Always $k$ will denote a field. I will frequently write $x_0, \ldots, x_n$ for a system of homogeneous coordinates on $\mathbb{P}^n$ and write $S = k[x_0, \ldots, x_n]$ for the associated polynomial ring.

(1) Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring. If $I$ is an ideal in $R$ show that $R/I$ has finite length if and only if it has finite dimension.

(2) Let $S$ and $T$ be multiplicatively closed sets in the rings $A$ and $R$ respectively, and let $\theta : A \to R$ be a ring homomorphism such that $\theta(S) \subset T$. Show that $\theta$ extends to a homomorphism $A[S^{-1}] \to R[T^{-1}]$.

(3) Let $S$ be a multiplicatively closed set in $R$ and $T$ its image in $R/I$ where $I = \{ a \in R \mid as = 0 \text{ for some } s \in S \}$. Show that the natural map $R[S^{-1}] \to (R/I)[T^{-1}]$ is an isomorphism.

(4) Let $S$ be a multiplicatively closed set in $R$ and suppose that every element of $S$ is a unit in $R$. Show that the map $R \to R[S^{-1}]$ is an isomorphism.

(5) Suppose $I$ and $J$ are ideals in a ring $R$. Give an example to show that $I \cap J = 0 \neq \sqrt{I} \cap \sqrt{J} = 0$. Show that if $I + J = R$ and $I \cap J = 0$ then $\sqrt{I} \cap \sqrt{J} = 0$. Hence show that if $R = I \oplus J$, then $R = \sqrt{I} \oplus \sqrt{J}$.

(6) Let $C$ and $D$ be degree two curves in $\mathbb{P}^2$. Show that their scheme theoretic image can not be contained in line.

(7) Let $\mathbb{Z}_2$ act on $\mathbb{C}^2 \times \mathbb{C}^2$ by having the non-identity element $\sigma$ act by $\sigma(p, q) = (q, p)$. Thus $\mathbb{Z}_2$ acts on $R = \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^2)$ by automorphisms. Determine the ring of invariants $R^\mathbb{Z}_2 := \{ f \in R \mid f^\sigma = f \}$. You might try to do this by placing a grading on $R$ such that each homogenous component $R_n$ is finite dimensional and stable under $\mathbb{Z}_2$. If you make the right choice it should not be hard to compute $R^\mathbb{Z}_2$. Now write $R = S/I$ where $I$ is an ideal in a polynomial ring $S$ and find a set of generators for $I$.

How will you show that $I$ is the whole kernel of the map $S \to R$? One way to do this is to use Hilbert series. If $V$ is a graded vector space such that $\dim V_n < \infty$ for all $n$ and $V_n = 0$ for $n \ll 0$ its Hilbert series is the formal Laurent series

$$H_V(t) := \sum (\dim V_n) t^n.$$ 

If $0 \to U \to V \to W \to 0$ is an exact sequence of graded vector spaces in which the arrows preserve degree then

$$H_V(t) = H_U(t) + H_W(t)$$

so, in particular, if $S$ is a graded $k$-algebra and $f \in S_n$ is a regular element then $H_{S/fS}(t) = (1 - t^n)H_S(t)$. Also, a surjective degree-preserving map $V \to W$ is an isomorphism if and only if $H_V(t) = H_W(t)$.

Finally, show that $\text{Spec} \ R$ is singular (using the Jacobian criterion perhaps). What is the singular locus of $(\mathbb{C}^2 \times \mathbb{C}^2)/\mathbb{Z}_2$?

(8) Show that the localization $A[S^{-1}] = 0$ if and only if $0 \in S$.  


