Chapter 1

Rings, modules, and algebras

This chapter fixes terminology and notation, and provides some gentle reminders.

The first two sections provide a refresher course on the basic language of rings and modules. This should be familiar. The notion of an algebra is recalled in section 1.3. Free algebras and tensor algebras are discussed in Section 1.4. Section 1.5 is a short account of the differences between the commutative and non-commutative worlds. Section 1.6 says a little about non-commutative algebraic geometry.

1.1 Rings

All rings are required to contain an identity element, denoted 1. Homomorphisms between rings are required to send the identity to the identity. Thus, the map which sends an element α of a field k to the 2×2 matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is not a ring homomorphism—it respects multiplication and addition but the image of the identity in k is not the identity in $M_2(k)$. Given a pair of rings $R \subset S$ having the same identity, we call R a subring of S and S an overring of R.

If R is a ring, $M_n(R)$ denotes the ring of $n \times n$ matrices with entries from R. We will denote by e_{ij} the matrix with 1 in the ij^{th} position and zeroes elsewhere. We call the e_{ij} matrix units.

Let R be a ring. Its opposite is the ring R^{op} that is R as an abelian group but endowed with a new multiplication $x * y = yx$ where yx is the product in R. A ring homomorphism $f : R^{\mathrm{op}} \to S$ is an additive map $f : R \to S$ satisfying $f(xy) = f(y)f(x)$; we call such a map a ring anti-homomorphism from R to S.

The center of a ring R is the set of all $z \in R$ such that $za = az$ for all $a \in R$. It is a subring having the same identity and is usually denoted by $Z(R)$ (from the German 'zentrum').

1.2 Modules

Let R be a ring and M a left R-module.

An R-module is also be called a representation of R.

We say that M is generated by a subset $\mathcal{S}$ if every element in it can be written as $r_1 m_1 + \cdots r_n m_n$ for some $r_i \in R$ and $m_i \in \mathcal{S}$. We write $Rm_1 + \cdots Rm_n$ for the submodule of M generated by $\{m_1, \ldots, m_n\}$. If M is generated by a finite set we say it is finitely generated. If M is generated by a single element we say it is cyclic; if $M = Rm$, then the map $\varphi : R \to M$ defined by $\varphi(m) = rm$ is surjective so $M \cong R/\ker\varphi$. Hence the cyclic R-modules are, up to isomorphism, the quotients of R by left ideals.

Since M is an abelian group, we have the ring of all abelian group homomorphisms from M to itself, namely $\mathrm{Hom}_{\mathbb{Z}}(M, M)$. The R-module action on M is a map $R \times M \to M$; define $\rho : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ by

$$\rho(r)(m) := r.m. \tag{2-1}$$

The module axioms imply that ρ is a ring homomorphism. Conversely, if $\rho : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is a ring homomorphism, formula (2-1) endows M with an R-module structure. Thus a left R-module structure on M is the same thing as a ring homomorphism $R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$.

The preceding sentence is false for right R-modules: if M is a right R-module, the map $\rho : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ defined by $\rho(x)(m) := m.x$, for $m \in M$ and $x \in R$, is a ring *anti-homomorphism* because $\rho(xy) = \rho(y) \circ \rho(x)$. Thus, a right R-module structure on M is the same thing as a ring anti-homomorphism $R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$, and this is the same thing as a ring homomorphism $R^{\mathrm{op}} \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$. Right (resp. left) R-modules are the same thing as left (resp. right) R^{op}-modules.

1.2.1 Module Homomorphisms

Let M and N be left R-modules. We write $\mathrm{Hom}_R(M, N)$ for the set of R-module homomorphisms from M to N. It is an abelian group under addition: $(\varphi + \psi)(m) := \varphi(m) + \psi(m)$. When $M = N$ we write $\mathrm{End}_R M = \mathrm{Hom}_R(M, M)$ and call this the endomorphism ring of M; its addition has just been defined and its multiplication is given by composition of functions, namely $\varphi\psi = \varphi \circ \psi$; the identity map $\mathrm{id}_M : M \to M$ is the identity element in $\mathrm{End}_R M$.

Left multiplication by $x \in R$ is an endomorphism of R as a right R-module. This leads to an isomorphism $R \cong \mathrm{End}(R_R)$. Similarly there is an isomorphism $R^{\mathrm{op}} \cong \mathrm{End}({}_R R)$.

1.2.2 Annihilators

The annihilator of $m \in M$ is

$$\mathrm{Ann}(m) := \{x \in R \mid xm = 0\}.$$

This set is the kernel of the R-module homomorphism $\varphi : R \to M$ defined by $\varphi(x) = xm$, so it is a left ideal of R and $Rm \cong R/\mathrm{Ann}(m)$.

The annihilator of an R-module M is

$$\mathrm{Ann}_R M := \{x \in R \mid xm = 0 \text{ for all } m \in M\}.$$

It is easy to see that $\mathrm{Ann}_R M = \ker \rho$, where $\rho : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is the ring map corresponding to the module structure. In particular, $\mathrm{Ann} M$ is a two sided ideal of R. If I is a two-sided ideal contained in $\ker \rho$ then ρ factors through the quotient R/I, thus giving M an R/I-module structure. In particular, M is an $R/\mathrm{Ann} M$-module.

1.2.3 Simple modules

A simple R-module, or an irreducible representation of R, is a module whose only submodules are 0 and itself. Clearly, M is simple if and only if $Rm = M$ for all $0 \neq m \in M$. If M is simple and $0 \neq m \in M$, then $M \cong R/\mathrm{Ann}(m)$, and $\mathrm{Ann}(m)$ is a maximal left ideal of R.

Proposition 2.1 *Up to isomorphism, the simple left R-modules are the quotients R/I as I runs over the maximal left ideals of R.*

Theorem 2.2 (Schur's Lemma) *If M is a simple R-module, then $\mathrm{End}_R M$ is a division ring.*

1.2.4 Bimodules

Let R and S be rings.

An S-R-bimodule is an abelian group M that is simultaneously a left S-module and a right R-module in such a way that the actions satisfy $s.(m.r) = (s.m).r$ for all $s \in S$, $r \in R$, $m \in M$. We write ${}_S M_R$ to indicate that M is an S-R-bimodule.

Let M be an R-S-bimodule. We make M into a left $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$-module, by defining $(r \otimes s) \cdot m = rms$. This *is* a left module structure because

$$\begin{aligned}\Big((r \otimes s) \cdot (r' \otimes s')\Big) \cdot m &= (rr' \otimes s's) \cdot m \\ &= rr'ms's \\ &= r(r'ms')s \\ &= (r \otimes s) \cdot \Big((r' \otimes s') \cdot m\Big).\end{aligned}$$

It is easy to show that R-S-bimodules are the same things as left $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$-modules.

A ring R is an R-R bimodule through the multiplication in R, the condition in the definition being satisfied as a consequence of the associativity of multiplication in R. Every two sided ideal of R is an R-R-bimodule. These are the only sub-bimodules of ${}_R R_R$, so R is simple as a bimodule (i.e. has no sub-bimodules apart from itself and 0) if and only if it is a simple ring.

Let M be a right R-module, and set $E = \mathrm{End}_R M$. Make M a left E-module by defining $\vartheta \cdot m = \vartheta(m)$ for $\vartheta \in E$, and $m \in M$. Now M can be given the

structure of an E-R-bimodule by defining $\vartheta \cdot m \cdot r = \vartheta(mr) = \vartheta(m)r$ for $m \in M$, $r \in R$ and $\vartheta \in E$. Another point of view on this is that the maps $\psi_x : M \to M$ defined by $\psi_x(m) = mx$ for $x \in R$ are E-module endomorphisms of M, so the map $\Psi : R \to \mathrm{End}_E M$ defined by $\Psi(x) = \psi_x$ is a ring homomorphism.

