Graded Rings and Geometry

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Abstract. blah, blah

CHAPTER 1

Quotient categories

1. Introduction

You are already familiar with the formation of quotient groups and quotient rings. There is an analogous notion of a quotient category. Roughly speaking, one forms a quotient category by making some collection of objects in a category isomorphic to zero. For example, if Z is a closed subvariety of a variety X, one can form a quotient of $\mathsf{Qcoh}X$ in which all the $\mathcal{O}_Z$-modules become isomorphic to zero—see Theorem 2.1. Just as a quotient group has a universal property[1], so does a quotient category.

Given an abelian category A and a suitable subcategory T, there is an abelian category A/T and functors

$$\mathsf{A} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathsf{A}/\mathsf{T}$$

with the following properties:

- π^* is left adjoint to π_*;
- π^* is an exact functor;
- $\pi^* M \cong 0$ if and only if M is in T;
- $\pi^*\pi_* \cong \mathrm{id}$;
- if B is an abelian category and $F : \mathsf{A} \to \mathsf{B}$ a functor such that $F(M) \cong 0$ for all M in T, then there is a unique functor $\alpha^* : \mathsf{A}/\mathsf{T} \to \mathsf{B}$ such that $F = \alpha^*\pi^*$.

Localizations of rings and modules provide important examples of quotient categories. Let R be a commutative ring and $z \in R$. The natural homomorphism $\phi : R \to R_z$ induces functors

$$\mathsf{Mod}R \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathsf{Mod}R_z$$

where $f^*M = M \otimes_R R_z$ and f_*N is simply N viewed as an R-module via ϕ. Now π^*M is zero if and only if every element of M is annihilated by a power of f. Let T be the full subcategory of $\mathsf{Mod}R$ consisting of such modules. Then $\mathsf{Mod}R_z$ is equivalent to the quotient $(\mathsf{Mod}R)/\mathsf{T}$. You already know the following:

- the functor $- \otimes_R R_z$ is left adjoint to $\mathrm{Hom}_{R_z}(R_z, -) = f_*$;

[1] If N is a normal subgroup of G and $\theta : G \to G'$ is a group homomorphism such that $\theta(N) = \mathrm{id}$, there is a unique homomorphism $\alpha : G/N \to G'$ such that $\theta = \alpha\pi$ where $\pi : G \to G/N$ is the quotient map.

- localization of modules is an exact functor;
- if $N \in \mathsf{Mod} R_z$, then $N_z = N$;
- if $\psi : R \to S$ is a ring homomorphism such that $\psi(f)$ is a unit in S, then there is a unique homomorphism $\rho : R_z \to S$ such that $\psi = \rho\phi$.

In this situation, if $X = \operatorname{Spec} R$ and Z is the zero locus of f, then $\mathsf{Mod} R = \mathsf{Qcoh} X$ and $\mathsf{Mod} R_z = \mathsf{Qcoh}(X - Z)$.

The fact that $X - Z$ is an affine scheme is because Z is the zero locus of a single function.

In contrast, $\mathbb{C}^2 \backslash \{0\}$ is not an affine variety, so there is no ring obtained by inverting elements of $\mathbb{C}[X, Y]$ that adequately captures the geometry of $\mathbb{C}^2 \backslash \{0\}$. Nevertheless, we can take a quotient of $\mathsf{Mod}\mathbb{C}[X, Y]$ to obtain the modules on $\mathbb{C}^2 \backslash \{0\}$.

2. Application

We will use the universal property of the quotient category to prove the following result.

THEOREM 2.1. *Let Z be a closed subscheme of a scheme X. Then there is an equivalence of categories*

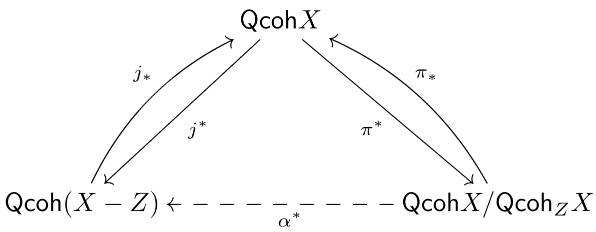
$$\mathsf{Qcoh}(X - Z) \equiv \frac{\mathsf{Qcoh} X}{\mathsf{Qcoh}_Z X}$$

where the right-hand-side denotes the quotient category by the full subcategory consisting of the modules whose support is contained in Z.

Proof. Associated to the open immersion

$$j : X - Z \to X$$

are its inverse and direct image functors, j^* and j_*, which fit into the following diagram:



We must prove the existence of an equivalence α^*.

Since j^* is left adjoint to j_* there are natural transformations

$$\varepsilon : \mathrm{id} \to j_* j^* \qquad \text{and} \qquad \eta : j^* j_* \to \mathrm{id}\,.$$

If $\mathcal{F}$ is a sheaf on $X - Z$, then its extension to X followed by its restriction to $X - Z$ gives back $\mathcal{F}$ so

$$j^* j_* \cong \mathrm{id}\,.$$

On the other hand, for each $M \in \mathsf{Qcoh} X$ there is an exact sequence

$$0 \longrightarrow \ker \varepsilon_M \longrightarrow M \xrightarrow{\varepsilon_M} j_* j^* M \longrightarrow \operatorname{coker} \varepsilon_M \longrightarrow 0.$$

Because j is an open immersion, j^* is an exact functor, so applying it to this sequence produces an exact sequence

$$0 \longrightarrow j^*\big(\ker \varepsilon_M\big) \longrightarrow j^*M \xrightarrow{j^*(\varepsilon_M)} j^*j_*j^*M \longrightarrow j^*\big(\operatorname{coker} \varepsilon_M\big) \longrightarrow 0.$$

in $\mathsf{Qcoh}(X-Z)$. However, $j^*j_* \cong \mathrm{id}$, so $j^*(\varepsilon_M)$ is an isomorphism. The exactness of the sequence therefore implies that

$$j^*\big(\ker \varepsilon_M\big) \;=\; j^*\big(\operatorname{coker} \varepsilon_M\big) = 0.$$

Since the restrictions of $\ker(\varepsilon_M)$ and $\operatorname{coker}(\varepsilon_M)$ to $X-Z$ are zero they belong to $\mathsf{Qcoh}_Z X$.

We now apply the universal property of the quotient category to deduce the existence of a functor α^* such that

$$j^* = \alpha^* \pi^*.$$

It follows that

$$\alpha^* \circ \pi^* j_* = j^* j_* \cong \mathrm{id}\,. \tag{2-1}$$

Because $\ker \varepsilon_M$ and $\operatorname{coker} \varepsilon_M$ are supported on Z, π^* vanishes on both these modules. Because π^* is exact it follows that $\pi^*(\varepsilon_M) : \pi^*M \to \pi^*j_*j^*M$ is an isomorphism. In other words,

$$\pi^* j_* j^* \cong \pi^*.$$

Hence

$$\pi^* j_* \circ \alpha^* \cong \pi^* j_* \circ \alpha^* \circ \pi^* \pi_* \cong \pi^* j_* \circ j^* \pi_* \cong \pi^* \pi_* \cong \mathrm{id}\,. \tag{2-2}$$

It follows from (2-1) and (2-2) that α^* is an equivalence of categories. □

3. A look ahead: projective schemes

The projective space $\mathbb{P}^n_k$ over a field k is by definition the set of lines through the origin in k^{n+1}. A little more formally, it is the orbit space

$$\frac{k^{n+1} - \{0\}}{k^\times}$$

for the natural action of the multiplicative group $k^\times$.

Let $S = k[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables with its standard $\mathbb{Z}$-grading, i.e.,

$$\deg x_i = 1$$

for all i. If k is infinite, e.g., if it is algebraically closed, a function $f \in S$ is constant on the orbits if and only if it is homogeneous. In a similar fashion an S-module is "constant on the orbits" if and only if it is a graded S-module. The graded S-modules form an abelian category, $\mathsf{GrMod}S$. Associated to a graded S-module M is a quasi-coherent sheaf $\widetilde{M}$ on $\mathbb{P}^n$. However, if M is supported at zero, i.e., if every element of M is annihilated by a suitably high power of $\mathfrak{m} = (x_0, \dots, x_n)$, then $\widetilde{M} = 0$. Thus, if T is the subcategory of $\mathsf{GrMod}S$ consisting of such M the functor $M \mapsto \widetilde{M}$ factors through the quotient category $\mathsf{GrMod}S/\mathsf{T}$. Quotienting out T is the algebraic analogue of removing the origin from k^{n+1} before forming $\mathbb{P}^n$.

THEOREM 3.1 (Serre). [2] *There is an equivalence of categories*

$$\mathsf{Qcoh}\mathbb{P}^n \equiv \frac{\mathsf{GrMod}S}{\mathsf{T}}.$$

In other words, there is an adjoint pair of functors

$$\mathsf{GrMod}S \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathsf{Qcoh}\mathbb{P}^n$$

with π^* exact and $\pi^*\pi_* = \mathrm{id}$. Hence, to prove results about sheaves $\mathcal{F}$ on $\mathbb{P}^n$ one can often work with the graded S-module $\pi_*\mathcal{F}$. Likewise, results about graded S-modules give results about quasi-coherent sheaves on $\mathbb{P}^n$. For example, let $S(i)$ denote the graded S-module which is S but with $1 \in S$ placed in degree i. In the category of graded S-modules we have an isomorphism $S(i) \otimes_S S(j) \cong S(i+j)$ which, after applying π^*, gives

$$\mathcal{O}(i) \otimes \mathcal{O}(j) \cong \mathcal{O}(i+j).$$

In particular, $\mathcal{O}_{\mathbb{P}^n} = \pi^*S$. The cohomology sheaves $H^i(\mathbb{P}^n, \mathcal{F})$ are by definition the right derived functors of the global section functor $\Gamma(\mathbb{P}^n, -)$ which is, of course, equal to $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, -)$. But $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{F}) = \mathrm{Hom}(\pi^*S, \mathcal{F})$ which is isomorphic to $\mathrm{Hom}_S(S, \pi_*\mathcal{F})$ by the adjointness. Now, $\mathcal{F} = \pi^*M$ for some graded S-module M so, since $\mathrm{Hom}_S(S, -)$ is an exact functor, the $H^i(\mathbb{P}^n, \mathcal{F})$s can be computed in terms of the right derived functors of $\pi_*\pi^*$ applied to M. For $i \geq 1$,

$$H^i(\mathbb{P}^n, \mathcal{F}) \cong H^{i+1}_{\mathfrak{m}}(M)$$

the local cohomology of M at the maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$.