2 HOMEWORK

(9) Let $A = k[x_1, \ldots, x_n]$ and $S = k[x_0, \ldots, x_n]$. Give $S$ its standard grading, $\deg x_i = 1$ for all $i$. View $S$ as the homogeneous coordinate ring of $\mathbb{P}^n$ and $A$ as the coordinate ring of the copy of $\mathbb{A}^n$ that consists of the points $(1, a_1, \ldots, a_n) \in \mathbb{P}^n$. Give both $\mathbb{A}^n$ and $\mathbb{P}^n$ their Zariski topologies. Show that the Zariski topology on $\mathbb{A}^n$ is the restriction of the Zariski topology on $\mathbb{P}^n$. If $J$ is a graded ideal in $S$ and $V(J)$ its zero locus in $\mathbb{P}^n$, what is “the” ideal $I$ in $A$ for which $V(I) = V(J) \cap \mathbb{A}^n$?

Consider the homogenization map

$$A \to S_{\text{homog}}, \quad f \mapsto f^* := x_0^{\deg f} f(x_1/x_0, \ldots, x_n/x_0).$$

If $I$ is an ideal in $A$, define $I^* := \{ f^* \mid f \in I \}$. Show that

(a) $I^*$ is a graded ideal in $S$ by proving the more general result that an ideal generated by homogeneous elements is graded;

(b) $V(I^*)$ is the closure in $\mathbb{P}^n$ of $V(I)$, the zero locus of $I$ in $\mathbb{A}^n$.

(10) Give either a proof or a counterexample to the following statement: if $I$ is a homogeneous ideal in $S$ and $X \subset \mathbb{P}^n$ its zero locus, then the minimal primes over $I$ are homogeneous and their zero loci are the irreducible components of $X$.

(11) Let $X$ and $Y$ be subvarieties of $\mathbb{P}^n$ cut out by the homogeneous ideals $I$ and $J$ respectively. Suppose that $X \cap Y$ does not meet the hyperplane at infinity, $x_0 = 0$. So we can think of $X$ and $Y$ as subvarieties of $\mathbb{A}^n = \mathbb{P}^n - \{ x_0 = 0 \}$. Determine their scheme-theoretic intersection in terms of $I$ and $J$. Does this result suggest a definition of a projective scheme? Does it suggest a way to define the scheme-theoretic intersection of two projective varieties that do not involve using the affine definition?

(12) Suppose that $I$ and $J$ are homogeneous ideals in $S$. Find an algebraic condition on $I$ and $J$ that is equivalent to the condition that the projective varieties $V(I)$ and $V(J)$ are equal. Hint: Look first at when $V(I) = \emptyset$ and also do the case $I \subset J$ first. Is there a largest ideal $J$ such that $V(I) = V(J)$? If so, what is $J$ in terms of $I$?

(13) Let $k$ be an algebraically closed field and $R$ a finitely generated commutative $k$-algebra. Show there is no $k$-algebra homomorphism $\theta : k[x, y] \to R$ such that the induced map $\text{Max } R \to \text{Max } k[x, y]$ between the sets of maximal ideals sends $\text{Max } R$ homeomorphically onto $\text{Max } (k[x, y]) - \{ (x, y) \}$. This shows that $\mathbb{A}^2 - \{ 0 \}$ is not an affine scheme.

(14) Let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Suppose that $I \subset J$ are homogeneous ideals in $S$. Find an algebraic condition on $I$ and $J$ that is equivalent to the condition that the natural morphism $\text{Proj } (S/J) \to \text{Proj } (S/I)$ is an isomorphism. In other words, find conditions that are equivalent to the condition that the maps

$$S/I[x_i^{-1}]_0 \to S/J[x_i^{-1}]_0$$

are isomorphisms for $i = 0, \ldots, x_n$.

(15) Let $I$ and $J$ be graded ideals in a graded ring $S$. Assume $S_0 = k$ and $S$ is generated by $S_1$ as a $k$-algebra. Let $z \in S_1$ and define $R = S[z^{-1}]_0$. Define $I_z := I[z^{-1}]_0$ and $J_z$ similarly. Either give proofs or counter-examples to the following statements: $I_z + J_z = (I+J)_z$, $I_z J_z = (IJ)_z$, $I_z \cap J_z = (I \cap J)_z$. 

