Solutions

6.30

(i) Suppose that A, B, C are modules over R.

For any $z \in B$, $p(z) \in C$.

So there are $y_i \in Y$, $r_i \in R$ s.t

$$p(z) = \sum r_i y_i$$

Now $p(z - \sum r_i y_i') = p(z) - \sum r_i y_i = 0$.

Hence $z - \sum r_i y_i' \in \ker p = \text{Im}\, i$

Since $\text{Im}\, i$ is generated by $i(X)$.

we must have $z - \sum r_i y_i' = \sum s_j i(x_j)$ for some $s_j \in R$, $x_j \in X$

i.e $z = \sum r_i y_i' + \sum s_j i(x_j)$

This proves the claim.

(ii) This is a direct consequence of (i)

6.32 Proof:

"$\Leftarrow$" We have to construct a R-module map $j: C \to B$ s.t $pj = id_C$.

For any $x \in C$, choose $x' \in B$ s.t $p(x') = x$.

Define $j(x) = x' - iq(x')$.

① j is well-defined

i.e we need to show if $p(x') = 0$, then $x' - iq(x') = 0$,

Notice that q is injective, and

$q(x' - iq(x')) = q(x') - qiq(x') = 0$.

Hence $x' - iq(x') = 0$.

② j is an R-module map.

This is clear from definition.

③ $pj = id_C$.

$pj(x) = p(x' - iq(x')) = p(x') = x$.

"$\Rightarrow$" Now we have to construct $q: B \to A$ with $qi = id_A$. For any $x \in B$

Define $q(x) = i^{-1}(x - jp(x))$.

This is well-defined because $x - jp(x) \in \ker p = \operatorname{Im} i$

① q is an R-module map

This is clear.

② For any $y \in A$.

$qi(y) = i^{-1}(i(y) - jpi(y)) = i^{-1}(i(y)) = y$.

Problem A

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n \to 0.$$

Induction on n. If $n \leq 3$, we get a short exact sequence

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \to 0$$

In this case, the result follows by the basic fact that

$$\dim V_2 = \dim(\mathrm{Im} f_1) + \dim(\ker f_1)$$
$$= \dim V_3 + \dim V_1$$

Now assume $n > 3$.

Notice that $\ker f_{n-1} = \mathrm{Im}\, f_{n-2}$ by the exact-ness.

So we have two exact sequences

(1) $0 \to V_1 \to V_2 \to \cdots \to V_{n-2} \xrightarrow{f_{n-2}} \mathrm{Im}\, f_{n-2} \to 0.$

(2) $0 \to \ker f_{n-1} \to V_{n-1} \to V_n \to 0$

By induction hypothesis.

(1) gives $\sum_{i=1}^{n-2} (-1)^i \dim V_i + (-1)^{n-1} \dim(\mathrm{Im}\, f_{n-2}) = 0$

(2) gives $\dim(\ker f_{n-1}) = \dim V_{n-1} - \dim V_n$

Combine them together we get

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

tem (b)

I be an ideal of $k[X]$. WLOG we assume $I \neq 0$.

Pick $0 \neq f(x) \in I$ with smallest degree

If $\deg f(x) = 0$, then $I = R$ since $f(x)$ is a ~~const~~ scalar.

So assume $\deg f(x) \geq 1$.

For any $g(x) \in I$, $\exists\ h(x), r(x) \in k[X]$ such that

$g(x) = h(x) f(x) + r(x)$

with $\deg r(x) < \deg f(x)$ or $r(x) = 0$.

Now $r(x) = g(x) - h(x) f(x) \in I$.

Hence by the choice of $f(x)$, we must have $r(x) = 0$.

(otherwise $r(x)$ will be a non-zero element in I with degree strictly smaller than $f(x)$).

This shows $g(x) = h(x) f(x)$.

i.e. I is generated by $f(x)$.