SOLUTIONS TO SELECTED HOMEWORK PROBLEMS MATH 504 FALL 2012

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Abstract. Solutions....

1. Homework 1

Throughout k is an algebraically closed field and A is a k-algebra.

1.1. *Show every non-zero finite dimensional representation V of a k-algebra A has an irreducible subrep. Show by example this is false if $\dim_k V = \infty$.*

Let W be a non-zero subrepresentation of V having minimal dimension. Since W is non-zero and its only non-zero submodule is itself, W is irreducible.

Let $A = k[x]$, the polynomial ring on one variable. Let $V = A$ with A acting on itself by multiplication. The irreducible representations of A are

$$\frac{k[x]}{(x-\lambda)}, \qquad \lambda \in k.$$

Hence if W is a simple A-module there is a non-zero $w \in W$ such that $(x-\lambda)w = 0$ for some $\lambda \in k$. But a product of non-zero elements of A is non-zero so V does not contain a simple submodule.

1.2.

1.2.1. *(a). Show that if V is an irreducible finite dimensional representation of A and $z \in Z(A)$, there is an element $\chi_V(z) \in k$ such that z acts on V by multiplication by $\chi_V(z)$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the* <u>central character</u> *of V.*

If $z \in Z(A)$, define $f_z : V \to V$ by $f_z(v) = zv$. Since z is central, f_z is an A-module homomorphism, i.e., $f_z \in \operatorname{End}_A(V)$.

Define $f : Z(A) \to \operatorname{End}_A(V)$ by $f(z) = f_z$. Then f is a ring homomorphism.

By Schur's Lemma, $\operatorname{End}_A(V)$ is a division ring. It contains a copy of k acting on V by scalar multiplication. And $\operatorname{End}_A(V) \subset \operatorname{End}_k(V) \cong M_n(k)$, the $n \times n$ matrix algebra over k where $n = \dim_k V$. In particular, $\operatorname{End}_A(V)$ has finite dimension. If $\phi \in \operatorname{End}_A(V)$, then $\{\phi, \phi^2, \ldots\}$ is linearly dependent. Hence ϕ is algebraic over k. But $k = \bar{k}$ so $\phi \in k$. Hence the natural map $k \to \operatorname{End}_A(V)$ is an isomrophism.

1.2.2. *(b). Show that if V is an indecomposable finite dimensional representation of A then for any z ? Z(A), the operator ?(z) by which z acts in V has only one eigenvalue ?V (z), equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus ?V : Z(A) ? k is a homomorphism, which is again called the central character of V .*

If λ is an eigenvalue of V, define

$$V_\lambda := \{v \mid (z-\lambda)^n v = 0 \quad \text{for some} n \geq 1\}.$$

It is easy to see that V_λ is an A-submodule of V and that V is the sum of all the V_λ. Follows that V is the direct sum of them.

1.2.3. *(c). Does $\rho(z)$ in (b) have to be a scalar operator?*

1.3. *Let I be an ideal of $R = k[x,y]$ containing $R_{\geq n}$ for some n. Show R/I is an indecomposable module.*

For the moment let R be any ring, M any simple left R-module and J any two-sided ideal in R. Then JM is a submodule of M so is either M or $\{0\}$. If $JM = M$, then $J^n M = M$ for all M so $J^n \neq 0$ for all n.

Let $\mathfrak{m}$ denote the image of (x,y) in R/I. We will show that $\mathfrak{m}$ is the only maximal ideal in R/I. Since $(x,y)^n = R_{\geq n}$, $\mathfrak{m}^n = 0$ in R/I. Hence $\mathfrak{m}^n V = 0$ for every simple R-module V, whence $\mathfrak{m}V = 0$. In particular, if $\mathfrak{n}$ is a maximal ideal of R/I, then

$$\mathfrak{m}\left(\frac{R}{\mathfrak{n}}\right) = 0.$$

Hence $\mathfrak{m} \subset \mathfrak{n}$. Therefore $\mathfrak{m} = \mathfrak{n}$ so $\mathfrak{m}$ is the only maximal ideal in R/I.

If L and M are ideals in R not equal to R, then $L \subset \mathfrak{m}$ and $M \subset \mathfrak{m}$, whence $L + M \subset \mathfrak{m}$. Hence R/I is indecomposable.