(16) Now let \( z, S, \) and \( R \) be as in the previous question and define \( R_{\leq n} := S_nz^{-n} \).

If \( f \in R_{\leq n} - R_{\leq n-1} \) let \( f^* \in S_n \) be the unique element such that \( f = f^*z^{-n} \).

Let \( I \) and \( J \) be ideals in \( R \) and define \( I^* \) to be the ideal generated by \( \{ f^* \mid f \in I \} \). Either give proofs of, or counter-examples to, the following statements: \( I^* + J^* = (I + J)^*, I^*J^* = (IJ)^*, I^* \cap J^* = (I \cap J)^* \).

(17) Let \( k \) be an algebraically closed field and \( k[x, y] \) the polynomial ring in 2 variables. Let \( f \) be an irreducible polynomial, \( C \) the curve in \( \mathbb{A}^2 \) it cuts out, \( p \in C \), and \( \mathfrak{m}_p \) the maximal ideal in \( \mathcal{O}(C) \) vanishing at \( p \).

**Terminology.** We call \( p \) a simple point of \( C \) if either

\[
\frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0.
\]

If \( p \) is not a simple point it is called a multiple point and its multiplicity is defined as the minimal \( n \) such that \( f \in \mathfrak{m}_p^n \). We write \( m_p(C) \) for the multiplicity of \( p \in C \).

Thus \( p \) is a singular point of \( C \) if and only if \( m_p(C) \geq 2 \).

(a) Show that \( p \) is a simple point if and only if \( \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = 1 \). Hint: expand \( f \) in a Taylor series around \( p \). It might make the notation simpler to assume that \( p \) is chosen so that \( p = (0, 0) \).

(b) How can the Taylor series expansion of \( f \) at \( p \) be used to compute \( m_p(C) \)? Is there a more algebraic way of saying this if we choose coordinates such that \( p = (0, 0) \)?

(18) (Continuation of the previous exercise.) Let \( L \) be a line in \( \mathbb{A}^2 \) passing through \( p \). Suppose that \( p = (\alpha, \beta) \) is a simple point of \( C \). Show that \( \text{I}(L, C, p) > 1 \) if and only if \( L \) is the line given by

\[
\frac{\partial f}{\partial x}(p)(x - \alpha) + \frac{\partial f}{\partial y}(p)(y - \beta) = 0.
\]

In other words, the tangent line to \( C \) at \( p \) is given by this equation.

(19) (Continuation of the previous exercises.) Show for every line \( L \) through \( p \) that \( \text{I}(C, L, p) \geq m_p(C) \).

(20) (Continuation of the previous exercises.) Show for every other curve \( D \) through \( p \) that \( \text{I}(C, D, p) \geq m_p(C)m_p(D) \).

(21) Let \( C \subset \mathbb{P}^2 \) be the zero locus of a positive degree irreducible homogeneous polynomial \( F \in k[X, Y, Z] \). Show that \( p \) is a simple point of \( C \) if and only if at least one of the partial derivatives of \( F \) does not vanish at \( p \), and in that case the line

\[
\frac{\partial F}{\partial X}(p)X + \frac{\partial F}{\partial Y}(p)Y + \frac{\partial F}{\partial Z}(p)Z = 0
\]

is the tangent line to \( C \) at \( p \).

(22) (\( \text{char} \ k = 0 \)) This is a rather open-ended question. Let \( f \in k[t] \). There is a simple criterion for whether or not \( f \) has a multiple zero: it has no multiple zeroes if and only if \( \gcd\{f, f'\} = 1 \). Is there a similar simple criterion for whether or not a form \( F \in k[x, y] \) has a multiple zero on \( \mathbb{P}^1 \). You might need to think about (i) the relation between \( \gcd\{F, G\} \) and \( \gcd\{F_*, G_*\} \); (ii) the relation between derivatives of \( F \) with respect to \( x \) and \( y \) and derivatives of \( f_* \) with respect to \( t \) where \( t \) is a ratio of two linear forms.