EXERCISES

2.1 Show that the transpose mapping, $a \mapsto a^{\mathsf{trans}}$, is an isomorphism $M_n(R) \to M_n(R^{\mathrm{op}})^{\mathrm{op}}$.

2.2 Show that the ring $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{smallmatrix}\right)$ is right artinian and right noetherian but not left artinian or left noetherian.

2.3 Show that R-S-bimodule structures on an abelian group are the same things as pairs of maps $f : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$, $g : S \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ such that (i) f is a ring homomorphism, (ii) g is an anti-homomorphism and (iii) $f(R)$ commutes with $g(S)$.

2.4 Show that if M is an R-S-bimodule, then M may be made into a left $R \otimes S^{\mathrm{op}}$-module, and conversely.

1.3 Algebras

Let k be a commutative ring and R a ring.

A ring homomorphism $f : k \to R$ whose image is in the center of R is said to give R the structure of a k-algebra; we call f the structure map of the algebra. Every ring has the structure of a $\mathbb{Z}$-algebra, the structure map being $f(n) = n.1$ for $n \in \mathbb{Z}$. Notice that f is not required to be injective.

We are mainly interested in algebras over fields, in which case the structure map $f : k \to R$ must be injective.

It is common practice to identify k with its image in R; from this point of view a k-algebra is a just a ring containing a central subring that is isomorphic to (a quotient of) k. A ring can be an algebra over several fields at once: $\mathbb{C}$ is an algebra over $\mathbb{Q}$, over $\mathbb{R}$, and over $\mathbb{C}$ itself.

The commutative polynomial ring, $k[t_1, \ldots, t_n]$, and the ring $M_n(k)$ of $n \times n$ matrices are k-algebras. The ring of real quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is an $\mathbb{R}$-algebra but not a $\mathbb{C}$-algebra even though its subalgebra $\mathbb{R} \oplus \mathbb{R}i$ is isomorphic to $\mathbb{C}$; the point is that this subalgebra is not in the center of $\mathbb{H}$.

If R_1 and R_2 are k-algebras, with structure maps $f_i : k \to R_i$, then a k-algebra homomorphism $g : R_1 \to R_2$ is a ring homomorphism satisfying $gf_1 = f_2$. That is the composition $k \to R_1 \to R_2$ gives R_2 a k-algebra structure which coincides with that obtained from $f_2 : k \to R_2$. When working with k-algebras we usually consider only the k-algebra maps between them. This is often not made explicit.

Let R be a k-algebra with structure map $f : k \to R$.

Every quotient ring R/I inherits a k-algebra structure and the natural map $R \to R/I$ is a k-algebra homomorphism. If $g : R \to S$ is a ring homomorphism, then S inherits the structure of a k-algebra if the image of $g \circ f$ lies in the center of S. Every ring is an algebra over its center.

Every R-module, left or right, becomes a k-module, with k-action given by $c.m = f(c)m$ for $c \in k$ and $m \in M$, where $f : k \to R$ is the structure map. Elements of R act on M as k-linear maps because k is in the center of R. The map ρ defined by (2-1) therefore has its image in $\mathrm{Hom}_k(M, M)$. Moreover, $\mathrm{Hom}_k(M, M)$ is a k-algebra, and ρ is a k-algebra homomorphism. Hence for a k-algebra R, specifying an R-module M is the same thing as specifying a k-algebra homomorphism $\rho : R \to \mathrm{Hom}_k(M, M)$.

Every R-module map $\varphi : M \to N$ is k-linear, so there is an inclusion $\mathrm{Hom}_R(M, N) \subset \mathrm{Hom}_k(M, N)$. In particular, $\mathrm{End}_R M \subset \mathrm{End}_k M$.

If M is an R-module, then multiplication by $z \in Z(R)$ is an R-module homomorphism $\lambda_z : M \to M$, $\lambda_z(m) = zm$. The rule $z \mapsto \lambda_z$ is a ring homomorphism $\lambda : Z(R) \to \mathrm{End}_R M$; the image of λ is contained in the center of $\mathrm{End}_R M$. Hence the image of $\lambda \circ f : k \to \mathrm{End}_R M$ belongs to the center of $\mathrm{End}_R M$, and $\mathrm{End}_R M$ becomes a k-algebra.

1.3.1 Tensor Product of k-algebras

If R and S are k-algebras, then $R \otimes_k S$ is made into a ring by defining

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = r_1 r_2 \otimes s_1 s_2,$$

and extending linearly in the obvious way. The identity element is $1 \otimes 1$. The map $\alpha \mapsto \alpha \otimes 1 = 1 \otimes \alpha$ is a homomorphism $k \to R \otimes_k S$ giving $R \otimes_k S$ a k-algebra structure. The natural maps $R \to R \otimes_k S$, $r \mapsto r \otimes 1$, and $S \to R \otimes_k S$, $s \mapsto 1 \otimes s$, are k-algebra homomorphisms. Notice that $R \otimes k \cong R$ as k-algebras.

If k is a field the maps $R \to R \otimes_k S$ and $S \to R \otimes_k S$ are injective, so we view R and S as subalgebras of $R \otimes_k S$. As such, R and S commute with one another, so $R \otimes S$ is commutative if both R and S are. The intersection of R and S inside $R \otimes_k S$ is $k \otimes k = k$.

Given a k-algebra T, subalgebras R and S are linearly disjoint if whenever $\{r_\lambda \mid \lambda \in \Lambda\}$ and $\{s_\mu \mid \mu \in \Gamma\}$ are linearly independent subsets of R and S, then $\{r_\lambda s_\mu \mid \lambda \in \Lambda, \mu \in \Gamma\}$ is linearly independent.

Lemma 3.1 *Let T be a k-algebra. Suppose that T contains subalgebras R and S such that $R \cap S = k$ and elements of R commute with elements of S. Then the map $\psi : R \otimes S \to T$ given by $\psi(r \otimes s) = rs$ is an algebra homomorphism. Furthermore ψ is an isomorphism onto its image if and only if R and S are linearly disjoint.*

Proposition 3.2 $M_n(k) \otimes M_m(k) \cong M_{nm}(k)$.

Proof. Let e_{ij} denote the usual matrix units. If $1 \le i, j \le n$ and $1 \le k, l \le m$ define $\epsilon_{(ik)(jl)} := e_{ij} \otimes e_{kl}$. Then the $\epsilon_{(ik)(jl)}$ are a k-basis for $M_n(k) \otimes M_m(k)$. Moreover, their multiplication is given by

$$\begin{aligned} \epsilon_{(ik)(jl)} \cdot \epsilon_{(pr)(qs)} &= (e_{ij} \otimes e_{kl}) \cdot (e_{pq} \otimes e_{rs}) \\ &= \delta_{jp}\delta_{lr} e_{iq} \otimes e_{ks} \\ &= \delta_{(jl),(pr)} \epsilon_{(ik)(qs)} \end{aligned}$$

which is precisely the multiplication table for $M_{nm}(k)$ with matrix units $\{\epsilon_{(ik)(jl)} \mid 1 \leq i, j \leq n \text{ and } 1 \leq k, l \leq m\}$. □

Example 3.3 Let R be a k-algebra. We write $M_n(R)$ for the ring of $n \times n$ matrices with entries in R. It can be characterized as being a free left R-module with basis $\{e_{ij} \mid 1 \leq i, j \leq n\}$ and multiplication given by

$$\left(\sum_{i,j} r_{ij} e_{ij}\right) \cdot \left(\sum_{p,q} s_{pq} e_{pq}\right) = \sum_{i,j,p,q} \delta_{jp} r_{ij} s_{pq} e_{iq}.$$

Then $R \otimes M_n(k) \cong M_n(R)$. ◊

Example 3.4 Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be indeterminates over the field k. Then the tensor product of the polynomial rings $k[X_1, \ldots, X_m] \otimes k[Y_1, \ldots, Y_n]$ is isomorphic to the polynomial ring in $m + n$ indeterminates $k[t_1, \ldots, t_{m+n}]$ via the map $X_i \otimes 1 \mapsto t_i$ and $1 \otimes Y_j \mapsto t_{m+j}$. Much more is true. If X and Y are affine varieties over the field k, then $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y)$. Thus the tensor product of algebras is the algebraic analogue of the cartesian product of varieties. ◊

If M is a left R-module and N a left S-module, then $M \otimes_k N$ is a left $R \otimes_k S$-module under the action is $(r \otimes s) \cdot (m \otimes n) := rm \otimes sn$.