Looking further ahead, we will find that this idea of passing back and forth between sheaves and graded modules is fruitful in a wide range of situations. For example, smooth toric varieties when the grading group is $\mathbb{Z}^n$ and certain smooth Deligne-Mumford stacks where the grading group is now allowed to have torsion.

For example, to compute the Picard group of the moduli stack $\mathcal{M}_{1,1}$ of pointed elliptic curves reduces to a computation involving the polynomial ring in two variables of degrees 4 and 6. The answer is $\mathbb{Z}/12$ and one eventually sees that the proof of this uses the fact that the polynomial ring is a UFD and that $lcm(4, 6) = 12$.

4. Serre subcatgeories

Throughout this section A denotes an abelian category.

Definition 4.1. A non-empty full subcategory T of an abelian category A is a Serre subcategory if, for all short exact sequences $0 \to M' \to M \to M'' \to 0$ in A, M belongs to T if and only if both M' and M'' do. Objects in T are said to be torsion and an object is torsion-free if the only subobject of it belonging to T is zero[3]. ◊

[2] Serre's theorem is more general than this: there is an equivalence in which S is replaced by any quotient S/I in which I is a graded ideal $\mathbb{P}^n$ is replaced by the scheme-theoretic zero locus of I.

[3] Thus zero is the only object which is both torsion and torsion-free.

For the rest of this section T will denote a Serre subcategory of A.

Since T is closed under subobjects and quotients the inclusion $\mathsf{T} \to \mathsf{A}$ preserves kernels and cokernels; in other words kernels and cokernels in T agree with those in A. Since T is closed under extensions it is closed under finite direct sums and products, and these agree with those in A. It follows that T is an abelian category, and the inclusion functor is exact.

If M_1 and M_2 are submodules of M that are torsion, so is their sum since it is a quotient of $M_1 \oplus M_2$.

Example 4.2. Let $\mathcal{S}$ be a multiplicatively closed subset of a commutative ring R. We say that a module is $\mathcal{S}$-torsion if every element in it is annihilated by an element of $\mathcal{S}$. The $\mathcal{S}$-torsion modules form a Serre subcategory of $\mathsf{Mod}R$.

More generally, suppose that R is a ring having ring of fractions $\operatorname{Fract} R$ and S is an intermediate ring, $R \subset S \subset \operatorname{Fract} R$. If ${}_RS$ is flat, then $\{M \in \mathsf{Mod}R \mid M \otimes_R S = 0\}$ is a Serre subcategory of $\mathsf{Mod}R$. ♢

The general principle behind Example 4.2 is that if $F : \mathsf{A} \to \mathsf{B}$ is an exact functor, then the full subcategory of A consisting of those M such that $FM = 0$ is a Serre subcategory.

5. Direct limits of abelian groups

A direct limit is a generalization of the union of an ascending chain of sets but the maps from one set to the next need not be injective.

A directed system of abelian groups is a collection of abelian groups G_n, $n \in \mathbb{Z}$, and homomorphisms $\phi_n : G_n \to G_{n+1}$. When $m \le n$ we define

$$\phi_{mn} := \phi_{n-1} \circ \ldots \circ \phi_m : G_m \to G_n.$$

A direct limit of (G_n, ϕ_n) is an abelian group G together with homomorphisms $\Phi_n : G_n \to G$ such that

(1) $\Phi_m = \Phi_n \circ \phi_{mn}$ whenever $m \le n$;
(2) if $\Phi'_n : G_n \to G'$, $n \in \mathbb{Z}$, are homomorphisms such that $\Phi'_m = \Phi'_n \circ \phi_{mn}$ whenever $m \le n$, then there is a unique homomorphism $\rho : G \to G'$ such that $\Phi'_n = \rho \Phi_n$ for all n.

The direct limit, which exists by the next result, is denoted by

$$\varinjlim G_n.$$

Proposition 5.1. *Direct limits of abelian groups exist and are unique up to unique isomorphism.*

Proof. Let (G_n, ϕ_n) be a directed system.

Write

$$P := \bigoplus_{n \in \mathbb{Z}} G_n$$

and let $\iota_n : G_n \to P$ be the natural inclusion. Let N be the subgroup of P generated by all elements of the form

$$\iota_n(g) - \iota_{n+1}\phi_n(g), \quad g \in G_n, \ n \in \mathbb{Z}.$$

Let $\pi : P \to P/N$ be the natural map.

Define $\Phi_n := \pi \iota_n$. We will show that $(P/N, \Phi_n)$ is a direct limit.

Let $m \leq n$. Since $\pi\iota_{j+1}\phi_j = \pi^*\iota_j$, it follows that

$$\Phi_n\phi_{mn} = \pi\iota_n\phi_{n-1}\dots\phi_m = \pi^*\iota_{n-1}\phi_{n-2}\dots\phi_m = \dots = \pi\iota_m = \Phi_m.$$

Now we show that $(P/N, \Phi_n)$ has the required universal property. Suppose that (G', Φ'_n) satisfies the hypothesis in condition (2) above. By the universal property of the direct sum there is a unique homomorphism $\psi : P \to G'$ such that $\Phi'_n = \psi\iota_n$. Now $\psi(N) = 0$ because

$$\psi\Big(\iota_n(g) - \iota_{n+1}\phi_n(g)\Big) = \Phi'_n(g) - \Phi'_{n+1}\phi_n(g) = 0.$$

Hence there is a homomorphism $\rho : P/N \to G'$ such that $\psi = \rho\pi$. It follows that $\Phi'_n = \psi\iota_n = \rho\pi\iota_n = \rho\Phi_n$.

We leave the proof of uniqueness to the reader. □

There is nothing special about the role of the integers as the indexing set. It can be replaced by any directed set[4]. For example, let $\mathfrak{p}$ be a prime ideal in a commutative ring R and make $R - \mathfrak{p}$ a directed set by declaring that $f \leq g$ if $f|g$; there is a homomorphism $R_f \to R_g$ whenever $f \leq g$ and it is easy to see that $R_\mathfrak{p}$ is the direct limit of the rings R_f where f ranges over the elements in $R - \mathfrak{p}$.

We will need to take direct limits over directed sets.

Proposition 5.2. *Let (G_n, ϕ_n) be a directed system. Then*

(1) *every element in $\varinjlim G_n$ is the image of an element in some G_i;*
(2) *the image in $\varinjlim G_n$ of an element $g \in G_n$ is zero if and only if $\phi_{nm}(g) = 0$ for $m \gg n$.*

Proof. We retain the notation used in the previous proof.

(1) Let $x \in \varinjlim G_n$. Then x is the image of an element in the direct sum $\oplus G_n$, so there are elements $g_j \in G_j$, $m \leq j \leq n$, such that

$$x = \pi(\cdots, 0, 0, g_m, \dots, g_n, 0, 0, \cdots) = \sum_{j=m}^{n} \pi^*\iota_j(g_j) = \sum_{j=m}^{n} \Phi_j(g_j).$$

But $\Phi_j(g_j) = \Phi_n\phi_{jn}(g_j)$, so

$$x = \Phi_n\Bigg(\sum_{j=m}^{n} \phi_{jn}(g_j)\Bigg).$$

(2) If $\phi_{nm}(g) = 0$, then $\Phi_n(g) = \Phi_m\phi_{nm}(g) = 0$; i.e., the image of g in $\varinjlim G_n$ is zero.

To prove the converse, let $g \in G_n$ and suppose that $\Phi_n(x) = 0$. Then $\iota_n(g) \in N$ so there is a finite set of elements $g_j \in G)j$ such that

$$\begin{aligned}\iota_n(g) &= \sum \iota_j(g_j) - \iota_{j+1}\phi_j(g_j)\\ &= \sum \iota_j(g_j) - \iota_j\phi_{j-1}(g_{j-1})\\ &= \sum \iota_j\big(g_j - \phi_{j-1}(g_{j-1})\big).\end{aligned}$$

It follows that

$$\iota_n(g) = \iota_n\big(g_n - \phi_{n-1}(g_{n-1})\big) \qquad \text{and} \qquad \iota_j\big(g_j - \phi_{j-1}(g_{j-1})\big) = 0 \text{ when } j \neq n.$$

[4]A partially ordered set I is directed if given $i, j \in I$, there is $k \in I$ such that $i \leq k$ and $j \leq k$.

But ι_j is injective, so $g_j = \phi_{j-1}(g_{j-1})$ when $j \neq n$. Also $g_{j-1} = 0$ for $j \ll 0$, so $g_j = 0$ for all $j \leq n-1$. Hence $\iota_n(g) = \iota_n(g_n)$. It follows that $g_{n+1} = \phi_n(g_n) = \phi_n(g)$ and, by induction, $g_{n+k} = \phi_{n,n+k}(g)$ for all $k \geq 1$. However, $g_{n+k} = 0$ for $k \gg 0$, so $\phi_{nm}(g) = 0$ for $m \gg n$. □

6. The quotient functor

LEMMA 6.1. *Fix M and N in A and let I be the set*[5] *of all pairs (M', N') of submodules $M' \subset M$ and $N' \subset N$, such that M/M' and N' are torsion. Then I is a directed set with respect to the partial order*

$$(M', N') \leq (M'', N'') \qquad \textit{if } M'' \subset M' \textit{ and } N' \subset N''.$$

Proof. If (M_1, N_1) and (M_2, N_2) belong to I so does $(M_1 \cap M_2, N_1 + N_2)$; but this element of I is $\geq$ both (M_1, N_1) and (M_2, N_2). □

Exercise: Show that $M/M_1 \cap M_2$ is torsion if M/M_1 and M/M_2 are.