2. Homework 2

2.1. *Let R be a ring, not necessarily with 1, such that $x^2 = x$ for all $x \in R$. Show R is commutative.*

Let $x, y \in R$. Then

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$$

so $x + x = 0$. We also have

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$$

so $xy + yx = 0$. Adding xy to both sides and using the fact that $xy + xy = 0$, we obtain $yx = xy$.

2.2.

3. Homework 3

Rotman: 6.25, 6.26, 6.27, 6.30, 6.31, 6.32

3.1. *Let $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ be an exact sequence of finite dimensional vector spaces. Show that*

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

3.2. *Let k be a field. Use the division algorithm for polynomials to show every ideal in the polynomial ring $k[x]$ is principal, i.e., generated by one element.*

3.3. *Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the coordinate ring of the unit circle at the origin. Show that the maximal ideal $\mathfrak{m} = (x, y - 1)$ is not principal. Show that $\mathfrak{m} \oplus \mathfrak{m} \cong R \oplus R$.*

3.3.1. This is a tricky question but illustrates some important points. Although $\mathfrak{m}$ needs two generators, its direct sum with itself needs only two. That is often a surprise to those who haven't seen such a phenomenon before. Although $\mathfrak{m}$ is not free it is projective. It can be difficult to determine a minimal set of generators for an ideal (or module). It can be difficult to decide if two modules are isomorphic. It is also nice to notice that $\mathfrak{m}^2$ *is* principal because

$$\begin{aligned} \mathfrak{m}^2 &= (x^2, x(y-1), y^2 - 2y + 1) \\ &= (1 - y^2, x(y-1), y^2 - 2y + 1) \\ &= (1 - y^2, x(y-1), -2y + 2) \\ &= (y - 1). \end{aligned}$$

3.3.2. Before addressing question 3.1, let's make some observations.

There is a vector space decomposition

$$\mathbb{R}[x,y] = (x^2+y^2-1) \oplus \mathbb{R}[x] \oplus \mathbb{R}[x]y$$

so $R = \mathbb{R}[x] \oplus \mathbb{R}[x]y$.

The ring R is a domain (some people say integral domain), by which one means that a product of non-zero elements in R is non-zero: if $a,b,c,d \in \mathbb{R}[x] \subset R$, then

$$(a+by)(c+dy) = ac + bd(1-x^2) + (ad+bc)y;$$

this is zero only if $ad+bc = ac+bd(1-x^2) = 0$; but that implies $bd^2(x^2-1) = -adc = -bc^2$ whence $b(c^2+d^2(x^2-1)) = 0$; however, the value of $c^2+d^2(x^2-1)$ at $x = \frac{1}{2}$ is not zero (think about the signs of c^2, d^2, and x^2-1); since $c^2+d^2(x^2-1) \neq 0$, $b = 0$ because $\mathbb{R}[x]$ is a domain; therefore $ad = ac = 0$ from which we deduce that either a is zero, in which case $a+by = 0$, or $c = d = 0$, in which case $c+dy = 0$. We have just shown that if a product in R is zero, at least one of the factors is zero. Hence R *is* a domain.

Elements of $\mathbb{R}[x,y]$ are, of course, functions $\mathbb{R}^2 \to \mathbb{R}$. Elements in the ideal (x^2+y^2-1) are precisely the polynomial functions that vanish on the unit circle which I will denote by S^1. If $f,g \in \mathbb{R}[x,y]$ belong to the same coset of (x^2+y^2-1), i.e., if there difference belongs to (x^2+y^2-1), they take the same value at every point on S^1. Therefore elements of R are well-defined functions $S^1 \to \mathbb{R}$. But must not try to evaluate elements of R at other points of $\mathbb{R}^2$ because they are not well-defined. Still, an element in $\mathbb{R}[x] \oplus \mathbb{R}[x]y$ is a well-defined function $\mathbb{R}^2 \to \mathbb{R}$. So we can decide if two elements of R are different by picking a representative of each in $\mathbb{R}[x] + \mathbb{R}[x]y$ and observing that those representatives take different values at some point in $\mathbb{R}^2$.