Extending the base. Let A be a k-algebra and K a commutative k-algebra. Then the natural map $K \to A \otimes_k K$ makes $A \otimes_k K$ a K-algebra. We say that $A \otimes_k K$ is obtained from A by extending the base or by extending scalars.

1.3.2 Algebras over fields

Suppose that k is a field and A a k-algebra.

Every A-module becomes a k-vector space, and an A-module is said to be finite dimensional if it is finite dimensional as a vector space over k.

A k-algebra A is a finite dimensional k-algebra if $\dim_k A < \infty$. It is clear that a finite dimensional k-algebra is artinian. Its simple modules are finite dimensional so its artinian modules are precisely the finite dimensional ones. If $\dim_k M = n < \infty$, then $\mathrm{End}_A M \subset M_n(k)$, and $\mathrm{End}_A M$ is a finite dimensional k-algebra.

Let A be a finite dimensional k-algebra, and M a simple A-module. Since $\dim_k M < \infty$, $\mathrm{End}_A M$ is a finite dimensional k-algebra. Hence the center of the division ring $\mathrm{End}_A M$ is a finite dimensional field extension of k. Thus, if k is algebraically closed, that field extension must be k itself. This proves the following result.

Proposition 3.5 *Let A be a finite-dimensional k-algebra and M a simple A-module. If k is algebraically closed, then* $\mathrm{End}_A M \cong k$.

EXERCISES

3.1 Let k be a commutative ring, and suppose that R is a ring endowed with the structure of a k-module. Show that this structure makes R a k-algebra if and only if the multiplication map $R \times R \to R$ is a k-module homomorphism.

3.2 Give an example of a k-algebra R, with structure map $f : k \to R$, and a ring homomorphism $g : R \to S$ for which the composition $gf : k \to S$ does not give S a k-algebra structure.

3.3 Let A, B, C, D be k-algebras, M an A-B-bimodule and N a C-D-bimodule; in brief, ${}_AM_B$ and ${}_CN_D$. Show that $\mathrm{Hom}_k(M, N)$ is a right A-module, a left B-module, a left C-module and a right D-module. In other words $\mathrm{Hom}_k(M, N)$ is a $B \otimes_k C$-$A \otimes_k D$-bimodule or a left module over $A^{\mathrm{op}} \otimes B \otimes C \otimes D^{\mathrm{op}}$. Explicitly, the action is given by

$$(a \otimes b \otimes c \otimes d.f)(m) = cf(amb)d.$$

3.4 Consider the rings $\mathbb{C}$ and $\mathbb{H}$ (the quaternions) as $\mathbb{R}$-algebras (so that $\otimes$ denotes $\otimes_{\mathbb{R}}$ in the following exercises). Make $R = \mathbb{C} \otimes \mathbb{C}$ into a ring as above. Show that R is isomorphic to the semisimple ring $\mathbb{C} \oplus \mathbb{C}$. [*Hint:* find idempotents in R.]

3.5 Show that $\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$. [*Hint:* find matrix units.]

3.6 Show that $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$.

3.7 Let k^2 have ordered basis e_1, e_2, and let $k^2 \otimes k^2$ have ordered basis $v_1 = e_1 \otimes e_1$, $v_2 = e_1 \otimes e_2$, $v_3 = e_2 \otimes e_1$, $v_4 = e_2 \otimes e_2$. Consider the linear transformations h and f of k^2, which are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis e_1, e_2. What are the 4×4 matrices which represent the linear transformations

$$f \otimes 1,\ 1 \otimes f,\ f \otimes f,\ h \otimes f,\ f \otimes f,\ h \otimes h,\ h \otimes 1,\ 1 \otimes h$$

with respect to the ordered basis $v_1, \ldots, v_4$ of $k^2 \otimes k^2 \cong k^4$.

3.8 Let V be a finite dimensional vector space and let V^* be its dual. Give $V \otimes_k V^*$ a ring structure by defining

$$(u \otimes \alpha).(v \otimes \beta) := \alpha(v)u \otimes \beta,$$

whenever $u \otimes \alpha, v \otimes \beta \in V \otimes V^*$, and extend this to finite sums using distributivity. If $\{u_i\}$ is a basis for V, and α_i is its dual basis, show that the element $\sum u_i \otimes \alpha_i$ is the identity for this ring structure, and therefore is independent of the choice of basis. Make $V \otimes V^*$ a k-algebra through the map $k \to V \otimes V^*$, $\alpha \mapsto \alpha.1$. Show that the map $\Phi : V \otimes V^* \to \mathrm{End}_k(V)$ defined by $\Phi(\sum_i v_i \otimes f_i)(u) = \sum_i f_i(u)v_i$ is a k-algebra isomorphism.

3.9 If U and V are k-vector spaces show that there is a k-algebra isomorphism $\mathrm{End}_k U \otimes \mathrm{End}_k V \to \mathrm{End}_k(U \otimes V)$.

3.10 Let K/k be an extension field. Let $k[X_1, \ldots, X_n]$ be the commutative polynomial ring. Show that $K \otimes_k k[X_1, \ldots, X_n] \cong K[X_1, \ldots, X_n]$.

1.4 Free algebras and tensor algebras

Definition 4.1 Given a set X, write X^* for the set of all finite sequences of elements from X, including the empty sequence which is denoted by 1. The

elements of X^* are called words. Juxtaposition of words defines an associative multiplication on X^* making it a monoid with an identity element 1. Let $k\langle X\rangle$ denote the k-vector space with basis the elements of X^*. The multiplication in X^* is extended to a k-linear multiplication on $k\langle X\rangle$ which is associative and distributive. Thus $k\langle X\rangle$ becomes a ring.

The map $k \to k\langle X\rangle$, $\alpha \mapsto \alpha.1$ is an injective ring homomorphism. We identify k with its image in $k\langle X\rangle$, and write α for $\alpha.1$. Since 1 is an identity in X^* it is also an identity in $k\langle X\rangle$. Thus $k\langle X\rangle$ is a ring with identity, and is a k-algebra. We call $k\langle X\rangle$ the free k-algebra on X.

If $X = \{x_1, \dots, x_n\}$ it is usual to write $k\langle X\rangle$ as $k\langle x_1, \dots, x_n\rangle$, and to call it the free algebra on $x_1, \dots, x_n$. The elements of $k\langle x_1, \dots, x_n\rangle$ are called non-commutative polynomials in $x_1, \dots, x_n$. ◊

Example 4.2 If $X = \emptyset$, then $X^* = \{1\}$, so $k\langle X\rangle = k.1$ and is isomorphic to the base ring k.