Definition 6.2. Let A be an abelian category and T a Serre subcategory. The quotient category A/T is defined as follows:

- its objects are the objects of A;
- if M and N are A-modules then

(6-1) $$\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, N) := \varinjlim \operatorname{Hom}_{\mathsf{A}}(M', N/N'),$$

  where the direct limit is taken over the set I in Lemma 6.1;
- the composition of morphisms in A/T is induced by that in A. ◊

PROPOSITION 6.3. *Definition 6.2 makes sense.*

Proof. First, the direct limit in (6-1) exists. Fix M and N in $\mathsf{A}$and let I be the directed set in Lemma 6.1. If $(M', N') \leq (M'', N'')$, the natural maps $M'' \to M'$ and $N/N' \to N/N''$ induce maps

$$\operatorname{Hom}_{\mathsf{A}}(M', N/N') \to \operatorname{Hom}_{\mathsf{A}}(M'', N/N') \to \operatorname{Hom}_{\mathsf{A}}(M'', N/N'').$$

Thus $\operatorname{Hom}(M', N/N')$ is a directed system of abelian groups indexed by I so has a direct limit.

By the Remark at the end of section 5, every morphism in $\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, N)$ is the image of a morphism in $\operatorname{Hom}_{\mathsf{A}}(M', N/N')$ for some $(M', N') \in I$.

The composition of morphisms

$$\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(N, Z) \times \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, N) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, Z)$$

in A/T is defined as follows. Let $\bar{f} \in \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(N, Z)$ and $\bar{g} \in \operatorname{Hom}_{\mathsf{A},\mathsf{T}}(M, N)$. By the previous paragraph, $\bar{f}$ and $\bar{g}$ are images of morphisms $g : M' \to N/N'$ and $f : N'' \to Z/Z'$ in A where M/M', N', N/N'', and Z' belong to T. Define $M'' := g^{-1}(N' + N''/N')$, check that M/M'' is torsion, and define

$$g' : M'' \to N' + N''/N'$$

to be the restriction of g to M''. Both $f(N' \cap N'')$ and $Z'' := Z' + f(N' \cap N'')$ are torsion. Now define

$$f' : N''/N' \cap N'' \to Z/Z''$$

[5] One needs a hypothesis that A has a small set of generators to ensure that I really is a set—that hypothesis implies that the collection of subobjects of a given object is a small set.

to be the map induced by f. Define h to be the composition

$$M'' \xrightarrow{\;g'\;} \frac{N'+N''}{N'} \xrightarrow{\;\sim\;} \frac{N''}{N'\cap N''} \xrightarrow{\;f'\;} \frac{Z}{Z''},$$

where the middle map is the natural isomorphism. Finally, one checks that $\bar{h}$, the image of h in $\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M,Z)$, depends only on $\bar{f}$ and $\bar{g}$ and not on a choice of representatives f and g.

Third, $\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M,M)$ contains an identity morphism, namely the image of id_M in the direct limit. □

Since $(M,0) \in I$, $\operatorname{Hom}_{\mathsf{A}}(M,N)$ is one of the terms in the directed system. There is therefore a homomorphism

$$\operatorname{Hom}_{\mathsf{A}}(M,N) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M,N)$$

of abelian groups. It is straightforward to check that this map respects the composition of morphisms in A and A/T. In particular, it sends identity maps in A to identity maps in A/T.

Definition 6.4. Let T be a Serre subcategory of A. The quotient functor

$$\pi^* : \mathsf{A} \to \mathsf{A}/\mathsf{T}$$

is defined by $\pi^* M = M$ on objects, and $\pi^* f =$ the image of f in the direct limit, on morphisms. ◊

It follows from the definitions that A/T is an additive category and that π^* is an additive functor.

Lemma 6.5. *Let f be a morphism and M an object in A. Then*
(1) *$\pi^* f = 0$ if and only if the image of f is torsion;*
(2) *$\pi^* M \cong 0$ if and only if M is in T.*

Proof. (1) Let $f \in \operatorname{Hom}_{\mathsf{A}}(M,N)$. By Proposition 5.2, $\pi^* f = 0$ if and only if the image of f in some later term $\operatorname{Hom}_{\mathsf{A}}(M', N/N')$ of the directed system is zero. The image of f in this is the composition

$$M' \longrightarrow M \xrightarrow{\;f\;} N \longrightarrow \frac{N}{N'}$$

which is zero if and only if $fM' \subset N'$. Hence $\pi^* f = 0$ if and only if there are subobjects $M' \subset M$ and $N' \subset N$ such that M/M' and N' are torsion and $fM' \subset N'$; i.e., if and only if there is $M' \subset M$ such that M/M' and fM' are torsion. But fM/fM' is torsion whenever M/M' is torsion so the condition that fM' is torsion is equivalent to the condition that fM is torsion. Hence $\pi^* f = 0$ if and only if fM is torsion.

(2) An object in an abelian category is zero if and only if the identity map on it is zero. But $\mathrm{id}_{\pi^* M} = \pi^*(\mathrm{id}_M)$ so $\pi^* M = 0$ if and only if $\pi^*(\mathrm{id}_M) = 0$; by (1) this happens if and only if M is torsion. □

Proposition 6.6. *Let $f : M \to N$ be a morphism in A. Then*
(1) *$\pi^* f$ is epic if and only if* $\operatorname{coker} f$ *is torsion;*
(2) *$\pi^* f$ is monic if and only if* $\ker f$ *is torsion;*
(3) *the kernel and cokernel of $\pi^* f$ are* $\pi^*(\ker f)$ *and* $\pi^*(\operatorname{coker} f)$ *respectively;*
(4) *$\pi^* f$ is an isomorphism if and only if both* $\ker f$ *and* $\operatorname{coker} f$ *are torsion.*

Proof. Let $\iota : K \to M$ and $\eta : N \to N/fM$ be the kernel and cokernel of f.

(1) ($\Rightarrow$) Suppose $\pi^* f$ is epic. Since $\eta f = 0$, $\pi^*\eta \circ \pi^* f = 0$; hence $\pi^*\eta = 0$. It follows that the image of η is torsion.

($\Leftarrow$) Suppose N/fM is torsion. Let $\phi : \pi^* N \to \pi^* P$ be a map such that $\phi \circ \pi^* f = 0$. To show $\pi^* f$ is epic we must show that $\phi = 0$. We can write $\phi = \pi^* g$ where $g : N' \to P/P'$ and N/N' and P' are torsion. Let f' denote the restriction of f to $f^{-1}N'$. Since $f'M' = N' \cap fM$, the composition gf' makes sense and $\pi^*(gf') = \pi^* g \circ \pi^* f' = 0$. Hence $gf'M'$ is torsion. But $gN'/gf'M'$ is isomorphic to a quotient of $N'/N' \cap fM$ which is torsion because N/fM is. It follows that gN' is torsion, whence $\pi^* g = 0$.

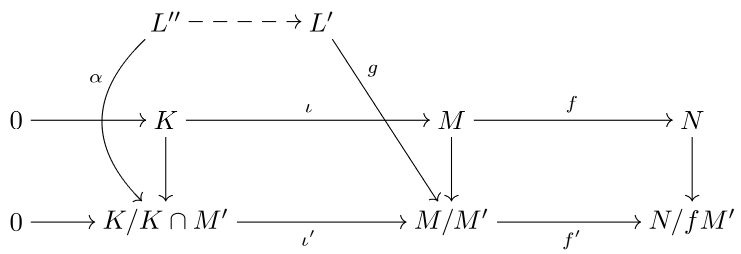
(2) The proof is analogous to that of (1). It can be found on page 366 of Gabriel's paper [**?**, p. 366].

(3) We already know that the composition

$$\pi^* K \xrightarrow{\pi^*\iota} \pi^* M \xrightarrow{\pi^* f} \pi^* N$$

is zero, so to show that $\pi^* K$ is the kernel of $\pi^* f$ we must prove the following: if $\phi : \pi^* L \to \pi^* M$ is such that $\pi^* f \circ \phi = 0$, then there is a unique morphism $\psi : \pi^* L \to \pi^* K$ such that $\phi = \pi^*\iota \circ \psi$.

Now $\phi = \pi^* g$ where $g : L' \to M/M'$ and L/L' and M' are torsion. There is a commutative diagram



with exact rows. Since $\pi^* f \circ \pi^* g = 0$, $f'gL'$ is torsion. Let $L'' = g^{-1}(\operatorname{im}\iota')$. Then L'' is the kernel of the composition

$$L' \to M/M' \to \frac{M/M'}{\operatorname{im}\iota'} = \frac{M/M'}{\ker f'} \cong f'(M/M')$$

so $L'/L'' \cong f'gL'$ which is torsion. Since L/L' is also torsion, L/L'' is torsion. It follows that $\pi^* g$ is equal to $\pi^*(g|_{L''})$. But $gL'' \subset \operatorname{im}\iota'$ so $g|_{L''}$ factors through $K/K \cap M'$; say $g|_{L''} = \iota' \circ \alpha$. Hence $\phi = \pi^* g = \pi^*\iota' \circ \pi^*\alpha$.

But the vertical arrows $K \to K/K \cap M'$ and $M \to M/M'$ become isomorphisms after applying π^* because M' and $K \cap M'$ are torsion. Hence $\pi^*\iota' = \pi^*\iota$, so ϕ factors through $\pi^* K$, as required. This completes the proof that $\pi^* K$ is the kernel of $\pi^* f$.

The proof that $\pi^* C$ is the cokernel of $\pi^* f$ is somewhat similar and we leave it to the diligent reader.

(4) A morphism in an abelian category is an isomorphism if and only if it is both epic and monic. However, we don't know yet that A/T is abelian. Certainly, if $\pi^* f$ is an isomorphism it is both monic and epic so the kernel and cokernel of f are torsion.

To prove the converse, suppose that the kernel and cokernel of f are torsion. By the first isomorphism theorem there is an isomorphism g fitting into the diagram

$$M \xrightarrow{\ \alpha\ } M/K \underset{g^{-1}}{\overset{g}{\leftrightarrows}} \operatorname{im} f \xrightarrow{\ \beta\ } N$$

where α and β are the obvious maps and $f = \beta g^{-1}\alpha$. By hypothesis and (3), $\pi^*\alpha$ and $\pi^*\beta$ are isomorphisms, so $\pi^* f$ is also an isomorphism. □

Theorem 6.7. *Let* T *be a Serre subcategory of* A*. Then* A/T *is abelian and the quotient functor* $\pi^* : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ *is exact.*

Proof. Proposition 6.6 showed that every morphism in A/T has a kernel and a cokernel so it remains to show that the natural map from the coimage to the image of a morphism is an isomorphism.

