Math 504, Homework 1, October 5, 2001

In the exercises below $k$ denotes a field.

1. Think of six interesting questions about the fields $\mathbb{F}_p$ and $\mathbb{Q}(\sqrt{d})$.

2. The field of rational functions in one variable, denoted $k(x)$, consists of all ratios $p/q$ where $p$ and $q$ are polynomials in $x$ having coefficients in $k$, and $q \neq 0$. We add and multiply these in the obvious way. The inverse of a non-zero element $p/q$ is $q/p$. This is the field of rational functions on the affine line over $k$. Likewise, the field $k(x,y)$ of rational functions on the affine plane over $k$ consists of all ratios $p/q$ where $p$ and $q$ are polynomials in the variables $x$ and $y$, and $q \neq 0$. Are the fields $k(x)$ and $k(x,y)$ isomorphic? What does the word “isomorphic” mean in this context?

3. Let $X$ be a set, and $R$ the set of all functions $f : X \to k$. If $f$ and $g$ belong to $R$, how do you suggest we define the sum $f + g$, and the product $fg$? List what you think are the important properties of the sum and product? Is there an element of $R$ that deserves the name zero? Is there an element of $R$ that deserves the name one? If so, say what that element is, and what its properties are that warrant it being given that name?

4. Write $C(X)$ instead of $R$ for the set of all $k$-valued functions on $X$. For each subset $Z$ of $X$, define

$$I(Z) := \{ f \in C(X) \mid f(x) = 0 \text{ for all } x \in Z \}.$$ 

State all the properties of $I$ that you think are important. For example, how does it behave with respect to the sum and product in $R$? Is there a special name for subsets of $R$ having these properties?

5. Let $Z$ be a subset of $X$. Define $\psi : C(X) \to C(Z)$ by

$$\psi(f) = f|_Z.$$ 

That is, if $f : X \to k$, $\psi(f)$ is the restriction of $f$ to $Z$. What are the properties of $\psi$ with respect to the addition and multiplication operations in $C(X)$ and $C(Z)$? How do the elements you labelled one and zero behave under $\psi$?

6. Let $R$ be any ring of functions $X \to k$. Associate to each subset $Z$ of $X$ the subset

$$I(Z) := \{ f \in R \mid f|_Z = 0 \}.$$ 

If $Z' \subset Z$, what is the relation between $I(Z)$ and $I(Z')$? How are $I(Z \cap Z')$ and $I(Z \cup Z')$ related to $I(Z)$ and $I(Z')$?

7. Let $R$ be any ring of functions $X \to k$. Associate to each ideal $I$ in $R$ the subset

$$Z(I) = \{ z \in X \mid f(z) = 0 \text{ for all } f \in I \}.$$
What is the relation between the notions of inclusion, sum and product of ideals, and the notions of inclusion, intersection, and union of subsets of $X$?

8. Let $R$ be any ring of functions $X \to k$. The previous exercises give functions $I(-)$ and $Z(-)$ between subsets of $X$ and ideals of $R$. What can you say about the compositions $I \circ Z$ and $Z \circ I$?

9. Let $X$ and $Y$ be two sets, and let $\alpha : Y \to X$ be any function. Define $\psi : C(X) \to C(Y)$ by

$$\psi(f) = f \circ \alpha;$$

that is, if $y \in Y$, then $\psi(f)(y) = f(\alpha(y))$. What are the properties of $\psi$ with respect to the operations of addition and multiplication in $C(X)$ and $C(Y)$? How do the elements you labelled one and zero behave under $\psi$?

10. Let $\beta : Z \to Y$ and $\alpha : Y \to X$ be maps between sets. Let $C(\beta) : C(Y) \to C(Z)$ and $C(\alpha) : C(X) \to C(Y)$ be the induced maps, namely $C(\beta)(g) = g \circ \beta$ and $C(\alpha)(f) = f \circ \alpha$. Show that $C(\alpha \beta) = C(\beta) \circ C(\alpha)$. Show that if $\alpha$ is the identity map, then $C(\alpha)$ is also the identity map.
Math 504, Homework 2, October 12, 2001

In the exercises below $k$ denotes a field.

1. State four interesting questions about $\mathbb{C}[x, y, z]/(f)$.

2. State four interesting questions about $\mathbb{C}[x, y, z]/(f, g)$.

3. Show that every non-constant homogeneous polynomial in $\mathbb{C}[x, y]$ factors as a product of linear polynomials. Hence show that the zero locus of a non-constant homogeneous polynomial in $\mathbb{C}[x, y]$ is a union of 1-dimensional subspaces of $\mathbb{C}^2$; that is, a union of complex lines through the origin.

4. Suppose that $f_1, \ldots, f_r$ are homogeneous polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Show that their common zero locus,
   \[ V(f_1, \ldots, f_r) := \{ p \in \mathbb{C}^n \mid f_1(p) = \cdots = f_r(p) = 0 \}, \]
   is a union of lines (i.e., complex lines through the origin).