Lemma 3.1. *Let R be any ring and M a left R-module. Let K and L be submodules of a module M. The sequence*

$$0 \longrightarrow K \cap L \xrightarrow{f} K \oplus L \xrightarrow{g} K + L \longrightarrow 0$$

in which $K \oplus L$ is the external direct sum,[1] *$f(x) = (x,x)$ and $g(x,y) = x-y$ is exact.*

Proof. It is clear that f and g are R-module homomorphisms. Certainly f is injective and g is surjective. It is also clear that $\ker(g) = \operatorname{im}(f)$. $\square$

3.3.3. The exact sequence in Lemma 3.1 is useful. One nice use is when K and L are left ideals of R such that $K + L = R$. In that case, let's define $h : R \to K \oplus L$ by $h(r) = (rk, r\ell)$ where $k \in K$ and $\ell \in L$ are any elements such that $k - \ell = 1$. It is clear that $gh = \mathrm{id}_R$ so the sequence splits and we get $K \oplus L \cong (K \cap L) \oplus R$.

3.3.4. (Remember $\mathfrak{m}$, $\mathfrak{m} = (x, y-1)$.) For example, if $\mathfrak{n}$ is the maximal ideal $(x, y+1)$ in $R = \mathbb{R}[x,y]/(x^2+y^2-1)$, then $\mathfrak{m} + \mathfrak{n} = R$.

<u>Claim:</u> $\mathfrak{m} \cap \mathfrak{n} = xR$. <u>Proof:</u> Certainly, $xR \subset \mathfrak{m} \cap \mathfrak{n}$. On the other hand, $R/xR \cong \mathbb{R}[y]/(y^2-1)$ so $\dim_{\mathbb{R}}(R/xR) = 2$. But $\dim R/\mathfrak{m}\cap\mathfrak{n} = 2$ also, because, for example, $(\mathfrak{m}+\mathfrak{n})/\mathfrak{m} \cong \mathfrak{n}/(\mathfrak{m}\cap\mathfrak{n})$, so $\mathfrak{m} \cap \mathfrak{n} = xR$. $\lozenge$

It now follows from §3.3.3 that $\mathfrak{m} \oplus \mathfrak{n} \cong xR \oplus R \cong R \oplus R$.

[1] We are *not* saying that $K \cap L$ is $\{0\}$.

3.3.5. Next we observe that $\mathfrak{m}$ and $\mathfrak{n}$ are isomorphic as R-modules: the map $\Phi : \mathfrak{m} \to \mathfrak{n}$ defined by $\Phi(r) = r(1+y)/x$ is an R-module homomorphism with inverse $\Phi^{-1}(s) = sx/(1+y)$. You need to check that the image of Φ is $\mathfrak{n}$; it isn't even obvious at first that $r(1+y)/x$ belongs to R if $r \in \mathfrak{m}$; however, $\Phi(x) = 1+y$ and $\Phi(1-y) = x$ so, since Φ sends the generators x and $1-y$ of $\mathfrak{m}$ to the generators $1+y$ and x of $\mathfrak{n}$, Φ is indeed an isomorphism of R-modules. Since $\mathfrak{m} \cong \mathfrak{n}$,

$$R \oplus R \cong \mathfrak{m} \oplus \mathfrak{n} \cong \mathfrak{m} \oplus \mathfrak{m}.$$

A little more work is needed to write down an explicit isomorphism. I'll leave you to do that though all the details are implicit in the above argument.

3.3.6. Now I turn to the question of whether $\mathfrak{m}$ is principal. I tried without success to do this by brutal calculation. I have written out a proof that $\mathfrak{m}$ is not principal at some time in the (distant?) past but haven't been able to lay my hands on it - or remember it. It *should* be possible to prove it with a not-too-brutal calculation, but since I couldn't do that I'll take another route.

You probably know that every ideal in the polynomial ring $\mathbb{C}[u]$ is principal: if I is a non-zero ideal let f be a non-zero element in I of minimal degree and a any element in I; by the division algorithm, there are polynomials q and r such that $a = qf + r$ and $\deg(r) < \deg(f)$; I contains f and a so contains $a - qf = r$; but f is a non-zero element in I of minimal degree so $r = 0$; hence $a = fq$ and $I = (f)$.

The ring $\mathbb{C}[u, u^{-1}]$ consists of all Laurent polynomials $\sum_{j=-n}^{n} a_j u^j$ with the usual multiplication. It too has the property that all its ideals are principal: one shows every ideal in $\mathbb{C}[u, u^{-1}]$ is generated by its intersection with the subring $\mathbb{C}[u]$.