Let $f : M \to N$. Let $\iota : K \to M$ and $\eta : N \to N/fM$ be the kernel and cokernel of f. By definition, $\operatorname{coim}(\pi^* f) = \operatorname{coker}(\pi^*\iota)$ which is equal to $\pi^*(\operatorname{coker}\iota)$ by Proposition 6.6. Similarly, $\operatorname{im}(\pi^* f) = \pi^*(\ker\eta)$. Applying π^* to the natural isomorphism $\operatorname{coim} f \to \operatorname{im} f$ therefore produces an isomorphism $\operatorname{coim}\pi^* f \to \operatorname{im}\pi^* f$. Hence A/T is abelian.

The fact that $\pi^*(\ker f) = \ker\pi^* f$ and $\pi^*(\operatorname{coker} f) = \operatorname{coker}\pi^* f$ implies at once that π^* sends exact sequences to exact sequences. □

Theorem 6.8. *Let* A *and* B *be abelian categories and* $\mathsf{T} \subset \mathsf{A}$ *a Serre subcategory.*

(1) *Let* $F : \mathsf{A} \to \mathsf{B}$ *be an exact functor such that* $FM = 0$ *for all torsion* M*. Then there is a functor* $G : \mathsf{A}/\mathsf{T} \to \mathsf{B}$*, unique up to natural isomorphism, such that the diagram*

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{\ \pi^*\ } & \mathsf{A}/\mathsf{T} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathsf{B} & \end{array}$$

commutes.

(2) *A functor* $G : \mathsf{A}/\mathsf{T} \to \mathsf{B}$ *is exact if and only if* $G\pi^*$ *is exact.*

Proof. (1) The main thing is to define G on morphisms. To do that it suffices to show for a fixed M and N in A that there are maps

$$\operatorname{Hom}_{\mathsf{A}}(M', N/N') \to \operatorname{Hom}_{\mathsf{B}}(FM, FN)$$

as M' and N' run over all submodules of M and N such that M/M' and N' are torsion that are compatible with the directed system.

Since M/M' is torsion F vanishes on it and applying F to the inclusion $M' \to M$ produces an isomorphism $FM' \to FM$. Similarly, the natural map $FN \to F(N/N')$ is an isomorphism. Hence there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{A}}(M, N) & \longrightarrow & \operatorname{Hom}_{\mathsf{B}}(FM, FN) \\ \downarrow & & \downarrow{\scriptstyle\beta} \\ \operatorname{Hom}_{\mathsf{A}}(M', N/N') & \xrightarrow[\alpha]{} & \operatorname{Hom}_{\mathsf{B}}(FM', F(N/N')). \end{array}$$

But β is an isomorphism so we get a map $\beta^{-1}\alpha : \mathrm{Hom}_{\mathsf{A}}(M', N/N') \to \mathrm{Hom}_{\mathsf{B}}(FM, FN)$. It is easy to see that these are compatible with the directed system so there is a map

$$\varinjlim \mathrm{Hom}_{\mathsf{A}}(M', N/N') \to \mathrm{Hom}_{\mathsf{B}}(FM, FN).$$

We now use this map to define $G : \mathsf{A}/\mathsf{T} \to B$ on morphisms. On objects we set $G\pi^*M := FM$.

The uniqueness of G is left to the reader.

(2) Left to the reader. □

Proposition 6.9. [?, Corollaire 1, page 368] *If $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ is an exact sequence in A/T, then there is an exact sequence $0 \to L \to M \to N \to 0$ in A, and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^*L & \longrightarrow & \pi^*M & \longrightarrow & \pi^*N & \longrightarrow & 0 \end{array}$$

in which the vertical maps are isomorphisms.

The quotient functor preserves direct sums.

Example 6.10. The quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{T}$ need not preserve products. Let Ab denote the category of abelian groups and T be the subcategory of torsion groups. Then the product of all $\mathbb{Z}/n\mathbb{Z}$, $n \geq 2$, is not torsion because it contains a copy of $\mathbb{Z}$, but each $\mathbb{Z}/n\mathbb{Z}$ is torsion. ♢

Proposition 6.11. *Let $G : \mathsf{A} \to \mathsf{B}$ be an exact functor between abelian categories, and let $\mathsf{T} = \ker G$. Let $\pi^* : \mathsf{A} \to \mathsf{A}/\mathsf{S}$ be the quotient functor, and let $\overline{G} : \mathsf{A}/\mathsf{S} \to \mathsf{B}$ be the unique functor such that $G = \overline{G}\pi^*$.*

1. *If F is a left adjoint to G, then $\overline{G}$ is an equivalence of categories if and only if the unit $\mathrm{id}_{\mathsf{B}} \to GF$ is an isomorphism.*
2. *If H is a left adjoint to G, then $\overline{G}$ is an equivalence of categories if and only if the co-unit $HG \to \mathrm{id}_{\mathsf{A}}$ is an isomorphism.*

7. The torsion submodule

Definition 7.1. If an object M has a largest torsion subobject, that subobject is denoted by τM and is called the torsion submodule of M. We will often indicate the existence of a largest torsion submodule by saying *"suppose τM exists"*. ♢

Lemma 7.2. *If τN exists, then $\mathrm{Hom}_{\mathsf{A}}(M, N/\tau N) = 0$ for all torsion modules M. In particular, $N/\tau N$ is torsion-free.*

Proof. Suppose that M is in T and that $f : M \to N/\tau N$. Write N' for the kernel of the composition $N \to N/\tau N \to \operatorname{coker} f$. Then there is an exact sequence $0 \to \tau N \to N' \to N'/\tau N \cong \operatorname{im} f \to 0$. Since M is torsion so is $\operatorname{im} f$, and hence so is N' as T is Serre. Since τN is the largest torsion submodule of N, $N' \subset \tau N$. Therefore $\operatorname{im} f = 0$, whence $f = 0$ as required. □

Lemma 7.3. *Let M and N be A-modules. If τN exists, then*

$$\mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(M, N) = \varinjlim \mathrm{Hom}_{\mathsf{A}}(M', N/\tau N) \tag{7-1}$$

where the direct limit is taken over

$$J := \{(M', \tau N) \mid M' \subset M \text{ and } M/M' \text{ is torsion}\}.$$

Proof. It is easy to see that J is cofinal in the set I defined in Lemma 6.1. □

A direct sum of torsion modules need not be a torsion module. For example, consider the category of k-vector spaces and declare a vector space to be torsion if it has finite dimension.

LEMMA 7.4. *The following conditions on a Serre subcategory* $\mathsf{T} \subset \mathsf{A}$ *are equivalent:*

(1) *every* A*-module has a largest torsion subobject;*
(2) *the inclusion functor* $\mathsf{T} \to \mathsf{A}$ *has a right adjoint;*
(3) *every direct sum of torsion objects is torsion.*

Proof. Let $i_* : \mathsf{T} \to \mathsf{A}$ denote the inclusion functor.

(1) ⇒ (2) We construct a right adjoint τ to i_* as follows. If M is in A, then τM is defined to be the largest torsion subobject of M. If $f : M \to N$, then $f(\tau M)$ is a quotient of τM so is torsion, and therefore contained in τN. We define $\tau f : \tau M \to \tau N$ to be the restriction of f. It is easy to check that τ is a functor $\mathsf{A} \to \mathsf{T}$. It is a right adjoint to i_* because if M is torsion the image of any map $f : M \to N$ is contained in τN. In other words, the natural map $\operatorname{Hom}_{\mathsf{A}}(M, \tau N) \to \operatorname{Hom}_{\mathsf{A}}(M, N)$ is an isomorphism, so

$$\operatorname{Hom}_{\mathsf{A}}(i_* M, N) = \operatorname{Hom}_{\mathsf{A}}(M, N) \cong \operatorname{Hom}_{\mathsf{A}}(M, \tau N) = \operatorname{Hom}_{\mathsf{T}}(M, \tau N).$$

(2) ⇐ (1) Let $i^! : \mathsf{A} \to \mathsf{T}$ be a right adjoint to i_*. For every N in A, the map $\varepsilon_N : i_* i^! N \to N$ is monic so $i^! N$, which is torsion, embeds in N.

Let $M \subset N$ be torsion. The inclusion of M in N can be viewed as an element of $\operatorname{Hom}_{\mathsf{A}}(i_* M, N)$. However, the adjunction isomorphism $\nu : \operatorname{Hom}_{\mathsf{T}}(M, i^! N) \to \operatorname{Hom}_{\mathsf{A}}(i_* M, N)$ satisfies $\nu(\alpha) = \varepsilon_N \circ i_*(\alpha)$, so every map $i_* M \to N$ factors as a composition

$$i_* M \longrightarrow i_* i^! N \xrightarrow{\varepsilon_N} N.$$

In particular, the inclusion of M in N factors in this way. Therefore M is contained in $i^! N$ and we conclude that $i^! N$ is the largest torsion submodule of N.

(1) ⇒ (3) Suppose M_i, $i \in I$, are in T. Let N be the largest subobject of $\oplus M_i$ that is torsion. It must contain each M_i and hence their sum; but that sum is $\oplus M_i$.

(3) ⇒ (1) If M_i, $i \in I$, is the set of all torsion submodules of a module M, then their sum is a quotient of their direct sum, so is torsion. Ths sum must, of course, be the largest torsion submodule of M. □

Definition 7.5. Let T be a Serre subcategory of A. If the inclusion functor $\mathsf{T} \to \mathsf{A}$ has a right adjoint, then that adjoint is called the torsion functor and is denoted by τ. ♢

8. Localizing subcategories

Definition 8.1. A Serre subcategory T of an abelian category A is a localizing subcategory if the quotient $\pi^* : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ has a right adjoint. We write π_* for the right adjoint when it exists. ♢

The key result is that when A has injective envelopes T is a localizing category if and only if it is closed under direct sums (Theorems 8.3 and 8.7).

PROPOSITION 8.2. *Suppose* T *is a localizing subcategory of* A. *Let* $\pi^* : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ *be the quotient functor and and* π_* *its right adjoint. Let* $\mathcal{F}$ *be an* A/T*-module. Then*

1. $\pi_*\mathcal{F}$ *is torsion-free;*
2. *if* $f \in \operatorname{Hom}_{\mathsf{A}}(M,N)$ *and* $\pi^* f$ *is an isomorphism, then the map*
$$\operatorname{Hom}(f,\pi_*\mathcal{F}) : \operatorname{Hom}_{\mathsf{A}}(N,\pi_*\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F})$$
is an isomorphism;
3. *the map* $\pi^* : \operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* M,\pi^*\pi_*\mathcal{F})$ *is an isomorphism for all* A*-modules* M*;*
4. *if* Z *is torsion, every exact sequence of the form* $0 \to \pi_*\mathcal{F} \to N \to Z \to 0$ *splits;*
5. $\pi^*\pi_* \cong \operatorname{id}_{\mathsf{A}/\mathsf{T}}$*;*
6. π_* *is fully faithful.*

Proof. (1) If M is torsion, then $\operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F}) \cong \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* M,\mathcal{F}) = 0$ so $\pi_*\mathcal{F}$ is torsion-free.