If $X = \{x\}$, then $X^* = \{1, x, xx, xxx, \dots\}$. It is usual to denote these elements by $\{1, x, x^2, x^3, \dots\}$. Thus $k\langle X\rangle$ is isomorphic to the polynomial ring in one variable.

If $X = \{x, y\}$, then X^* is

$$\{1, x, y, xx, xy, yx, yy, xxx, xxy, xyx, xyy, yxx, yxy, yyx, yyy, \dots\}. \qquad (4\text{-}2)$$

The multiplication in X^* is not commutative so $k\langle X\rangle$ is not commutative. It is usual to write a word such as $xxxyyxyyy$ as $x^3y^2xy^3$. The elements of $k\langle X\rangle$ are finite k-linear combinations of words. For example, if α and β are in k, then $1 + \alpha x^2 - \beta yxy^3$ is a typical element of $k\langle X\rangle$. ◊

Proposition 4.3 *Let A be a k-algebra and $\psi : X \to A$ a map of sets. Then there is a unique k-algebra homomorphism $\Psi : k\langle X\rangle \to A$ such that $\Psi(x) = \psi(x)$ for all $x \in X$.*

Proof. (Existence) Since X^* is a basis for $k\langle X\rangle$, there is a unique k-linear map $\Psi : k\langle X\rangle \to A$ defined by $\Psi(1) = 1$, and

$$\Psi(x_1 \dots x_n) := \psi(x_1) \dots \psi(x_n)$$

whenever $x_1, \dots, x_n \in X$. It follows from the definitions of $k\langle X\rangle$ and Ψ, that Ψ is a ring homomorphism. Since Ψ is k-linear and $\Psi(1) = 1$, it is also a k-algebra map.

(Uniqueness) For any k-algebras A and B, two ring homomorphisms $A \to B$ that agree on k and a set of k-algebra generators for A agree everywhere. Hence Ψ is completely determined by the fact that $\Psi(x) = \psi(x)$ for all $x \in X$. □

Thus $X \mapsto k\langle X\rangle$ is left adjoint to the forgetful functor k-algebras$\to$ Set.

Corollary 4.4 *Every k-algebra is a quotient of a free k-algebra.*

Proof. In Proposition 4.3 take $X = A$ and $\psi = \mathrm{id}_A$. □

This result is marvellously inefficient: usually we are interested in finitely generated algebras and then, if A is generated as a k-algebra by $x_1, \dots, x_n$, A is a quotient of the free algebra $k\langle x_1, \dots, x_n\rangle$.

Definition 4.5 If $x_1, \dots, x_n \in X$, the degree of $w = x_1 \cdots x_n \in X^*$ is defined to be $\deg w = n$. If $0 \neq f \in k\langle X\rangle$ the degree of f is

$$\deg f := \max\{\deg w \mid w \in X^* \text{ appears in } f \text{ with a non-zero coefficient}\}$$

and we define $\deg 0 = -\infty$. ◊

This notion of degree has the usual properties, namely $\deg fg = \deg f + \deg g$ and $\deg(f+g) \leq \max\{\deg f, \deg g\}$.

Definition 4.6 Fix an ordering on X. For words of a fixed degree, n say, define $x_1 \cdots x_n < y_1 \cdots y_n$ if either $x_1 < y_1$ or $x_1 \cdots x_m = y_1 \cdots y_m$ but $x_{m+1} < y_{m+1}$ for some m. This ordering is called the degree and dictionary ordering. ◊

For example, if $X = \{a, b, c, d, \dots\}$ with $a < b < c < d < \cdots$, then the words in increasing order are

$$1, a, b, c, \dots, aa, ab, ac, \dots, aaa, aab, aac, \dots, aba, abb, abc, \dots .$$

Thus, the words are ordered first by degree, then in alphabetical order. In particular, if $X = \{x, y\}$ with $x < y$ the elements of X^* are listed in (4-2) in increasing order.

Definition 4.7 Fix an ordering on X and give X^* the induced degree and dictionary ordering. Let f be a non-zero element of $k\langle X\rangle$. Suppose that a word w appears in f with a non-zero coefficient $\alpha \in k$. We call αw the leading term of f if all the words occurring in $f - \alpha w$ with non-zero coefficient are strictly less than w. We call α the leading coefficient of f. ◊

Just as for the commutative polynomial ring, we can use a leading term argument to show that $k\langle X\rangle$ is a domain if k is (cf. Exercise 1).

Proposition 4.8 *Let $R = k\langle X\rangle/(\mathcal{F})$. Write $\mathcal{F} = \{f_1, f_2, \dots\}$. For each i, let $w_i \in X^*$ be such that $\alpha_i w_i$ is the leading term of f_i where $0 \neq \alpha_i \in k$. Then R is spanned as a k-module by the images of*

$$\{w \in X^* \mid w \textit{ does not contain any } w_i \textit{ as a subword}\}.$$

Proof. Write $\bar{g}$ for the image of $g \in k\langle X\rangle$ under the natural map $k\langle X\rangle \to R$. Certainly R is spanned as a k-module by $\{\bar{w} \mid w \in X^*\}$.

By hypothesis, we can write $f_i = \alpha_i w_i + g_i$ with all the words w_λ that appear in g_i being $< w_i$. If w_i is a subword of w there are words u and v such that $w = uw_iv$, and hence

$$\bar{w} = -\alpha_i \bar{u}\bar{g}_i\bar{v};$$

this is a k-linear combination of terms of the form $\bar{u}\bar{w}_\lambda\bar{v}$ with $w_\lambda < w_i$. Notice that $uw_\lambda v < uw_i v = w$. Repeating this argument, the result follows by induction. □

1.4.1 The tensor algebra

We now present a basis free appoach to the free algebra.

Definition 4.9 Let V be a k-module. The tensor algebra on V, denoted $T(V)$, is defined as follows: write $V^{\otimes 0} = k$,

$$V^{\otimes n} = \underbrace{V \otimes_k \cdots \otimes_k V}_{n \text{ copies}},$$

and

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = k \ \oplus \ V \ \oplus \ (V \otimes V) \ \oplus \cdots .$$

Make $T(V)$ into a ring by defining the product to be the k-linear extension of

- $\alpha \in V^{\otimes 0} = k$ and $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ to be

$$(\alpha).(v_1 \otimes \ldots \otimes v_n) = (\alpha v_1) \otimes \ldots \otimes v_n = (v_1 \otimes \ldots \otimes v_n).(\alpha);$$

- $u_1 \otimes \ldots \otimes u_m \in V^{\otimes m}$ and $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ to be

$$(u_1 \otimes \ldots \otimes u_m).(v_1 \otimes \ldots \otimes v_n) = u_1 \otimes \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes v_n.$$

This gives $T(V)$ the structure of a ring with identity $1 \in k = V^{\otimes 0}$; $T(V)$ becomes a k-algebra through the natural map $k \to V^{\otimes 0} = k$. ◊

Proposition 4.10 *Let X be a set, and V a free k-module. Every set map $\psi : X \to V$ extends to a unique k-algebra homomorphism $\Psi : k\langle X\rangle \to T(V)$. The map Ψ is injective if and only if $\{\psi(x) \mid x \in X\}$ is linearly independent, and is surjective if and only if $\psi(X)$ spans V. In particular, Ψ is an isomorphism if and only if ψ maps X injectively to a basis for V.*

Proof. The existence of Ψ is guaranteed by Proposition 4.3. By definition, $k\langle X\rangle$ has basis X^* and, if $x_1, \ldots, x_n \in X$,

$$\Psi(x_1 \ldots x_n) = \psi(x_1) \ldots \psi(x_n) = \psi(x_1) \otimes \ldots \otimes \psi(x_n) \in V^{\otimes n}.$$

Hence $\psi(X)$ is linearly independent if and only if $\{\Psi(x_1 \ldots x_n) \mid x_1, \ldots, x_n \in X\}$ is linearly independent. The theorem now follows in a straightforward way. □

Let $\mathrm{GL}(V)$ denote the group of invertible k-linear maps $V \to V$. The action of $\mathrm{GL}(V)$ on V extends to an action of $\mathrm{GL}(V)$ as k-algebra automorphisms of

$T(V)$. In particular, if V has basis $x_1, \ldots, x_n$ then the symmetric group action on $\{x_1, \ldots, x_n\}$ extends to an action of S_n as automorphisms of $T(V)$.