Notice that $\mathbb{C}[x, y]/(x^2 + y^2 - 1) = \mathbb{C}[u, u^{-1}]$ with $u = x + iy$ and $u^{-1} = x - iy$, i.e., $x = \frac{1}{2}(u + u^{-1})$ and $y = \frac{1}{2i}(u - u^{-1})$. We can think of R as the subring of $\mathbb{C}[u, u^{-1}]$ generated by $\mathbb{R}$ and the elements $\frac{1}{2}(u + u^{-1})$ and $\frac{1}{2i}(u - u^{-1})$ which we will, of course, denote by x and y respectively. More formally, one can show there is a homomorphism $\Phi : \mathbb{R}[x, y] \to \mathbb{C}[u, u^{-1}]$ such that $\Phi(x) = \frac{1}{2}(u + u^{-1})$ and $\Phi(y) = \frac{1}{2i}(u - u^{-1})$ and $\ker(\Phi) = (x^2 + y^2 - 1)$.

OK, back to business, $R = \mathbb{R}[u+u^{-1}, u-u^{-1}] \subset S = \mathbb{C}[u, u^{-1}]$ and $\mathfrak{m} = (x, y-1)$. The ideal $S\mathfrak{m}$ is principal, say $S\mathfrak{m} = fS$. Both x and $y-1$ are multiples of f so $u+u^{-1}$ and $u^{-1}(u^2 - 2iu - 1)$ are multiples of f. Since $u+u^{-1} = u^{-1}(u+i)(u-i)$ and $u^{-1}(u^2 - 2iu - 1) = u^{-1}(u-i)^2$ are in fS, $u - i \in (f)$. It follows that $(f) = (u - i)$ so we can, and will, assume $f = u - i$.

Suppose $\mathfrak{m}$ is principal, i.e., there is an element $g \in R$ such that $\mathfrak{m} = gR$. Since $fS = \mathfrak{m}S = gS$, f and g are unit multiples[2] each other; hence $g = \lambda u^n(u - i)$ for some $\lambda \in \mathbb{C} - \{0\}$ and $n \in \mathbb{Z}$. But $g \in R$ so g is a well-defined function $S^1 \to \mathbb{R}$. In particular, its values at $p = (1, 0) \in S^1$ and at $p' = (-1, 0) \in S^1$ are real numbers. Since $u(p) = (x + iy)(p) = 1$ and $u(p') = (x + iy)(p') = -1$, $g(p) = \lambda(1 - i)$ and $g(p') = \lambda(-1)^n(-1 - i)$. Notice that $(-1)^n g(p)/g(p') = (1 - i)/(-1 - i)$ and this complex number is *not* in $\mathbb{R}$ so the hypothesis that $\mathfrak{m}$ is principal leads to a contradiction. Therefore $\mathfrak{m}$ is not principal.

3.3.7. *A better proof that* $\mathfrak{m}$ *is not principal.* Suppose $\mathfrak{m} = gR$. Then $\mathfrak{m}^2 = g^2 R$. We observed above that $\mathfrak{m}^2 = (y - 1)$ so g^2 and $y - 1$ are unit multiples of each other. Write $g = a + by$ where $a, b \in \mathbb{R}[x] \subset R$. Without loss of generality we can,

[2] Aaaarggghhh, I haven't talked about units. An element a in a ring A is a unit of A if A contains an element a^{-1} such that $aa^{-1} = a^{-1}a = 1$.

and will, assume b is monic. Thus $g^2 = a^2 + b^2(1-x^2) + 2aby = \lambda(y-1)$ for some unit $\lambda \in R$. It follows that $a = -\lambda/2b$ and hence that $(\lambda^2/4b^2) + b^2(1-x^2) = \lambda$. A little manipulation shows that this implies

$$(\lambda - 2b^2 - 2b^2x)(\lambda - 2b^2 + 2b^2x) = 0.$$

If either factor is zero, then λ is divisible by b so b is a unit in R. But $b \in \mathbb{R}[x]$ and the fact that it is a unit in $R = \mathbb{R}[x] + \mathbb{R}[x]y$ implies it is a unit in $\mathbb{R}[x]$, i.e., $b \in \mathbb{R}$. However, b is monic so $b = 1$. Hence $\lambda = -2a \in \mathbb{R}[x]$. However, the only elements of $\mathbb{R}[x]$ that are units in R are the elements in $\mathbb{R} - \{0\}$. Hence $\lambda \in \mathbb{R}$ which is absurd because $(\lambda - 2 - 2x)(\lambda - 2 + 2x) = 0$. From this absurdity we conclude that $\mathfrak{m}$ is *not* principal.