(2) By adjointness there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{A}}(N,\pi_*\mathcal{F}) & \longrightarrow & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* N,\mathcal{F}) \\ {\scriptstyle \operatorname{Hom}(f,\pi_*\mathcal{F})}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(\pi^* f,\mathcal{F})} \\ \operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F}) & \longrightarrow & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* M,\mathcal{F}) \end{array}$$

in which the horizontal maps are isomorphisms. By Proposition 6.6(4), $\pi^* f$ is an isomorphism, so the right-hand vertical map is an isomorphism; hence the left-hand map is an isomorphism.

(3) Since $\pi_*\mathcal{F}$ is torsion-free, $\tau(\pi_*\mathcal{F})$ exists—it is zero. Thus, by (7-1), the map $f \mapsto \pi^* f$ is the natural map

$$\operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F}) \to \varinjlim \operatorname{Hom}_{\mathsf{A}}(M',\pi_*\mathcal{F}) \tag{8-1}$$

where the direct limit is taken over the $M' \subset M$ such that M/M' is torsion. By (2), all the maps $\operatorname{Hom}_{\mathsf{A}}(M',\pi_*\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}}(M'',\pi_*\mathcal{F})$ in the direct system are isomorphisms, whence so is (8-1).

(4) Let $f : \pi_*\mathcal{F} \to N$ be the map in the exact sequence. Then the map $\operatorname{Hom}(f,\pi_*\mathcal{F}) : \operatorname{Hom}_{\mathsf{A}}(N,\pi_*\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}}(\pi_*\mathcal{F},\pi_*\mathcal{F})$ is an isomorphism by (2), so there exists $g : N \to \pi_*\mathcal{F}$ such that $f \circ g = \operatorname{id}_N$.

(5) Let $\varepsilon : \pi^*\pi_* \to \operatorname{id}_{\mathsf{A}/\mathsf{T}}$ be the counit. We must show that $\varepsilon_{\mathcal{F}} : \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for each $\mathcal{F}$ in A/T. By Yoneda's Lemma, it suffices to prove that

$$\operatorname{Hom}(\mathcal{G},\varepsilon_{\mathcal{F}}) : \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G},\pi^*\pi_*\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G},\mathcal{F})$$

is an isomorphism for all $\mathcal{G}$ in A/T. Such a $\mathcal{G}$ is equal to $\pi^* M$ for some A-module M, so we must show show that the bottom map in the following diagram is an isomorphism:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{A}}(M,\pi_*\mathcal{F}) & \xrightarrow{\ \nu\ } & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* M,\mathcal{F}) \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G},\pi^*\pi_*\mathcal{F}) & \longrightarrow & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G},\mathcal{F}) \end{array} \tag{8-2}$$

This diagram commutes by (6-7) in Proposition **??**.6.5, and the left-hand vertical map is an isomorphism by (3), so the bottom map is an isomorphism too.

(6) This follows from (5) and Theorem **??**.6.6. □

Theorem 8.3. *Let* T *be a localizing subcategory of* A*. Then there is a torsion functor* $\tau : \mathsf{A} \to \mathsf{T}$ *and for each* M *in* A *there is an exact sequence*

$$0 \longrightarrow \tau N \longrightarrow N \xrightarrow{\eta_N} \pi_*\pi^* N \longrightarrow \operatorname{coker} \eta_N \longrightarrow 0$$

in which η_N *is an essential map, and* $\operatorname{coker} \eta_N$ *a torsion module.*

Proof. If $W = \ker \eta_M$ and $Z = \operatorname{coker} \eta_M$, then there is an exact sequence

$$0 \longrightarrow \pi^* W \longrightarrow \pi^* M \xrightarrow{\pi^*(\eta_M)} \pi^*\pi_*\pi^* M \longrightarrow \pi^* Z \longrightarrow 0$$

in A/T. By Proposition **??**.6.5, $\varepsilon_{\pi^* M} \circ \pi^*(\eta_M) = \mathrm{id}_{\pi^* M}$. However, part (5) of the previous result shows that $\varepsilon_{\pi^* M}$ is an isomorphism. Hence $\pi^*(\eta_M)$ is an isomorphism. Therefore $\pi^* W = \pi^* Z = 0$, whence W and Z are torsion.

By Proposition 8.2(1), $\pi_*\pi^* M$ is torsion-free, so W contains every torsion submodule of M. Thus W is the largest torsion submodule of M.

If T is a submodule of $\pi_*\pi^* M$ such that $T \cap \eta_M(M) = 0$, then T embeds in Z, so is torsion. But $\pi_*\pi^* M$ is torsion-free, so $T = 0$. Thus $\eta_M(M)$ is essential in $\pi_*\pi^* M$. □

Lemma 8.4. *An essential extension of a torsion-free module is torsion-free.*

Proof. Let Q be an essential extension of a torsion-free module N. If $M \subset Q$ is a torsion module, so is $M \cap N$. Therefore $M \cap N = 0$, whence $M = 0$. □

Example 8.5. An essential extension of a torsion module need not be torsion. Let R be a ring having a non-split extension $0 \to S \to M \to S' \to 0$ of two non-isomorphic simples (2×2 triangular matrices is such a ring). If T consists of all direct limits of finite length R-modules all of whose composition factors are isomorphic to S, then T is a localizing subcategory. Although S is torsion its essential extension M is not. ♢

The example also shows that applying π^* to an essential monic need not produce an essential monic.

Lemma 8.6. *Applying* π_* *to an essential monic produces an essential monic.*

Proof. Because it is a right adjoint π_* preserves monics. Let $\mathcal{L} \to \mathcal{M}$ be an essential monic in A/T. Suppose there is a direct sum $\pi_*\mathcal{L} \oplus N \subset \pi_*\mathcal{M}$. Applying π to this produces a direct sum $\mathcal{L} \oplus \pi^* N \subset \mathcal{M}$, so $\pi^* N = 0$. But N is torsion-free because $\pi_*\mathcal{M}$ is, so we deduce that $N = 0$. □

Theorem 8.7. *Let* T *be a Serre subcategory of* A*. Suppose that a torsion functor* $\tau : \mathsf{A} \to \mathsf{T}$ *exists. If* A *has injective envelopes, then*

1. T *is a localizing subcategory of* A*;*
2. *for each* N *in* A*,* $\pi_*\pi^* N$ *is isomorphic to the largest submodule of the injective envelope of* $N/\tau N$ *which extends* $N/\tau N$ *by a torsion module.*

Proof. To show that the quotient functor $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ has a right adjoint it suffices, by Proposition **??**.**??**, to show that the functor

$$M \mapsto \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^* M, \pi^* N)$$

is representable for each N in A. The representing object will be the module H we define next.

Fix N in A. Write $\bar{N} = N/\tau N$. Let H be the largest essential extension of a torsion module by $\bar{N}$. Explicitly, if $\alpha : \bar{N} \to E$ is the inclusion of N in an injective envelope, H is the kernel of the composition

$$E \to \operatorname{coker}\alpha \to \operatorname{coker}\alpha/\tau(\operatorname{coker}\alpha).$$

This gives rise to an exact sequence

$$0 \longrightarrow \tau N \longrightarrow N \xrightarrow{f} H \longrightarrow \operatorname{coker} f \longrightarrow 0$$

in which $\ker f$ and $\operatorname{coker} f$ are both torsion. In particular, $\pi^* f : \pi^* N \to \pi^* H$ is an isomorphism in $\mathsf{A/T}$. By Lemma 7.2, $\bar{N}$ is torsion-free, hence so is H by Lemma 8.4. Moreover, $E/H \cong \operatorname{coker}\alpha/\tau(\operatorname{coker}\alpha)$ is also torsion-free by Lemma 7.2.

Since $\pi^* f$ is an isomorphism so is the map

$$\operatorname{Hom}(\pi^* M, \pi^* f) : \operatorname{Hom}_{\mathsf{A/T}}(\pi^* M, \pi^* N) \to \operatorname{Hom}_{\mathsf{A/T}}(\pi^* M, \pi^* H).$$

Thus, it suffices to show that H is a representing object for the functor

$$M \mapsto \operatorname{Hom}_{\mathsf{A/T}}(\pi^* M, \pi^* H).$$

We will do this by showing that $\pi : \operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A/T}}(\pi^* M, \pi^* H)$ is an isomorphism.

Since H is torsion-free,

$$\operatorname{Hom}_{\mathsf{A/T}}(\pi^* M, \pi^* H) = \varinjlim \operatorname{Hom}_{\mathsf{A}}(M', H)$$

where the direct limit is taken over those $M' \subset M$ for which M/M' is torsion. We will show for such an M' that the natural map $\operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A}}(M', H)$ is an isomorphism. Since $\operatorname{Hom}_{\mathsf{A}}(-, H)$ is left exact and M/M' is torsion whereas H is torsion-free, it follows from Lemma 7.2 that this map is injective, so it remains to prove it is surjective. To see this, let $f' \in \operatorname{Hom}_{\mathsf{A}}(M', H)$ and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\big\downarrow & & & & & & \\
0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & E/H & \longrightarrow & 0.
\end{array}$$

Since E is injective there is a morphism $f : M \to E$ extending the composition $M' \to H \to E$. It follows that there exists a morphism $g : M/M' \to E/H$ making the diagram commute. But E/H is torsion-free and M/M' is torsion, so $g = 0$ by Lemma 7.2. Therefore the image of f is contained in H and f' is the restriction of f. Hence the map $\operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A}}(M', H)$ is surjective, and hence an isomorphism. □

COROLLARY 8.8. *Let T be a Serre subcategory of A. Suppose that A has direct sums and injective envelopes. Then T is localizing if and only if it is closed under arbitrary direct sums.*

Proof. ($\Rightarrow$) By hypothesis, π^* has a right adjoint so it commutes with direct sums (Corollary **??.??**). Therefore, if N_α are torsion modules, then $\pi^*(\oplus N_\alpha) \cong \oplus \pi^* N_\alpha = 0$, whence $\oplus N_\alpha$ is in T.