The construction of $T(V)$ is functorial; this is a consequence of Proposition 4.10; that is, a k-linear map $f : U \to V$ extends to a k-algebra homomorphism $T(U) \to T(V)$. Hence the automorphism action of $\mathrm{GL}(V)$ is a consequence of the functoriality of $T(-)$.

1.4.2 More about free algebras

Let k be a field.

Let $F = k\langle x_1, \ldots, x_n\rangle$ be the free algebra on variables $x_1 < x_2 < \cdots < x_n$. We we call the set of words in these variables the standard basis for F.

Proposition 4.11 *$k\langle x_1, \ldots, x_n\rangle$ is isomorphic to a subalgebra of $k\langle x, y\rangle$.*

Proof. Write $F = k\langle x_1, \ldots, x_n\rangle$ and $B = k\langle x, y\rangle$. Let $\mathcal{A}$ and $\mathcal{B}$ be the standard bases for A and B consisting of the words in the generators. Define $\varphi : A \to B$ by $\varphi(x_i) = xy^i$. Since φ sends $\mathcal{A}$ to $\mathcal{B}$ it suffices to show that φ is injective on $\mathcal{A}$. But this is clear, since we can recover each integer i_p from $w = \varphi(x_1^{i_1} \cdots x_n^{i_n})$ as the number of times $xy^p x$ occurs as a subword of w. □

It is useful to notice that a k-algebra that is a domain contains a copy of $k\langle x, y\rangle$ if it contains two non-zero left ideals having zero intersection (Lemma 2.5.6).

Proposition 4.12 *If $n \geq 2$, the center of $k\langle x_1, \ldots, x_n\rangle$ is k.*

Proof. It suffices to show that the only homogenous central elements are the scalars. Let $z = x_1 a_1 + \cdots + x_n a_n$ be a homogeneous central element of degree > 0. From the equality $zx_i = x_i z$. it follows that $a_j = 0$ for all $j \neq i$. By varying i we see that all a_j are zero. □

Notation. Each homogeneous element $f \in F$ of positive degree can be written uniquely as

$$f = \sum_{x \in X} f_x x.$$

Lemma 4.13 *Suppose that a_i and b_i are non-zero homogeneous elements of F such that*

- *$\deg a_i + \deg b_i$ is the same for all i,*
- *the elements are labelled so that $\deg a_1 \leq \cdots \leq \deg a_n$, and*
- *$a_1 b_1 + \cdots + a_n b_n = 0$.*

Then there are homogeneous elements c_i which are either zero, or of degree $\deg a_n - \deg a_i$, such that

$$a_n = a_1 c_1 + \cdots + a_{n-1} c_{n-1}.$$

Proof. If $b_n \in k$, the result is obvious. If not, then $(b_n)_x \neq 0$ for some $x \in X$. Now

$$0 = (a_1 b_1 + \cdots + a_n b_n)_x = a_1 (b_1)_x + \cdots + a_n (b_n)_x .$$

Now $\deg (b_n)_x < \deg b_n$, so the result follows by induction on the degree of b_n. □

It is clear from the standard basis for the free algebra that the right (and the left) ideal generated by the generators of the free algebra is a free module with those generators as a basis. Indeed, every right (and left) ideal is a free module.

Proposition 4.14 *Every left ideal in the free algebra is a free module.*

Proof. Let I be a left ideal. Index the elements of I by some ordinal in such a way that $\deg b_i \leq b_j$ whenever $i \leq j$. Let $\bar{b}_i$ denote the leading homogeneous component of b_i, and let J be the left ideal generated by those $\bar{b}_j$ which are not in the left ideal generated by $\{\bar{b}_i \mid i < j\}$; it follows from Lemma 4.13 that such a $\bar{b}_j$ satisfies $F\bar{b}_j \cap (\sum_{i<j} F b_i) = 0$, and hence that J is free on the $\bar{b}_j$. But $J = \operatorname{gr} I$, so by ????, I is freely generated by the corresponding b_j. □

EXERCISES

4.1 Show that the free algebra $k\langle x_1, \ldots, x_n \rangle$ is a domain.

4.2 In the free algebra $k\langle x, y \rangle$, show that the two-sided ideal (y) is not finitely generated as a left ideal.

4.3 Show that the free algebra $k\langle x, y \rangle$ is not noetherian by

(a) exhibiting an ascending chain of *left* ideals which does not terminate;

(b) exhibiting an ascending chain of *two-sided* ideals which does not terminate.

4.4 Show that the subalgebra of $k\langle x, y \rangle$ generated by $x, xy, xy^2, \ldots, xy^n$ is isomorphic to the free algebra on $n+1$ variables.

4.5 Let k be a field and R a k-algebra that is a domain. Suppose that $0 \neq a, b \in R$ are such that $aR \cap bR = 0$. Show that $k[a, b]$, the subalgebra generated by ab and b^2, is isomorphic to the free algebra on two variables.

4.6 Fix $0 \neq \lambda \in k$ and consider $R = k\langle x, y \rangle / (yx - \lambda xy)$. Show that $\{x^i y^j \mid i, j \geq 0\}$ is a basis for R. [*Hint:* use the result in Proposition 4.8.]

1.5 Differences from the commutative world

The theory of non-commutative rings is very different from that of commutative rings.

Let M and N be right modules over a ring R. If R is commutative, then $\operatorname{Hom}_R(M, N)$ can be given the structure of an R-module by defining

$$(\theta . r)(m) = \theta(m) . r$$

for $\theta : M \to N$, $m \in M$, and $r \in R$. If R is not commutative this will not work because $(\theta.r)(mx) = \theta(mx).r = \theta(m).xr$ need not be the same as $\theta(m).rx = ((\theta.r)(m)).x$. If R is commutative, and M and N are annihilated by ideals I and J, then $\mathrm{Hom}_R(M,N)$ is annihilated by $I+J$. Hence, the support of $\mathrm{Hom}_R(M,N)$ is contained in $(\mathrm{Supp}\, M) \cap (\mathrm{Supp}\, N)$. This extends to Ext-groups too; for example, if M and N have disjoint supports, then $\mathrm{Ext}^i_R(M,N) = 0$ for all i. It is obvious that this is a useful fact. When R is not commutative one cannot do this.

When R is commutative the tensor product $M \otimes_R N$ is an R-module. When R is not commutative one cannot even tensor together two modules unless one is a left module and the other a right module. Thus, when studying the category of right R-modules one of the basic operations, tensor product, can not be performed.

The polynomial ring $k[x_1, \ldots, x_n]$ is a free object in the category of commutative k-algebras in the sense that the left adjoint to the forgetful functor from k-algebras to k-vector spaces sends the n-dimensional vector space to the polynomial ring. More precisely, if $S(V)$ denotes the symmetric algebra on a k-vector space V and A is a k-algebra, then

$$\mathrm{Hom}_{k\text{-alg}}(S(V), A) \cong \mathrm{Hom}_k(V, A).$$

In the category of k-algebras the corresponding free object is the free algebra $k\langle x_1, \ldots, x_n\rangle$. We will make some simple comparisons between the polynomial ring $R = k[x_1, \ldots, x_n]$ and the free algebra $F = k\langle x_1, \ldots, x_n\rangle$, the main point of these comparisons being that the free algebra is not very much like the polynomial ring. Naturally, this leads to difficulties.