5. If $U$ is a vector space that is the direct sum of various subspaces $U_d$, $d \geq 0$, and $V$ is a subspace such that $V = \oplus_{d=0}^{\infty} (V \cap U_d)$, show that
   \[ U/V \cong \bigoplus_{d=0}^{\infty} \frac{U_d}{V \cap U_d}, \]
   and hence that
   \[ U/V = \bigoplus_{d=0}^{\infty} \frac{U_d + V}{V}. \]

6. Let $I$ be an ideal of $k[x_1, \ldots, x_n]$ that is generated by homogeneous elements. Show that
   \[ I = \bigoplus_{d=0}^{\infty} (I \cap k[x_1, \ldots, x_n])_d. \]

7. Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

8. Show that $k[x, y]$ is not a principal ideal domain.

9. Characterize the irreducible polynomials of degree two in $k[x]$ when $\text{char } k \neq 2$. What about when $\text{char } k = 2$?

10. Use the Euclidean algorithm to find the greatest common divisor in $\mathbb{Q}[x]$ of $nx^{n+1} - (n+1)x^n + 1$ and $x^n - nx + n - 1$. Express the greatest common divisor in the form $af + bg$ where $f$ and $g$ are the two given polynomials and $a$ and $b$ are suitable elements of $\mathbb{Q}[x]$. 
11. Let $C$ be the curve in $\mathbb{R}^2$ cut out by the equation $y^2 - x^3 = 0$. Consider $R = \mathbb{R}[x,y]/(x^3 - y^2)$ as a ring of functions $C \to \mathbb{R}$. Show that $R$ is isomorphic to the subring of the polynomial ring $k[t]$ consisting of those polynomials of the form

$$\alpha_0 + \alpha_2 t^2 + \cdots + \alpha_n t^n.$$ 

12. Continue the previous question. For each point $p \in C$, let $m_p$ denote the ideal of $R$ consisting of those functions that vanish at $p$. Show that $\dim_{\mathbb{R}} m_p/m_p^2 = 1$ if $p \neq (0,0)$, and that $\dim_{\mathbb{R}} m_q/m_q^2 = 2$ when $q = (0,0)$.

13. Continue the previous question. Decide exactly which ideals $m_p$ are principal.

14. Show that $1 + x_1^2 + \cdots + x_n^2$ is an irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ for all $n \geq 2$. 
Math 504, Homework 3, October 19, 2001

In the exercises below $k$ denotes a field.

1. Let $I$ be a two-sided ideal in a ring $R$. Prove there is a bijection between the set of two-sided ideals in $R$ that contain $I$ and the set of ideals in $R/I$. Under the bijection an ideal $J$ in $R$ corresponds to $J/I$.

Show that

$$R/J \cong \frac{R/I}{J/I}. $$

How do the sum and product of ideals correspond under this bijection?

2. Show that a finite domain is a field.

3. Let $R$ be a commutative domain containing a field $k$. Show that $R$ is a field if $\dim_k R < \infty$.

4. Let $R$ and $S$ be rings. Their product $R \times S$ is their cartesian product with component-wise addition and multiplication. This is a ring. If $I$ and $J$ are ideals in a ring $R$ such that $I + J = R$, show that $R/I \cap J \cong R/I \times R/J$.

5. In a PID show that $\gcd(f, g)$ generates the ideal $(f, g)$.

6. Let $R$ be a commutative ring. An ideal $p$ in $R$ is prime if $R/p$ is a domain. This is equivalent to the condition that a product $xy$ can belong to $p$ only if either $x$ or $y$ does. The spectrum of $R$, denoted $\text{Spec} R$, is the set of all prime ideals. Notice that every maximal ideal is prime so $\text{Spec} R$ contains Max $R$, the set of maximal ideals.

We make $\text{Spec} R$ a topological space by defining the closed sets to be

$$V(I) := \{ p \mid p \supseteq I \},$$

where $I$ runs over all two-sided ideals of $R$. Show this really does make $\text{Spec} R$ a topological space. This is called the Zariski topology.

7. Let $R$ be a PID. Show that $\text{Spec} R = \text{Max} R \cup \{ 0 \}$. Describe the closed subsets of $\text{Spec} R$. In particular, if $k$ is an algebraically closed field, what is $\text{Spec} k[t]$ and what is the topology on it? Think of the example of $k = \mathbb{C}$ and compare this to the usual topology.

8. If $\psi : R \to S$ is a homomorphism between commutative rings show that there is an induced map

$$\psi^\#: \text{Spec} S \to \text{Spec} R,$$

and that this map is continuous. Is this true if we replace $\text{Spec} R$ and $\text{Spec} S$ by $\text{Max} R$ and $\text{Max} S$?
You have just shown that the rule $R \mapsto \text{Spec } R$ and $\psi \mapsto \psi^R$ is a contravariant functor from the category of commutative rings to the category of topological spaces (contravariant because the arrows change direction). Actually you also need to show that $\text{id}_R^R = \text{id}_{\text{Spec } R}$ and $(\psi\phi)^R = \phi^R\psi^R$.