($\Leftarrow$) The direct sum of all the torsion submodules of a given module is torsion. But the sum of those submodules is a quotient of their direct sum, so is also torsion.

Hence every module has a largest torsion submodule. It follows from Lemma 7.4 and Theorem 8.7 that T is localizing. □

9. Cohomology in A and A/T

A comparison of homological issues in A and A/T requires an understanding of the relation between injectives in the two categories.

THEOREM 9.1. *Suppose* T *is a localizing subcategory of* A*, and that* A *has injective envelopes.*

(1) π_* *sends injectives to injectives, and injective envelopes to injective envelopes.*
(2) *The injectives in* A/T *are* $\{\pi^*Q \mid Q$ *is a torsion-free injective in* $\mathsf{A}\}$.
(3) A/T *has enough injectives.*
(4) *If* Q *is a torsion-free injective* A*-module, then* $Q \cong \pi_*\pi^*Q$.

Proof. (1) Because it is right adjoint to an exact functor π_* preserves injectives (Proposition **??**). Then by Lemma 8.6 it preserves injective envelopes.

(2) If Q is a torsion-free injective, then $\operatorname{Hom}_{\mathsf{A}}(-, Q)$ is an exact functor vanishing on T so, by Theorem 6.8, the rule

$$\pi^*M \mapsto \operatorname{Hom}_{\mathsf{A}}(M, Q) \tag{9-1}$$

defines an exact functor on A/T. By Theorem 8.7(2), $Q \cong \pi_*\pi^*Q$ so, by Proposition 8.2(3),

$$\operatorname{Hom}_{\mathsf{A}}(M, Q) \cong \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi^*M, \pi^*Q).$$

Therefore the functor defined by (9-1) is equivalent to $\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(-, \pi^*Q)$. But (9-1) is an exact functor, so π^*Q is injective.

Let $\mathcal{Q}$ be an injective in A/T. Then $\pi_*\mathcal{Q}$ is injective by (1), and is torsion-free by Proposition 8.2(1). Moreover, $\pi\pi_*\mathcal{Q} \cong \mathcal{Q}$ by Proposition 8.2(5), so every injective in A/T is of the form π^*Q for some injective A-module Q.

(3) Let $\mathcal{F}$ be an A/T-module, and let $f : \pi_*\mathcal{F} \to Q$ be the inclusion of $\pi_*\mathcal{F}$ in its injective envelope. Since $\pi_*\mathcal{F}$ is torsion-free, so is Q (Lemma 8.4). But π^*f is monic, so π^*Q is an injective containing $\pi\pi_*\mathcal{F} \cong \mathcal{F}$. Thus A/T has enough injectives. □

Next we show how that the right derived functors of τ and π_* are closely related when T is a localizing subcategory that is closed under injective envelopes.

Clearly, T is closed under injective envelopes if and only if every essential extension of a torsion module is torsion. This condition is sometimes described in the literature as a *stable torsion theory* (see [**?**, p. 46] and [**?**, p. 20] for example).

THEOREM 9.2. *Let* T *be a localizing subcategory of* A*. Suppose that* A *has enough injectives and that* T *is closed under injective envelopes. Then*

(1) *every injective in* A *is a direct sum of a torsion injective and a torsion-free injective;*
(2) *for* $i \geq 1$*, the right-derived functors of* τ *and* π_* *satisfy*

$$R^{i+1}\tau M \cong R^i\pi_*(\pi^*M)$$

for all A*-modules* M*;*
(3) *there is an exact sequence* $0 \to \tau M \to M \to \pi_*\pi^*M \to R^1\tau M \to 0$.

Proof. (1) Let E be an injective in A. Since E contains a copy of the injective envelope of τE, and since that injective is torsion by hypothesis, τE is injective. Therefore it is a direct summand of E, say $E = \tau E \oplus Q$. Clearly Q is a torsion-free injective.

(2) Let $M \to E^\bullet$ be an injective resolution of M. For each j, write I^j for the torsion submodule of E^j, and set $Q^j = E^j/I^j$. Then there is an exact sequence of complexes

$$0 \to I^\bullet \to E^\bullet \to Q^\bullet \to 0$$

which gives a long exact sequence

$$\cdots \to h^{i-1}(Q^\bullet) \to h^i(I^\bullet) \to h^i(E^\bullet) \to h^i(Q^\bullet) \to h^{i+1}(I^\bullet) \to \cdots$$

in homology. However, $h^i(I^\bullet) = R^i\tau M$, and $h^i(E^\bullet) = 0$ for $i \geq 1$. Therefore, for $i \geq 1$, $R^{i+1}\tau M \cong h^i(Q^\bullet)$.

By Theorem 9.1, π^*Q^j is injective in A/T, and $\pi_*\pi^*Q^j \cong Q^j$. Since π is exact, $\pi^*M \to \pi^*E^\bullet$ is an injective resolution in A/T. However, the complexes $\pi Q^\bullet$ and $\pi^*E^\bullet$ are isomorphic. Therefore, $\pi^*M \to \pi^*Q^\bullet$ is an injective resolution in A/T, so

$$R^i\pi_*(\pi^*M) \cong h^i(\pi_*\pi^*Q^\bullet) \cong h^i(Q^\bullet).$$

This completes the proof of (2), and (3) is given by the left-hand segment of the long homology sequence. □

Proposition 9.3. *If* T *is a stable torsion theory, then* π^* *sends a minimal injective resolution to a minimal injective resolution.*

Proof. Let $N \to E^\bullet$ be a minimal injective resolution in A. As in the previous proof let $I^\bullet$ be the torsion subcomplex of $E^\bullet$ and write $Q^\bullet = E^\bullet/I^\bullet$. Thus $\pi^*N \to \pi^*Q^\bullet$ is an injective resolution in A/T; the fact that this is a *minimal* resolution follows from the next paragraph.

Consider the following diagram in A, where E and E' are injective, and I and I' are their torsion submodules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \bar f} & & \\ 0 & \longrightarrow & I' & \longrightarrow & E' & \longrightarrow & Q' & \longrightarrow & 0 \end{array}$$

Suppose that fE is essential in E'; we will show that $\bar{f}Q$ is essential in Q'. It suffices to show that if $I' \subset M \subset E'$, and $M \neq I'$, then $M \cap (fE + I')$ is strictly larger than I'. Because the torsion theory is stable, I' is itself injective, so $M = I' \oplus C$ for some non-zero C. Since $fE \cap C \neq 0$, $M \cap (fE + I')$ is strictly larger than I'. Thus M/I' has non-zero intersection with the image of $\bar{f}$. □

Example 9.4. Let R be a commutative ring, and $\mathfrak{m}$ a maximal ideal in R. A module M is supported at $\mathfrak{m}$ if each element of M is killed by a power of $\mathfrak{m}$. Such modules form a Serre subcategory of $\mathsf{Mod}R$. This is a localizing subcategory, and the torsion functor τ is

$$\tau = \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^n, -).$$

The right derived functors of τ are therefore

$$R^i\tau = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{m}^n, -).$$

We write $H^i_{\mathfrak{m}}(M)$ for $R^i\tau M$, and call this the i^{th} local cohomology module of M with respect to τ.

The corresponding quotient category of $\mathsf{Mod}R$ is the category of quasi-coherent modules on the open complement in $\operatorname{Spec} R$ of $\mathfrak{m}$. This is called the punctured spectrum of R. If we write X for $\operatorname{Spec} R$, U for the punctured spectrum, and $j : U \to X$ for the inclusion map, then $j^* = \pi^*$ and $j_* = \pi_*$. Therefore, if $M \in \mathsf{Mod}R$, and $\mathcal{M} = j^*M$ is its restriction to U, then $R^ij_*\mathcal{M} \cong H^{i+1}_{\mathfrak{m}}(M)$ for $i \geq 1$.

For example, when $X = \mathbb{A}^2$, and $U = \mathbb{A}^2\backslash\{0\}$, one sees that $R^1j_*\mathcal{O}_U \neq 0$ because $H^2_{\mathfrak{m}}(R)$ is isomorphic to the injective envelope of $R/\mathfrak{m}$ (ref???). $\diamondsuit$

Lemma 9.5. *Let* T *be a localizing subcategory of* A*.*

(1) *If* M *is noetherian, so is* π^*M*.*
(2) *Suppose that every* A*-module is the union of its noetherian submodules. If* $\mathcal{M}$ *is a noetherian* A/T*-module, then there is a noetherian* A*-module* M *such that* $\mathcal{M} \cong \pi^*M$*.*

Proof. (1) Replacing M by $M/\tau M$, we may assume that M is torsion-free. Let $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \dots$ be an ascending chain of submodules of π^*M. Because π_* is left exact, $\pi_*\mathcal{N}_1 \subset \pi_*\mathcal{N}_2 \subset \dots$ is an ascending chain of submodules of $\pi_*\pi^*M$. Thus $\pi_*\mathcal{N}_1 \cap M \subset \pi_*\mathcal{N}_2 \cap M \subset \dots$ is an ascending chain of submodules of M. Since M is noetherian, it follows that this chain stabilizes. However, since π is left exact, it commutes with intersection. Thus, for large i,

$$\mathcal{N}_i = \pi\pi_*\mathcal{N}_i \cap \pi^*M = \pi(\pi_*\mathcal{N}_i \cap M) = \pi(\pi_*\mathcal{N}_{i+1} \cap M) = \mathcal{N}_{i+1}.$$

Hence the original chain stabilizes, and we conclude that π^*M is noetherian.

(2) By hypothesis, $\pi_*\mathcal{M}$ is the union of its noetherian submodules, say $\pi_*\mathcal{M} = \varinjlim M_i$, where each M_i is noetherian. Because π has a right adjoint, it commutes with direct limits, so $\mathcal{M} \cong \pi\pi_*\mathcal{M} = \varinjlim \pi^*M_i$. Each π^*M_i is a submodule of $\mathcal{M}$. By hypothesis, $\mathcal{M}$ is noetherian, so for some i, $\mathcal{M} = \pi^*M_i$. $\square$

APPENDIX A

Categories

These notes are a refresher course.