If $n \geq 2$, the center of F is k. Although R is noetherian, for $n \geq 2$, F is not. All simple R-modules are finite dimensional and are in bijection with the maximal ideals of R. If $n \geq 2$, then F has infinite dimensional simple modules. If $\mathcal{O}_p$ and $\mathcal{O}_q$ are non-isomorphic simple R-modules, then $\mathrm{Ext}^i_R(\mathcal{O}_p, \mathcal{O}_q) = 0$ for all integers i; this need not be true for F. The global homological dimension of R is n; the global homological dimension of F is 1 for all $n \geq 1$. Every right ideal of F is free. If $m > n$, then $k[x_1, \ldots, x_m]$ is not isomorphic to a subring of $k[x_1, \ldots, x_n]$. If $m > n \geq 2$, then $k\langle x_1, \ldots, x_m\rangle$ is isomorphic to a subring of $k\langle x_1, \ldots, x_n\rangle$.

One might hope that the differences between non-commutative and commutative rings would be less pronounced if one restricted attention to noetherian rings. Sadly, that is not so.

The enveloping algebras of finite dimensional Lie algebras are like polynomial rings in several ways. Let $\mathfrak{g}$ be a Lie algebra of dimension n over k, and $U(\mathfrak{g})$ its enveloping algebra. If $x_1, \ldots, x_n$ is a basis for $\mathfrak{g}$, the Poincaré-Birkhoff-Witt Theorem says that $U(\mathfrak{g})$ has a basis $\{x^I = x_1^{i_1} \ldots x_n^{i_n} \mid I = (i_1, \ldots, i_n) \in \mathbb{N}^n\}$. If $\mathfrak{g}$ is abelian, meaning that $[x,y] = 0$ for all x and y, then $U(\mathfrak{g}) \cong S(\mathfrak{g})$. The subspaces

$$U(\mathfrak{g})_d := k\text{-linear span of } \{x^I \mid |I| = i_1 + \cdots + i_n \leq d\}$$

provide an ascending filtration of $U(\mathfrak{g})$ and the associated graded ring is isomorphic to the symmetric algebra $S(\mathfrak{g})$. It follows that $U(\mathfrak{g})$ is a noetherian domain. The global homological dimension of $U(\mathfrak{g})$ is n.

If $\mathfrak{g}$ is the two-dimensional non-abelian Lie algebra over a field of characteristic zero, then the finite dimensional simple modules may be parametrized by the affine line $\mathbb{A}^1$. All these modules have dimension one. If $\mathcal{O}_p$ is the simple module corresponding to $p \in \mathbb{A}^1$, then

$$\mathrm{Ext}^1_{U(\mathfrak{g})}(\mathcal{O}_p, \mathcal{O}_p) \cong \mathrm{Ext}^1_{U(\mathfrak{g})}(\mathcal{O}_p, \mathcal{O}_{p+1}) \cong k.$$

If $\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra of 2×2 trace zero matrices over the complexes, then $U(\mathfrak{sl}(2,\mathbb{C}))$ has exactly one simple module of each finite dimension $n \geq 1$, and every finite-dimensional $U(\mathfrak{sl}(2,\mathbb{C}))$-module is semi-simple.

If $\mathfrak{g}$ is the 3-dimensional Heisenberg Lie algebra over a field of characteristic zero, and z is a non-zero central element in $\mathfrak{g}$, then $U(\mathfrak{g})/(z-1)$ has no non-zero finite-dimensional modules despite being the "same size" as the polynomial ring in two variables. Indeed, this ring has no two-sided ideals except itself and zero.

There are other ways to see that the restriction to noetherian rings will not be a panacea. For example, there are noetherian finitely generated k-algebras A and B such that $A \otimes_k B$ is not noetherian. A noetherian k-algebra A is *strongly noetherian* if $A \otimes_k C$ is noetherian for all commutative, noetherian k-algebras C. Hilbert's Basis Theorem implies that finitely generated commutative k-algebras are strongly noetherian. But non-commutative k-algebras are not; D. Rogalski has given an example, and his example is in many ways a very nice ring.

The previous paragraph shows that base change and fiber products for non-commutative spaces will not behave as well as they do in algebraic geometry. This is already apparent from categorical considerations. The tensor product $-\otimes_k -$ is the coproduct in the category of commutative k-algebras. However, the coproduct in the category of k-algebras, called the *free coproduct* and denoted by $*$, is such that $k[x] * k[y] \cong k\langle x, y\rangle$, the free algebra.

One of the grand themes of commutative algebra involves the interplay between the geometric notion of dimension (defined, for example, through Krull dimension) and various homological notions of dimension, defined, for example, in terms of projective dimension, or injective dimension, or depth, or grade. Much of this fails for non-commutative rings, and even when there are analogues of the commutative results they tend to be weaker and/or more complicated.

There are several notions of dimension for non-commutative rings. Each has strengths and weaknesses, and which is most useful tends to depend on the problem being considered. One can copy the commutative definition of Krull dimension in terms of chains of prime ideals but a non-commutative ring typically has rather few two-sided ideals so this definition is of limited utility. It is however useful for rings that are close to being commutative, for example, those that are finitely modules over their centers. P. Gabriel defined a notion of Krull dimension for objects in an arbitrary abelian category. For noetherian modules over a commutative noetherian ring Gabriel's definition agrees with the usual commutative definition (given in terms of the dimension of the support

of the annihilator of M). A big problem with Gabriel's Krull dimension is that it tends to be difficult to compute, and does not seem to be closely tied to homological notions of dimension unless the ring is close to commutative. Still, it is a useful inductive tool.

Gelfand-Kirillov dimension (introduced in the mid-sixties) is defined for algebras over a field, and for modules over such algebras. Although usually easier to compute than Krull dimension, GK-dimension can still be extremely difficult to compute for algebras defined in terms of generators and relations unless those relations have some special symmetries.

GK-dimension is defined in terms of how fast a ring can be generated—if A is a k-algebra and $V \subset A$ a finite dimensional subspace that contains 1 and generates A, then $\operatorname{GKdim} A = n$ if $\dim_k V^d$ grows like a polynomial of degree n; i.e., $\dim_k V^d \sim cd^n$ for some constant c. The commutative polynomial ring $k[x_1, \dots, x_n]$ has GK-dimension n, and similarly $\operatorname{GKdim} U(\mathfrak{g}) = \dim \mathfrak{g}$. If $n \geq 2$, then $\operatorname{GKdim} k\langle x_1, \dots, x_n\rangle = \infty$ because $\dim_k V^d$ grows like n^d.

If W is a finite dimensional subspace of an A-module M such that $M = WA$, then $\operatorname{GKdim} M$ is t if $\dim_k WV^d$ grows like a polynomial of degree t. The GK- and Krull dimensions of a finitely generated module over a finitely generated commutative k-algebra are the same. The GK-dimension of the ring $k\langle x, y\rangle/(xy - yx - 1)$ is two; its Krull dimension in Gabriel's sense is one when $\operatorname{char} k = 0$ and two when $\operatorname{char} k \neq 0$; its global homological dimension is one when $\operatorname{char} k = 0$ and two when $\operatorname{char} k \neq 0$; a basis for this ring is $\{x^i y^j \mid i, j \geq 0\}$. Should we consider $k\langle x, y\rangle/(xy - yx - 1)$ as being one- or two-dimensional?