3.4. Splitting. One of the questions is to prove the following result.

Lemma 3.2. *Let* $0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$ *be an exact sequence of left R-modules. There a homomorphism* $q : N \to M$ *such that* $pq = \mathrm{id}_N$ *if and only if there a homomorphism* $j : M \to L$ *such that* $ji = \mathrm{id}_L$.

Proof. □

4. Homework 4

Rotman 6.60, 6.61, 6.62, 6.69

4.1. *Show that a direct sum of projective modules is projective.*

4.2. *Show that a direct summand of a projective module is projective.*

4.3. *Show the following properties of a ring R are equivalent:*

(1) *every left R-module is projective;*
(2) *every left R-module is injective;*
(3) *every left R-module is a direct sum of simple modules;*
(4) *R is a direct sum of simple left ideals (necessarily a* finite *direct sum—why?)*

A ring having the above properties is said to be semisimple—we omit the adjective *left* because if the above properties hold for every left R-module they hold for every right R-module.

We will later prove *Wedderburn's Theorem (1908?)*: if R is semisimple it is isomorphic as a ring to a finite direct sum of matrix rings over division rings, i.e., $R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$ where $D_1, \ldots, D_r$ are division rings and $M_{n_1}(D_1)$ denotes the ring of $n_1 \times n_1$ matrices whose entries belong to D_1.

Lemma 4.1. *The following properties of a ring R are equivalent:*

(1) *every left R-module is projective;*
(2) *every left R-module is injective;*
(3) *every short exact sequence of left R-modules splits.*

Proof. (3) $\Rightarrow$ (1) Let Z be a left R-module. Hypothesis (3) implies that every exact sequence $0 \to X \to Y \to Z \to 0$ splits. Hence Z is projective.

(1) $\Rightarrow$ (2) Let X be a left R-module and $0 \to X \to Y \to Z \to 0$ an exact sequence. The sequence splits because Z is projective. Hence X is injective.

(2) $\Rightarrow$ (3) Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of left R-modules. Since X is injective the sequence splits. □

Lemma 4.2. *Suppose every short exact sequence of left R-modules splits. Then R is a finite direct sum of simple left ideals.*

Proof. Let I be the sum of all simple left ideals in R. Let S be a simple left ideal in R. The map $S \to Sx$, $r \mapsto rx$, is a homomorphism of left R-modules so its image, Sx, is a quotient of S and is therefore zero or simple. In either case $Sx \subset I$. Hence I is a right ideal in R; since I is by definition a left R-module it is in fact a two-sided ideal in R.

Suppose $I \neq R$. The ring R/I has a maximal left ideal (Zorn's lemma argument) which is of the form L/I where L is a maximal left ideal of R. The sequence $0 \to L/I \to R/I \to R/L \to 0$ splits. Since R/L is simple, R/I has a simple left ideal that is necessarily of the form J/I for some left ideal J that contains I. The sequence $0 \to I \to J \to J/I \to 0$ splits, so $J \cong I \oplus (J/I)$. Since J/I is simple J is a sum of simple modules. But I is the sum of *all* simple left ideals so $J = I$. This is a contradiction so we conclude that $I = R$.

Let Λ be an index set and write $R = \sum_{i \in \Lambda} S_i$ where each S_i is a simple left ideal. The identity 1 belongs to this sum so is a *finite* sum $1 = \sum_{i \in \Lambda} x_i$ where $x_i \in S_i$ and x_i is non-zero for all but finitely many i. Since $R = R.1$ it follows that R is contained in, hence equal to, a *finite* sum of simple left ideals.

We can therefore write $R = S_1 + \cdots + S_n$ where each S_i is simple. Choose the minimal such n. If the sum is not direct, then $0 = a_1 + \cdots + a_n$ where $a_i \in S_i$ and not all a_i are zero. Without loss of generality suppose $a_1 \neq 0$. Then $a_1 \in S_2 + \cdots + S_n$. Hence $S_1 = Ra_1 \subset S_2 + \cdots + S_n$. Thus $R = S_2 + \cdots + S_n$ contradicting the minimality of n. Hence the sum $S_1 + \cdots + S_n$ must be direct, i.e., $R = S_1 \oplus \cdots \oplus S_n$. □

The last two paragraphs in the proof of Lemma 4.2 showed that if R is a sum of simple left ideals it is a *finite* direct sum of simple left ideals.