Much of the action in algebraic geometry takes place within the category of quasi-coherent sheaves on a scheme. The quasi-coherent sheaves on a quasi-projective scheme X form an abelian category that is denoted by $\mathsf{Qcoh}X$.

Abelian categories are common place objects in algebra. The standard example of an abelian category is the category of modules over a ring. Abstracting the properties of this category leads to the definition of an abelian category. Every abelian category can be embedded as a full subcategory of a module category so the intuition one has from module categories carries over to abelian categories. There are some differences, and therefore some pitfalls. For example, a sum of simple modules in an abelian category need not be isomorphic to a direct sum of simples.

Although one's intuition from modules is useful one has to become accustomed to working without elements. Arrows are now an important part of one's equipment.

A Grothendieck category is a special kind of abelian category that is closer still to a module category. Not only can it be realized as a full subcategory of a module category, but this can be done in such a way that the embedding functor has an exact left adjoint. In other words, every Grothendieck category is a *localization* of a module category in the sense of Chapter ???. A Grothendieck category has enough injectives, meaning that every object embeds in an injective object. This allows one to do homological algebra. One of the axioms for a Grothendieck category is that it be cocomplete. In particular, it has direct limits. It turns out that a Grothendieck category is also complete. The sheaves of abelian groups on a topological space form a Grothendieck category, and so do the quasi-coherent $\mathcal{O}_X$-modules on a noetherian scheme X.

1. Special kinds of morphisms and objects

Definition 1.1. A morphism f in a category C is

- monic, or a monomorphism, if $g_1 = g_2$ whenever $fg_1 = fg_2$;
- epic, or an epimorphism, if $g_1 = g_2$ whenever $g_1 f = g_2 f$;
- an isomorphism if there exists g such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. If such a g exists it is unique, and is denoted by f^{-1}; we call it the inverse of f. Objects X and Y are isomorphic in C if there exists an isomorphism $f : X \to Y$ in C. ◊

Definition 1.2. Let X be an object in a category C.

A subobject of X is an equivalence class of pairs (A, α) consisting of an object A and a monomorphism $\alpha : A \to X$; two such pairs (A, α) and (A', α') are equivalent if there is an isomorphism $\iota : A' \to A$ such that $\alpha' = \alpha\iota$.

A quotient object of X is an equivalence class of pairs (B, β) consisting of an object B and an epimorphism $\beta : X \to B$; two such pairs (B, β) and (B', α') are equivalent if there is an isomorphism $\iota : B \to B'$ such that $\beta' = \iota\beta$. ♢

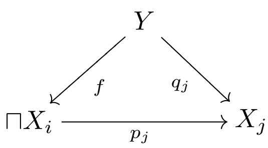
Definition 1.3. An object Z in a category C is
- an initial object if $\mathrm{Hom}_{\mathsf{C}}(Z, X)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a terminal object if $\mathrm{Hom}_{\mathsf{C}}(X, Z)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a zero object if it is both an initial and a terminal object.

A zero object is denoted by 0 and, for every pair of objects X and Y, the composition of morphisms $X \to 0 \to Y$ is called the zero morphism and is denoted by 0, or 0_{XY} if necessary. ♢

Initial, terminal, and zero objects are all unique up to unique isomorphism. Hence the definition of the zero morphism 0_{XY} does not depend on the choice of zero object.

2. Products and Coproducts

Definition 2.1. A product of objects X_i, $i \in I$, is an object $\sqcap X_i$ together with morphisms $p_j : \sqcap X_i \to X_j$, $j \in I$, such that for any morphisms $q_j : Y \to X_j$, $j \in I$, there is a unique morphism $f : Y \to \sqcap X_i$ making the diagrams

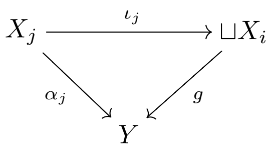


commute for all $j \in I$.

We write $X_1 \times \cdots \times X_n$ for a product of a finite set of objects $X_1, \ldots, X_n$ ♢

Example 2.2. If $X \times X$ exists its universal property implies the existence of a unique map $\Delta : X \to X \times X$ whose composition with each projection $X \times X \to X$ is the identity id_X. We call Δ the diagonal map. ♢

Definition 2.3. A coproduct of objects X_i, $i \in I$, is an object $\sqcup X_i$ together with morphisms $\iota_j : X_j \to \sqcup X_i$, $j \in I$, such that for any morphisms $\alpha_j : X_j \to Y$, $j \in I$, there is a unique morphism $g : \sqcup X_i \to Y$ making the diagrams



commute for all $j \in I$. ♢

The definitions do not assert that products and coproducts exist.

3. Functors

A functor $F : \mathsf{C} \to \mathsf{D}$ between two categories is a function that assigns to each object X in C an object FX in D and to each morphism $f : X \to Y$ in C assigns a morphism $Ff : FX \to FY$ in such a way that $F(f \circ g) = Ff \circ Fg$ and $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ for all f, g, and X in C.

Some have said that a mathematicians job is to find new functors.

Among the most famous examples is the fundamental group which is a functor from the category of topological spaces to groups. It allows one to obtain topological results by algebraic methods. For example, it follows easily from the definition that a functor sends isomorphic objects to isomorphic objects. Hence, two topological spaces having non-isomorphic fundamental groups are not homeomorphic.

A functor $F : \mathsf{C} \to \mathsf{D}$ is full if the map $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{D}}(FX, FY)$, $f \mapsto Ff$, is surjective and is faithful if this map is injective. If F is both full and faithful we say it is fully faithful. If F is fully faithful then D (almost!) contains a copy of C.

4. Natural transformations

Definition 4.1. Let $F, F' : \mathsf{A} \to \mathsf{B}$ be functors. A natural transformation $\tau : F \Rightarrow F'$ is a class of morphisms $\tau_M : FM \to F'M$, one for each object $M \in \mathsf{A}$, such that for each $f \in \operatorname{Hom}_{\mathsf{A}}(M, N)$ the diagram

$$\begin{array}{ccc} FM & \xrightarrow{Ff} & FN \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ F'M & \xrightarrow[F'f]{} & F'N \end{array} \tag{4-1}$$

commutes.

If each τ_M is an isomorphism, τ is said to be a natural equivalence or isomorphism, F and F' are said to be naturally equivalent or just isomorphic, and we write $F \cong F'$.

Categories C and D are equivalent if there are functors $F : \mathsf{C} \to \mathsf{D}$ and $F' : \mathsf{D} \to \mathsf{C}$ such that $FF' \cong \mathrm{Id}_{\mathsf{D}}$, and $F'F \cong \mathrm{Id}_{\mathsf{C}}$. ♢

5. Yoneda's Lemma

Let C be a category and D the category of contravariant functors $\mathsf{C} \to \mathsf{Sets}$. The Yoneda functor

$$\mathsf{C} \to \mathsf{D}$$

sends an object X to the functor $\operatorname{Hom}_{\mathsf{C}}(-, X)$ and a morphism $f : X \to Y$ to the natural transformation $\operatorname{Hom}_{\mathsf{C}}(-, f) : \operatorname{Hom}_{\mathsf{C}}(-, X) \to \operatorname{Hom}_{\mathsf{C}}(-, Y)$ defined by $\operatorname{Hom}_{\mathsf{C}}(-, f)(g) = f \circ g$ for $g \in \operatorname{Hom}_{\mathsf{C}}(S, X)$.

LEMMA 5.1. *The Yoneda functor is fully faithful, i.e., its "image" is equivalent to* C *and if* $\tau : \operatorname{Hom}_{\mathsf{C}}(-, X) \to \operatorname{Hom}_{\mathsf{C}}(-, Y)$ *is a natural transformation there is a unique morphism* $f : X \to Y$ *such that* $\tau_S = f \circ -$ *for all* S.

A functor F in D is representable if $F \cong \operatorname{Hom}_{\mathsf{C}}(-, X)$ for some X. Any two representing objects are isomorphic via a unique isomorphism, i.e., if we have two natural isomorphisms $\tau_i : F \to \operatorname{Hom}_{\mathsf{C}}(-, X_i)$, $(i = 1, 2)$, there is a unique isomorphism $f : X_1 \to X_2$ such that $\operatorname{Hom}_{\mathsf{C}}(-, f) \circ \tau_1 = \tau_2$.

The product and coproduct of a family of objects M_i in C can be characterized by the existence of isomorphisms

$$\operatorname{Hom}_{\mathsf{C}}(N, \sqcap M_i) \cong \sqcap \operatorname{Hom}_{\mathsf{C}}(N, M_i)$$

and

$$\operatorname{Hom}_{\mathsf{C}}(\sqcup M_i, N) \cong \sqcap \operatorname{Hom}_{\mathsf{C}}(M_i, N)$$

for all objects N in C. In other words, the existence of a product or coproduct can be phrased as the representability of a functor. For example, if the product exists it represents the functor

$$N \mapsto \sqcap \operatorname{Hom}_{\mathsf{C}}(N, M_i).$$

6. Adjoint pairs of functors

It has been said that categories were invented to define functors, that functors were invented to define natural transformations, and that natural transformations were invented to define adjoint pairs of functors.

Definition 6.1. Let $f^* : \mathsf{C} \to \mathsf{D}$ and $f_* : \mathsf{D} \to \mathsf{C}$ be functors. We say that f^* is a left adjoint of f_* and that f_* is a right adjoint of f^* if the functors $\operatorname{Hom}_{\mathsf{C}}(-, f_*-)$ and $\operatorname{Hom}_{\mathsf{D}}(f^*-, -)$, taking $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$, are naturally equivalent.

