Math 505 **Homework 1** **Winter 2014**

(1) One way of describing a ring is by generators and relations, i.e., as the quotient of a free algebra by a specified 2-sided ideal. Let k be a field. The free k-algebra on generators $x_1, \ldots, x_n$ has a k-basis consisting of all words in $x_1, \ldots, x_n$, including the empty word, and multiplication given by juxtaposition, or concatenation, of words, extended k-linearly. For example, $(x_1x_2 + x_4)(x_2x_3) = x_1x_2x_2x_3 + x_4x_2x_3$. Unsurprisingly, we denote this by $x_1x_2^2x_3 + x_4x_2x_3$. The empty word is the identity element.

We denote the free algebra by $k\langle x_1, \ldots, x_n\rangle$. We will write F for it.

Show that the two sided-ideal of F generated by $x_1, \ldots, x_n$ is a free left F-module with basis $x_1, \ldots, x_n$. (It is also a free right F-module with the same basis—once one knows it is a free left module the simplest way to see it is a free right module with the same basis is to appeal to the fact that there is an anti-automorphism of F that sends each word to its "reverse": just use the same letters in the opposite order.)

In fact, every left ideal in F is a free module!

(2) Let A be the algebra generated by x, x^*, y, y^* subject to the relations

$$x^*x + y^*y = 1,\ xx^* = yy^* = 1,\ xy^* = yx^* = 0.$$

By this I mean that A is the quotient of the free algebra $k\langle x, y, x^*, y^*\rangle$ by the two-sided ideal generated by the elements

$$x^*x + y^*y - 1,\ xx^* - 1,\ yy^* - 1,\ xy^*,\ yx^*.$$

Show there is an isomorphism of left A-modules $A \cong A \oplus A$, and hence $A^m \cong A^n$ (as left A-modules!) for all positive integers m and n.

Just as a linear map between finite-dimensional vector spaces can be represented by multiplication by a matrix a homomorphism between finite-rank free modules over any ring can be represented by matrix multiplication. For example, if $f : R^2 \to R^3$ is a homomorphism of left R-modules, write elements of R^2 and R^3 as row vectors and represent f as right multiplication by a 2×3 matrix with entries in R. If f is an isomorphism, its inverse is right multiplication by a 3×2 matrix.

(3) Let R be the polynomial ring $k[x, y, z]$. Write $\mathfrak{m} := (x, y, z)$ and ${}_Rk := R/\mathfrak{m}$. Show there is an exact sequence

$$0 \longrightarrow R \xrightarrow{\ \alpha\ } R^3 \xrightarrow{\ \beta\ } R^3 \xrightarrow{\ \gamma\ } R \xrightarrow{\ \varepsilon\ } {}_Rk \longrightarrow 0$$

where α, β, and γ, are right multiplication by

$$(x, y, z), \quad \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

respectively, and $\varepsilon : R \to {}_Rk$ is $f \mapsto f(0)$.

(4) Let V be the k-vector space consisting of all k-valued sequences $(a_0, a_1, \ldots)$. Let $S, T : V \to V$ be the linear maps $S(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$ and $T(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$. Let $\Phi : k\langle x, y\rangle \to \operatorname{End}_k(V)$ be the k-algebra homomorphism $\Phi(x) = S$ and $\Phi(y) = T$.

(a) Determine a single generator for $I := \ker(\Phi)$, i.e., an element such that I is the smallest two-sided ideal that contains it.

(b) Write down a k-basis for $A := k\langle x, y\rangle/I$ and describe the product of two basis elements.
(c) Describe A as a left $k[xy]$-module.
(d) Describe A as a left $k[yx]$-module.
(e) Describe A as a right $k[xy]$-module.

(5) Let k be a field. The Weyl algebra is the ring

$$A := \frac{k\langle x, y\rangle}{(xy - yx - 1)}.$$

(a) Show that $\{x^i y^j \mid i, j \geq 0\}$ is a basis for A.
(b) Find a basis for A as a left $k[yx]$-module. Notice that $k[xy] = k[yx]$.
(c) Show that the Weyl algebra is a domain, i.e., a product of non-zero elements is non-zero.
(d) Show that the Weyl algebra is a simple ring if $\text{char}(k) = 0$.
(e) Show that the center of the Weyl algebra is k if $\text{char}(k) = 0$.
(f) Show that the center of the Weyl algebra is generated by x^p and y^p if $\text{char}(k) = p > 0$.
(g) Show that the subalgebra of the Weyl algebra generated by x^p and y^p is isomorphic to the polynomial ring in 2 variables if $\text{char}(k) = p > 0$.

(6) Let k be a field of characteristic zero. Let $k[t]$ the polynomial ring in an indeterminate t with coefficients in k. Let $\text{End}_k(k[t^{\pm 1}])$ denote the ring of k-linear maps $k[t^{\pm 1}] \to k[t^{\pm 1}]$. Let $x = d/dt \in \text{End}_k(k[t])$ be the map $k[t] \to k[t]$ that sends a function to its derivative. Let $y \in \text{End}_k(k[t])$ be the map "multiplication by t. Let D be the k-subalgebra of $\text{End}_k(k[t])$ generated by x and y.
(a) Show that $D \cong A$ where A is the Weyl algebra.
(b) Find a composition series for $k[t^{\pm 1}]$ as a D-module.
(c) Decide if there are any isomorphisms between the different composition factors.
(d) Compute the annihilator of x^n for all $n \in \mathbb{Z}$.
(e) Compute the annihilator of each composition factor of $k[t^{\pm 1}]$.
(f) Is $k[t^{\pm 1}]$ a cyclic D-module?
(g) Compute the length of the module $D/D(yx - \lambda)$. Can you find a *natural* left D-module that has an D-submodule isomorphic to $D/D(yx - \lambda)$ when $k = \mathbb{R}$.