if $\alpha(L)$ is the only proper submodule of M.*

Proof. If the extension splits via $\gamma : N \to M$, then $\gamma(N)$ is a submodule distinct from $\alpha(L)$. Thus, if $\alpha(L)$ is the only proper submodule the extension can not split.

Suppose that K is a proper submodule distinct from $\alpha(L)$. Then L and K are the two composition factors of M, so K must be isomorphic to N. In particular, the restriction of β to K is an isomorphism $\psi : K \to N$. Hence $\gamma = \psi^{-1}$ splits the sequence. □

A non-split extension between non-isomorphic simples can only exist if the ring is non-commutative. This is a (the?) fundamental difference between commutative and non-commutative ring theory.

Lemma 6.3 *Let L and N be non-isomorphic simple R-modules.*

1. *If R is commutative then every extension of N by L splits.*
2. *If z is a central element of R that annihilates L but not N, then every extension of N by L splits.*

Proof. (1) This follows from (2).

(2) Since z is central, multiplication by z gives a map $M \to M$. By hypothesis, L is in the kernel, so the image is isomorphic to a quotient of N. Since $zN \neq 0$, the image is non-zero, so is isomorphic to N and is therefore a submodule of M that is not equal to L. The result now follows from the previous Lemma. □

Proposition 6.4 *Let N and L be R-modules. Then*

1. $\mathrm{Ext}^1_R(N, L) = 0$ *if and only if every exact sequence* $0 \to L \to M \to N \to 0$ *splits;*
2. $\mathrm{Ext}^1_R(N, L)$ *classifies the non-split extensions of N by L.*

Proof. Details can be found in books on homological algebra such as [?], [?], [?]. □

Example 6.5 Let $R = k[x_1, \dots, x_n]$ be the commutative polynomial ring. Write $\mathfrak{m} = (x_1, \dots, x_n)$ and $k = R/\mathfrak{m}$. By section 1.5,

$$\operatorname{Ext}^1_R(k,k) \cong (\mathfrak{m}/\mathfrak{m}^2)^* \cong k^n.$$

We can construct the non-split extensions explicitly as follows. Each codimension one subspace $I \subset \mathfrak{m}$ that contains $\mathfrak{m}^2$ is an ideal and $\mathfrak{m}/I \cong k$, so there is an exact sequence of R-modules

$$0 \to k \to R/I \to k \to 0. \tag{6-4}$$

This sequence is non-split because $\mathfrak{m}$ is the unique maximal ideal containing I. Since $R/I \cong R/I'$ if and only if $I = I'$, we obtain a family of non-isomorphic non-split extensions parametrized by the codimension one subspaces of $\mathfrak{m}/\mathfrak{m}^2$. Codimension one subspaces of $\mathfrak{m}/\mathfrak{m}^2$ correspond to lines in $(\mathfrak{m}/\mathfrak{m}^2)^*$ and to points in the projective space

$$\mathbb{P}((\mathfrak{m}/\mathfrak{m}^2)^*) = \mathbb{P}(\operatorname{Ext}^1_R(k,k)) \cong \mathbb{P}^{n-1}.$$

If we view R as the symmetric algebra $S(V^*)$, then

- $S(V^*)$ is naturally functions on V;
- there are natural isomorphisms $\operatorname{Ext}^1_R(k,k) \cong V \cong T_0 V$;
- the extensions R/I in (6-4) are parametized by points in $\mathbb{P}(V)$, equivalently by the tangent directions at 0 in V. ◊

Pairs of points in the plane. A pair of *distinct* points in the plane $\mathbb{C}^2$, say p and q, corresponds to a pair of distinct maximal ideal ideals, $\mathfrak{m}_p$ and $\mathfrak{m}_q$, and to the length two quotient $\mathbb{C}[x,y]/\mathfrak{m}_p\mathfrak{m}_q$. One can give the set consisting of pairs of distinct points in $\mathbb{C}^2$ the structure of a scheme: it is $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2$, the quotient by the group action $(p,q) \mapsto (q,p)$, minus the diagonal. This is a smooth variety, but to compactify it we need to put in (at least) the diagonal; but $X := \mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2$ is singular along the diagonal. A natural desingularization of X is given by the Hilbert scheme $\tilde{X}$ consisting of all length two closed subschemes of $\mathbb{C}^2$. The points of $\tilde{X}$ are in bijection with the quotients $\mathbb{C}[x,y]/I$ where I is an ideal of codimension two; the map $\pi : \tilde{X} \to X$ is given by sending $\mathbb{C}[x,y]/I$ to its semisimplification, the direct sum of its two composition factors viewed as an unordered pair $p,q) \in X$. In addition to ideals of the form $\mathfrak{m}_p\mathfrak{m}_q$, $\tilde{X}$ contains for each point p the ideals I such that $\mathfrak{m}_p^2 \subset I \subset \mathfrak{m}_p$ and $\dim \mathbb{C}[x,y]/I = 2$; these are the points in $\pi^{-1}(p,p)$. The previous example shows that $\pi^{-1}(p,p) \cong \mathbb{P}^1$ and the points in this parametrize the non-split extensions of $\mathbb{C}[x,y]/\mathfrak{m}_p$ by itself. One should think of the closed subscheme corresponding to I as a point p together with a tangent direction at p; in other words, as q approaches p, in the limit we get p together with the direction from which q approached. Such Is correspond to non-split extensions

$$0 \to \mathcal{O}_p \to \mathbb{C}[x,y]/I \to \mathcal{O}_p \to 0$$

so are parametrized by $\mathbb{P}(\mathrm{Ext}^1_{\mathbb{C}^2}(\mathcal{O}_p, \mathcal{O}_p)) \cong \mathbb{P}^1$.