We write $f^* \dashv f_*$ to denote the fact that f_* is right adjoint to f^*. We call (f^*, f_*) an adjoint pair if $f^* \dashv f_*$. We call $(f^*, f_*, f^!)$ an adjoint triple if $f^* \dashv f_*$ and $f_* \dashv f^!$. ♢

Proposition 6.2. *Let $f^* : \mathsf{C} \to \mathsf{D}$ and $f_* : \mathsf{D} \to \mathsf{C}$ be functors. Then (f^*, f_*) is an adjoint pair if and only if for all M in C and N in D there are bijections*

$$\nu_{MN} : \operatorname{Hom}_{\mathsf{C}}(M, f_*N) \to \operatorname{Hom}_{\mathsf{D}}(f^*M, N) \tag{6-1}$$

such that if $\alpha \in \operatorname{Hom}_{\mathsf{C}}(M, M')$ and $\beta \in \operatorname{Hom}_{\mathsf{D}}(N, N')$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{C}}(M', f_*N) & \xrightarrow{\nu_{M'N}} & \operatorname{Hom}_{\mathsf{D}}(f^*M', N) \\ {\scriptstyle (-)\circ\alpha}\downarrow & & \downarrow{\scriptstyle (-)\circ(f^*\alpha)} \\ \operatorname{Hom}_{\mathsf{C}}(M, f_*N) & \xrightarrow{\nu_{MN}} & \operatorname{Hom}_{\mathsf{D}}(f^*M, N) \\ {\scriptstyle (f_*\beta)\circ(-)}\downarrow & & \downarrow{\scriptstyle \beta\circ(-)} \\ \operatorname{Hom}_{\mathsf{C}}(M, f_*N') & \xrightarrow{\nu_{MN'}} & \operatorname{Hom}_{\mathsf{D}}(f^*M, N') \end{array} \tag{6-2}$$

commutes.

The commutativity of (6-2) is equivalent to the condition that

$$\nu(\lambda \circ \alpha) = \nu(\lambda) \circ f^*\alpha \tag{6-3}$$

and

$$\nu(f_*\beta \circ \lambda) = \beta \circ \nu(\lambda) \tag{6-4}$$

for all $\lambda : M' \to f_*N$. There are similar identities involving ν^{-1}.

The maps ν_{MN} give a morphism of bifunctors

$$\nu : \operatorname{Hom}_{\mathsf{C}}(-, f_*-) \to \operatorname{Hom}_{\mathsf{D}}(f^*-, -).$$

The commutativity of (6-2) says that this morphism is a natural transformation in each variable.

The paradigmatic algebraic example of an adjoint pair is provided by the tensor and Hom functors.

Example 6.3. If ${}_RB_S$ is an R-S-bimodule, then $- \otimes_R B$ is a left adjoint to $\operatorname{Hom}_S(B, -)$. In particular, if M is a right R-module and N is a right B-module, then the map that sends λ to the map $m \otimes b \mapsto (\lambda(m))(b)$ is an isomorphism

$$\operatorname{Hom}_R(M, \operatorname{Hom}_S(B, N)) \longrightarrow \operatorname{Hom}_S(M \otimes_R B, N).$$

One checks that the diagrams in Definition 6.1 commute by using the explicit form of the map. ♢

Example 6.4. A ring homomorphism $f : R \to S$ induces an adjoint triple of functors $(f^*, f_*, f^!)$ as defined in Example **??**. We mean that (f^*, f_*) and $f_*, f^!)$ are both adjoint pairs. The fact that (f^*, f_*) is an adjoint pair is a special case of Example 6.3 with the bimodule being ${}_RS_S$. The fact that (f_*, f^*) is an adjoint pair is also a special case of Example 6.3 with the bimodule being ${}_SS_R$. We call f^* and f_* the inverse image and direct image functors associated to f.

If $g : S \to T$ is another ring homomorphism, then $g_* \circ f_* = (g \circ f)_*$ and $f^* \circ g^* \cong (g \circ f)^*$. ♢

Proposition 6.5. *Let (f^*, f_*) be an adjoint pair of functors with $f^* : \mathsf{C} \to \mathsf{D}$ and $\nu : \mathrm{Hom}_{\mathsf{C}}(-, f_*-) \to \mathrm{Hom}_{\mathsf{D}}(f^*-, -)$ the associated isomorphism of bifunctors. There are natural transformations*

$$\varepsilon : f^*f_* \to \mathrm{id}_{\mathsf{D}} \qquad \eta : \mathrm{id}_{\mathsf{C}} \to f_*f^*$$

defined as follows. If M is in C and N is in D, then

$$\eta_M = \nu^{-1}(\mathrm{id}_{f^*M}) : M \to f_*f^*M \tag{6-5}$$

and

$$\varepsilon_N = \nu(\mathrm{id}_{f_*N}) : f^*f_*N \to N. \tag{6-6}$$

If $\alpha \in \mathrm{Hom}_{\mathsf{C}}(M, f_*N)$ and $\beta \in \mathrm{Hom}_{\mathsf{D}}(f^*M, N)$ then

$$\nu(\alpha) = \varepsilon_N \circ f^*(\alpha) \tag{6-7}$$

and

$$\nu^{-1}(\beta) = f_*(\beta) \circ \eta_M. \tag{6-8}$$

Theorem 6.6. *Let (f^*, f_*) be an adjoint pair of functors with associated counit $\varepsilon : f^*f_* \to \mathrm{id}$. Then*

1. *f_* is full if and only if every ε_M is split monic;*
2. *f_* is faithful if and only if every ε_M is epic;*
3. *f_* is fully faithful if and only if every ε_M is an isomorphism.*

7. Additive categories

Definition 7.1. A category is

- pre-additive if all its Hom sets are abelian groups, and composition of morphisms is bilinear,
- additive if it is pre-additive and has finite products and coproducts, and contains a zero object. ♢

Lemma 7.2. *Let f be a morphism in an additive category. Then*

1. *f is monic if and only if $fg = 0$ implies $g = 0$, and*
2. *f is epic if and only if $gf = 0$ implies $g = 0$.*

Remark. In an additive category a coproduct of the empty family is an initial object and a product of the empty family is a terminal object. Therefore the axiom that an additive category has a zero object follows from the other axioms.

7.1. Direct sum. A coproduct in an additive category is called a direct sum and is denoted by $\bigoplus$ rather than $\sqcup$.

The next result says, among other things, that in an additive category a finite direct sum of objects is canonically isomorphic to their product. Give an example to show this is not true of infinite direct sums.

PROPOSITION 7.3. *Let $\{M_i \mid i \in I\}$ be a small set of objects in an additive category, and suppose that their product and direct sum exist. Let*

$$\alpha_j : M_j \to \bigoplus M_i \qquad \textit{and} \qquad \rho_j : \sqcap M_i \to M_j$$

be the morphisms guaranteed by the definitions. For each pair of indices (i, j) define $\delta^i_j : M_i \to M_j$ by

$$\delta^i_j = \begin{cases} 0 & \textit{if } i \neq j, \\ \mathrm{id}_{M_i} & \textit{if } i = j. \end{cases}$$

Then

(1) *there are unique maps $\varepsilon_j : M_j \to \sqcap M_i$ such that $\rho_i \varepsilon_j = \delta^j_i$ for all $i, j \in I$;*
(2) *there are unique maps $\gamma_j : \bigoplus M_i \to M_j$ such that $\gamma_j \alpha_i = \delta^i_j$ for all $i, j \in I$;*
(3) *there is a unique map*

$$\Psi : \bigoplus M_i \to \sqcap M_i$$

such that $\Psi \alpha_i = \varepsilon_i$ for all $i \in I$;
(4) *$\rho_j \Psi = \gamma_j$ for all $j \in I$;*
(5) *if I is finite, then Ψ is an isomorphism.*

7.2. Kernels and cokernels.

Definition 7.4. Let $f : M \to N$ be a morphism in an additive category σ. A

- kernel of f is a pair (A, α), consisting of an object A and a morphism $\alpha : A \to M$ such that $f\alpha = 0$ and, if $\alpha' : A' \to M$ is a morphism for which $f\alpha' = 0$, then there is a unique morphism $\rho : A' \to A$ such that $\alpha' = \alpha\rho$.
- cokernel of f is a pair (B, β), consisting of an object B and a morphism $\beta : N \to B$ such that $\beta f = 0$ and, if $\beta' : N \to B'$ is a morphism for which $\beta' f = 0$, then there is a unique morphism $\rho : B \to B'$ such that $\beta' = \rho\beta$. $\diamondsuit$

If a kernel or cokernel exists it is unique up to unique isomorphism.

LEMMA 7.5. *Let $f : M \to N$ be a morphism in an additive category. Then*

(1) *f is a monomorphism if and only if $\ker f = (0 \to M)$;*
(2) *f is an epimorphism if and only if $\operatorname{coker} f = (N \to 0)$.*

PROPOSITION 7.6. *Let $f : M \to N$ be a morphism in an additive category. Then*

(1) *if $\ker f$ exists, it is a subobject of M;*
(2) *if $\operatorname{coker} f$ exists, it is a quotient object of N.*

7.3. Images and coimages. Let $f : M \to N$ be a morphism in an additive category having kernels and cokernels. Let $\iota : \ker f \to M$ and $\eta : N \to \operatorname{coker} f$ be the kernel and cokernel. The image and coimage of f are defined to be

$$\operatorname{im} f := \ker \eta$$
$$\operatorname{coim} f := \operatorname{coker} \iota.$$

We now show there is a canonical map $\phi : \operatorname{coim} f \to \operatorname{im} f$ making the rectangle in the diagram

$$\begin{array}{ccccccc} K = \ker f & \xrightarrow{\iota} & M & \xrightarrow{f} & N & \xrightarrow{\eta} & C = \operatorname{coker} f \\ & & \downarrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \\ & & \operatorname{coim} f & \overset{\phi}{\dashrightarrow} & \operatorname{im} f & & \end{array}$$

commute. Since $f\iota = 0$, f factors through $\operatorname{coker} \iota$; this gives a map $\bar{f} : \operatorname{coim} f \to N$ such that $\bar{f}\alpha = f$. Now $0 = \eta f = \eta \bar{f} \alpha$; but α is epic so $\eta \bar{f} = 0$; hence $\bar{f}$ factors through $\ker \eta$. This gives the map ϕ satisfying $\beta\phi = \bar{f}$.

The reason you may not have encountered the coimage before is that in an abelian category the map ϕ is always an isomorphism so one only uses the word "image".

8. Abelian categories

Definition 8.1. An additive category is abelian if every morphism has a kernel and a cokernel, and the natural map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism for all f. ◊

Every morphism $f : M \to N$ in an abelian category may be factored as $f = \beta \circ \alpha$ with α an epimorphism and β a monomorphism.

PROPOSITION 8.2. *A map in an abelian category is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Proof. If $f : M \to N$ is monic and epic, then its kernel and cokernel are zero, so $\operatorname{coim} f = M$ and $\operatorname{im} f = N$, whence $M \cong N$. □

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