Localization is a basic tool in commutative algebra: if $\mathcal{S}$ is a multiplicatively closed set of regular elements in a commutative ring R, there is a ring $R[\mathcal{S}^{-1}]$ and an injective ring homomorphism $R \to R[\mathcal{S}^{-1}]$ satisfying certain natural universal conditions. The ring $R[\mathcal{S}^{-1}]$ is a flat R-module and is not too much larger than R. Most of this fails in the non-commutative setting. But one has Goldie's theorem: a prime noetherian ring R has a ring of fractions $\operatorname{Fract} R$, meaning that there is an injective ring homomorphism $R \to \operatorname{Fract} R$ and every element of $\operatorname{Fract} R$ is of the form ab^{-1} for suitable elements $a, b \in R$; moreover, there is a division ring D and an integer n such that $\operatorname{Fract} R \cong M_n(D)$. Another way of looking at this is that prime noetherian rings are subrings of matrix rings over division rings. The real problem with localization in the non-commutative setting is this—even if x is a regular element in R, meaning that xy and yx are non-zero whenever y is non-zero, one cannot localize R at $\{1, x, x^2, \dots\}$ to obtain a ring $R[x^{-1}]$ that consists of elements ax^n, $a \in R$, $n \in \mathbb{Z}$, and contains a copy of R.

Nevertheless, one can always localize in the categorical sense.

For many purposes the most important object associated with a noncommutative ring is its category of modules; often one should think of the category of modules as the fundamental object and the ring as the secondary object. It is possible for non-isomorphic rings to have equivalent catgeories of modules. Such rings are said to be Morita equivalent. One quick way to see the flexibility provided by making the category rather than the ring as the primary object is to consider ring homomorphisms. There are not enough homomorphisms be-

tween non-commutative rings. For example, if R is a simple ring every non-zero homomorphism $R \to S$ is necessarily injective; if S is "more commutative" than R, there are *no* ring homomorphisms $R \to S$. This problem was recognized long ago. However, there are usually many functors between $\mathsf{Mod}S$ and $\mathsf{Mod}R$. For example, each R-S-bimodule ${}_RA_S$ gives an adjoint pair of such functors $-\otimes_R A$ and $\mathrm{Hom}_S(A,-)$.

If R and S are Morita equivalent then there is an R-S-bimodule A such that the equivalence is given by $-\otimes_R A : \mathsf{Mod}R \to \mathsf{Mod}S$. Bimodules and the adjoint pairs of functors associated to them play a prominent role in non-commutative algebra and non-commutative geometry.

The algebraic-geometric analogues of bimodules are correspondences. One can put aside the category of rings and ring homomorphisms and form instead the category of rings and bimodules. Doing so immediately gives a formalism that is close to that of affine algebraic geometry; to strengthen the analogy we introduce the notation $\mathrm{Sp}\,R = \mathsf{Mod}R$, and say that a morphism $\mathrm{Sp}\,R \to \mathrm{Sp}\,S$ is an S-R-bimodule A, and the composition of two such morphisms

$$\mathrm{Sp}\,R \xrightarrow{\ A\ } \mathrm{Sp}\,S \xrightarrow{\ B\ } \mathrm{Sp}\,T$$

is the T-R-bimodule $B\otimes_S A$. This is how to define a category of noncommutative affine spaces.

Because localization works poorly for noncommutative rings, noncommutative algebraic geometry is *not* the study of ringed spaces. There are, of course, situations in which that machinery is useful, most notably in the theory of $\mathcal{D}$-modules, differential operators on schemes. In that situation, $\mathcal{O}_X$ is a subsheaf of $\mathcal{D}_X$ and one localizes by inverting elements of $\mathcal{O}_X$, and this works well because sections of $\mathcal{O}_X$ act ad-nilpotently on sections of $\mathcal{D}_X$. Sheaves of non-commutative algebras also arise as $\mathcal{E}nd_X\mathcal{F}$ where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Important examples occur in this way, but to build non-commutative algebraic geometry on a foundation of ringed spaces turns out to be too restrictive.

Regular sequences are another standard tool of commutative algebra that is of limited use in the non-commutative setting because it is rather rare for a right ideal xR to also be a left ideal.

Despite the difficulties described, homological methods work rather well for non-commutative rings. So too do many of the tools of projective algebraic geometry. In some sense, these are global methods and one can say that global methods seem to be more appropriate for non-commutative algebra. To some extent this explains why the framework for non-commutative algebraic geometry is of a categorical nature: $\mathsf{Mod}R$ is a global object, and if one is interested only in operations on categories without being concerned whether the operations and new categories produced can be described in terms of rings, then one has a whole range of robust techniques.

1.6 Rapid non-commutative algebraic geometry

Imagine a race of brilliant aliens who have no need for theorems. Once these aliens know the definitions they immediately "know" all the consequences; these aliens have no need for theorems and proofs, all they need are the definitions. There are some amongst us who, although they appear in human form, are really such aliens—I could name names, but that might be unwise. Here is a fast course in non-commutative algebraic geometry for such aliens, with some commentary interspersed to enlighten we poor humans.

A Grothendieck category is a category that is equivalent to a quotient category of some $\mathsf{Mod}R$ by a localizing subcategory. A non-commutative algebraic space X, or space for short, is a Grothendieck category $\mathsf{Mod}X$. Objects in $\mathsf{Mod}X$ are called X-modules. The space is affine if $\mathsf{Mod}X$ is equivalent to $\mathsf{Mod}R$ for some ring R. We call R a coordinate ring of X. A closed subspace Y of X is a full subcategory $\mathsf{Mod}Y$ of $\mathsf{Mod}X$ such that every submodule and quotient of a Y-module is a Y-module, and such that the inclusion functor $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ has both a left adjoint, denoted i^*, and a right adjoint, denoted $i^!$. We write $i : Y \to X$, and call this the inclusion of Y in X.

The aliens know that the study of an algebraic variety, or scheme, X is for the most part the study of the category $\mathsf{Qcoh}X$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules. They also know that $\mathsf{Qcoh}X$ is a Grothendieck category for most reasonable X. And they know that X can be recovered from the category $\mathsf{Qcoh}X$, so for such an X one simply writes $\mathsf{Mod}X$ for $\mathsf{Qcoh}X$. A special case of this is that a commutative ring R can be recovered form $\mathsf{Mod}R$ as the ring of natural transformations of the identity functor, so, being brave and free, they take the liberty of writing $\operatorname{Spec} R \equiv \mathsf{Mod}R$.

The closed subspaces of an affine space $\mathsf{Mod}R$ are those of the form $\mathsf{Mod}R/I$ where I is a two-sided ideal, so this notion of closed agrees with the usual notion of closed subschemes in affine algebraic geometry.

A map $f : Y \to X$ of spaces is an adjoint pair of functors (f^*, f_*) such that $f_* : \mathsf{Mod}Y \to \mathsf{Mod}X$. We call f^* the inverse image functor, and f_* the direct image functor. Given an adjoint pair (f^*, f_*), the left adjoint of f_* is only determined up to natural equivalence, so one should really define a map as a natural equivalence class of adjoint pairs.

A ring homomorphism $\varphi : R \to S$ induces a map $f : \mathsf{Mod}S \to \mathsf{Mod}R$ with $f^* = - \otimes_R S$ and $f_* = \operatorname{Hom}_S(S, -)$; the functor f_* is "view an S-module as an R-module via φ". Thus a map of affine schemes in the usual sense gives a map in this new sense. More generally, any R-S-bimodule B induces a map $f : \mathsf{Mod}S \to \mathsf{Mod}R$ with $f^* = - \otimes_R B$ and $f_* = \operatorname{Hom}_S(B, -)$. The aliens also know that Watt's Theorem implies that every map $f : \mathsf{Mod}R \to \mathsf{Mod}S$ is induced by such a bimodule; one takes $B = f_*(R)$.