Lemma 4.3. *If R is a direct sum of simple left ideals, then every simple left R-module is projective.*

Proof. Let M be a simple left R-module. Let $m \in M - \{0\}$. Then $M = Rm$ and $M \cong R/L$ where $L = \{r \in R \mid rm = 0\}$. Write $R = S_1 \oplus \cdots \oplus S_n$ where each S_n is a simple left ideal of R. Let $\theta_i : S_i \to R/L$ be the composition $S_i \to R \to R/L$. Since $R/L \neq 0$ some $\theta_i \neq 0$; since S_i and R/L are simple a non-zero $\theta_i : S_i \to R/L$ must be an isomorphism. Hence $S_i \cong M$ for some i. But S_i is projective because it is a direct summand of the free module R. Hence M is projective. □

The proof of Lemma 4.3 showed that every simple left R-module is isomorphic to a simple left ideal in R. In particular, if R is sum of simple left ideals it has only finitely many simple left modules up to isomorphism.

Lemma 4.4. *Let $M = \sum_{i \in I} S_i$ be a sum of simple modules S_i, and $N \subset M$ a submodule. Then there is a subset $J \subset I$ such that*

$$M = \left(\bigoplus_{j \in J} S_j \right) \oplus N. \tag{4-1}$$

In particular, every submodule and every quotient module of M is a direct sum of simple modules.

Proof. By Zorn's lemma, there is a subset $J \subset I$ that is maximal with respect to the property that

$$M_J := \left(\sum_{j \in J} S_j \right) + N$$

is a direct sum of the S_js, $j \in J$, and N. Suppose $M_J \neq M$. Then there is $k \in I$ such that $S_k \not\subset M_J$. But S_k is simple so that implies $S_k \cap M_J = \{0\}$, whence $S_k + M_J$ is a direct sum. But $k \notin J$ so this contradicts the choice of J. Hence $M_J = M$.

Applying this to $N = 0$ we see that M is a direct sum of simple modules.

Let $\bar{M}$ be a quotient of M. Since M is a sum of simple modules so is $\bar{M}$. Hence $\bar{M}$ is a direct sum of simple modules.

Let N be a submodule of M. It follows from (4-1) that N is isomorphic to a quotient of M so N is a direct sum of simple modules. □

Proposition 4.5. *The following conditions on a module M are equivalent:*

1. *M is the sum of its simple submodules;*
2. *M is a direct sum of simple modules;*
3. *if N is a submodule of M there is a submodule L of M such that $M = L \oplus N$.*

Proof. It is clear that (2) implies (1)). It follows from Lemma 4.4 that (1) implies (2) and (3).

(3) $\Rightarrow$ (1) Let N be the sum of all simple submodules of M, and let N' be a submodule of M such that $N' \oplus N = M$. We will show that $N' = 0$.

Suppose to the contrary that $N' \neq 0$, and let $0 \neq m \in N'$. Because Rm is a finitely generated module it has a submodule K such that Rm/K is simple. By hypothesis, there is a submodule K' of M such that $M = K \oplus K'$. Since Rm contains K, it follows that

$$Rm = K \oplus (K' \cap Rm).$$

Hence $K' \cap Rm \cong Rm/K$ is simple, and therefore contained in N. But this is absurd because $K' \cap Rm \subset Rm \subset N'$. Hence N' is zero. □

If $M = L \oplus N$ we call L a complement to N in M.

Lemma 4.6. *Suppose R is a direct sum of simple left ideals. Then every left R-module is a direct sum of simple modules.*

Proof. Since R is a direct sum of simple left R-modules every free left R-module is a direct sum of simple left R-modules. Every left R-module is a quotient of a free left R-module so is the sum of its simple left modules. Now apply Proposition 4.5. □

Lemma 4.7. *Suppose R is a direct sum of simple left ideals. Then every left R-module is projective.*

Proof. By Lemma 4.6, every left R-module is a direct sum of simple modules. By Lemma 4.3 every simple left R-module is projective. Since a direct sum of projective modules is projective every left R-module is projective. □

This completes the solution of problem 4.3.

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