The aliens know that Grothendieck emphasised the point of view that algebraic geometry should be developed as a relative theory; that is, one has a base scheme, and then studies schemes over that base scheme. They therefore anticipate our next definition.

Let k be a commutative ring. A space X is called a k-space, or a space over

k if $\mathsf{Mod}X$ is a k-linear category, and there is a map $\pi : X \to \operatorname{Spec} k$ such that π_* and π^* are k-linear functors. We write $\mathcal{O}_X = \pi^* k$, and call it the structure module of X.

If R is a k-algebra, then the structure map $k \to R$ induces a map $\pi : \mathsf{Mod}R \to \mathsf{Mod}k \equiv \operatorname{Spec} k$, so $X = \mathsf{Mod}R$ is a k-space. With respect to the structure map π the structure module $\mathcal{O}_X$ is the coordinate ring viewed as a module because $\pi^* k = k \otimes_k R = R$.

A k-rational point of a k-space X is a closed subspace that is isomorphic to $\operatorname{Spec} k$.

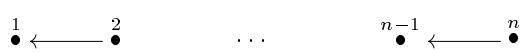
The only rings S for which $\mathsf{Mod}S$ is equivalent to $\mathsf{Mod}k$ are the matrix algebras $M_n(k)$, so the k-rational points of $\mathsf{Mod}R$ are the closed subspaces $\mathsf{Mod}R/I$ for which $R/I \cong M_n(k)$. Thus each k-rational point p determines a unique simple R-module, which we denote by $\mathcal{O}_p$, that is finite dimensional over k. If k is an uncountable algebraically closed field, and R is a finitely generated k-algebra, then there are natural bijections between the k-rational points of $\mathsf{Mod}R$, the isomorphism classes of finite dimensional simple R-modules, and the maximal two-sided ideals $\mathfrak{m}$ of R such that $\dim_k R/\mathfrak{m} < \infty$.

Because a Grothendieck category has injective resolutions, the usual homological tools can be used. In particular, if M and N are X-modules, one can define $\operatorname{Ext}^i_X(M, N)$ as the right derived functors of $\operatorname{Hom}_X(-, N)$, and compute them via injective resolutions of N. If X is a k-space via $\pi : X \to \operatorname{Spec} k$, then one can define cohomology groups $H^i(X, -)$ as $\operatorname{Ext}^i_X(\mathcal{O}_X, -)$, where $\mathcal{O}_X$ is the structure module $\pi^* k$.

The aliens have great powers of visualization, and having seen these definitions, they are able to conjure up mental images of non-commutative spaces. They like to think categorically, so to them a matrix algebra over a field k is "essentially commutative" because $\mathsf{Mod}M_n(k)$ is equivalent to $\mathsf{Mod}k$. Their first example of a non-commutative space is therefore $\mathsf{Mod}T$ where T is a lower triangular matrix ring.

Over a commutative ring R, the support of $\operatorname{Ext}^i_R(M, N)$ is contained in the intersection of the supports of M and N, so if p and q are distinct closed points of $\operatorname{Spec} R$, $\operatorname{Ext}^1_R(\mathcal{O}_p, \mathcal{O}_q) = 0$. A distinguishing feature of non-commutative spaces is that this fact is no longer true. The aliens pictures of non-commutative spaces contain this non-commutative data: if p and q are distinct points such that $\operatorname{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q) \neq 0$ they visualize an arrow $p \to q$ from the point p to the point q.

The $n \times n$ lower triangular matrix algebra over k has n simple modules $S_1, \ldots, S_n$ labelled so that the non-split extensions are $\operatorname{Ext}^1_T(S_{i+1}, S_i) \neq 0$. Their picture of $\mathsf{Mod}T$ is therefore

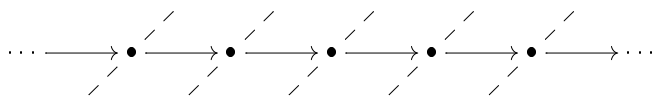


Another distinguishing feature of non-commutative, as opposed to commutative, algebra is that there can exist both finite dimensional and infinite dimensional simple modules. It is possible for a reasonable affine k-space $\mathsf{Mod}R$

to have no k-rational points at all. A simple example is the ring R of differential operators on the complex polynomial ring $\mathbb{C}[x]$.

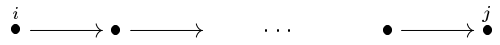
The aliens are not frightened by the prospect of a space with no points. They suggest we should be reassured by realizing that a space with no closed points has only one less closed point than the one-dimensional space Spec R where R is a discrete valuation ring. Also, as K. Schwede tells us, there are schemes having no closed points. The aliens are able to visualize a one-dimensional space with no points. They draw closed points in black, and use green, their favorite color, to draw one-dimensional spaces with no closed points. So their picture of Spec R consists of a short green curve with a single black point on it.

A similar example is given by the space $X = \mathsf{GrMod}k[x]$, the graded line, where the polynomial ring $k[x]$ is graded by giving x degree one. The closed points are the one-dimensional graded modules $k(n)$, $n \in \mathbb{Z}$, where $k = k[x]/(x)$, and $k(n)$ is k concentrated in degree $-n$. The exact sequence $0 \to (x)/(x^2) \to k[x]/(x^2) \to k[x]/(x) \to 0$ is non-split, so there are arrows $k(i) \to k(i-1)$. We make X a k-space in the obvious way, by setting $\mathcal{O}_X = k[x]$ with its generator in degree one. Each shift $\mathcal{O}_X(n)$ has a unique simple quotient, namely $k(n)$, so we can think of $\mathcal{O}_X(n)$ as representing a one-dimensional geometric object with a single point lying on it. So, the aliens' picture of X is



where the dashed lines would be green but for limitations of TEX.

The picture suggests the existence of various subspaces of X. One could pick any two points i and j and take the "subspace"



(6-3)

Define the subspace $Y \subset X$ by defining $\mathsf{Mod}Y$ to be the full subcategory of $\mathsf{Mod}X$ consisting of the modules M for which $M_n = 0$ if $n > j$ or $n < i$. This is a closed subspace. The left adjoint of the inclusion functor $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ sends a module M to M/N where N is minimal subject to M/N being in $\mathsf{Mod}Y$. Its right adjoint sends M to the largest submodule of M that is in $\mathsf{Mod}Y$.

It is clear that $\mathsf{Mod}Y$ is equivalent to the category of representations of the quiver in (6-3). This is equivalent to the category of modules over the $n \times n$ lower triangular matrices where $n = j - i + 1$.

The aliens favorite non-commutative spaces are ones that are close to commutative varieties. These have coordinate rings of the form $A = k[x_1, \ldots, x_n]$ generated by a finite number of non-commuting elements x_i and have the property that they are finite modules over their centers. Their centers are then of the form $R = k[z_1, \ldots, z_m]$ for some commuting elements $z_1, \ldots, z_m$. They tells us that the points of Sp A are the maximal ideals of A, and that the rule

$\mathfrak{m} \mapsto R \cap \mathfrak{m}$ gives a map $\operatorname{Sp} A \to \operatorname{Max} R = X$, where X denotes the variety corresponding to R. They also know that this map is bijective almost everywhere, and has finite fibers; the fiber over $\mathfrak{n} \in \operatorname{Max} R$ is $\operatorname{Sp} A/A\mathfrak{n}A$, and $A/A\mathfrak{n}A$ is a finite-dimensional k-algebra.

String theory is part of third-grade education for the aliens.