9. View $R = k[x_1, \ldots, x_n]$ as functions $k^n \to k$. Show that there is a natural injective map $k^n \to \text{Max } R \to \text{Spec } R$, so the Zariski topology induces a topology on $k^n$. What are the closed subsets of $k^n$? Show that every polynomial function $f : k^n \to k$, i.e. every $f \in R$, is a continuous map when $k^n$ and $k$ are both given the Zariski topologies. (The Zariski topology on $k$ is obtained from the inclusions $k \to \text{Max } k[t] \to \text{Spec } k[t]$).

10. A boolean ring is a commutative ring in which $x^2 = x$ for every element. If $R$ is boolean show that $\text{Max } R = \text{Spec } R$.

11. Let $R$ be a commutative domain and suppose that every non-zero non-unit is a product of irreducibles. Show $R$ is a UFD if and only if $(x)$ is a prime ideal for all irreducibles in $R$.

In particular, since $R = k[x_1, \ldots, x_n]$ is a UFD, this shows that $R/(f)$ is a domain if and only if $f$ is irreducible.
Math 504, Homework 4, October 26, 2001

In the exercises below \( k \) denotes a field.

1. A commutative ring is local if it has a unique maximal ideal. Let \( p \in \mathbb{Z} \) be prime. Show that the subring

\[
S := \{ a/b \mid a, b \in \mathbb{Z}, \ p \text{ does not divide } b \}
\]

of \( \mathbb{Q} \) is local.

2. Let \( p \) be an irreducible element in a PID \( R \). Show that the ring

\[
S := \{ a/b \mid a, b \in R, \ p \text{ does not divide } b \}
\]

is local.

3. Let \( J \) and \( K \) be ideals in \( k[x_1, \ldots, x_n] \). Define

\[
K : J := \{ x \in k[x_1, \ldots, x_n] \mid xJ \subset K \}.
\]

Show that \( V(K : J) = V(K) \setminus V(J) \).

4. In \( \mathbb{Z}[x] \) factor into irreducibles \( x^n - 1 \) for \( 3 \leq n \leq 10 \).

5. Are any two of the following rings isomorphic:

\[
\mathbb{Z}/(4), \mathbb{F}_2[x]/(x^2), \mathbb{F}_2[t]/(t^2 - 1), \mathbb{F}_2[y]/(y^2 + y + 1)\?
\]

Explain. You can sometimes show two rings are not isomorphic by showing that their (lattices of) ideals are different.

6. What is the integral closure of \( R = k[x, y]/(y^2 - x^2(x - 1)) \) in its field of fractions. Hint: find a subring of \( k[t] \) that is isomorphic to \( R \).
Math 504, Homework 5, November 2, 2001

In the exercises below $k$ denotes a field.

1. Let $R$ be a domain with field of fractions $F$. Let $S$ be a subset of $R$ consisting of non-zero elements and suppose that $st \in S$ whenever $s$ and $t$ belong to $S$. Let $S = R[S^{-1}]$ be the subring of $F$ generated by $R$ and the inverses of the elements in $S$. Every element in $S$ can therefore be written in the form $xy^{-1}$ where $x \in R$ and $y \in S$. Show that there is a natural 1-1 correspondence between the prime ideals of $S$ and the prime ideals $p$ of $R$ such that $p \cap S = \phi$.

2. Continue with the notation of the previous exercise. A previous week’s homework exercise showed that the inclusion map $\psi : R \to R[S^{-1}]$ induces a continuous map

$$\psi^b : \text{Spec } R[S^{-1}] \to \text{Spec } R$$

when the two spectra are given the Zariski topology. Show that this map is a homeomorphism onto the open subset of $\text{Spec } R$ that is the complement of the set

$$Z := \{p \in \text{Spec } R \mid p \cap S \neq \phi\}.$$ 

Remarks and hints: Let’s write $U$ for the complement to $Z$. The topology on $U$ is induced from that on $\text{Spec } R$—the closed sets of $U$ are exactly the subsets of the form $Y \cap U$ where $Y$ is a closed subset of $\text{Spec } R$. If $J$ is an ideal of $R[S^{-1}]$, then $J = (J \cap R)R[S^{-1}]$, i.e., as an ideal of $R[S^{-1}]$, $J$ is generated by its intersection with $R$. If $I$ is an ideal in $R$, then $IR[S^{-1}] = \{as^{-1} \mid a \in I, s \in S\}$. You can use these remarks without proof.

In general $Z$ will not be a closed subset of $\text{Spec } R$. When $R$ is the ring of integers and $S$ consists of all non-zero integers, then $Z \subset \text{Spec } Z$ consists of all non-zero ideals in $Z$, and this subset of $\text{Spec } Z$ is not equal to $V(I)$ for any ideal $I$ in $Z$. Remember the zero ideal belongs to $\text{Spec } Z$. In this case $R[S^{-1}] = \mathbb{Q}$, and the map $\text{Spec } \mathbb{Q} \to \text{Spec } Z$ sends the unique point in $\text{Spec } \mathbb{Q}$ to the zero prime ideal in $\text{Spec } Z$.

3. Let $L$ be a submodule of a left $R$-module $N$. Show that there is a bijection between the submodules of $N$ that contain $L$ and the submodules of $N/L$. Under the bijection a submodule $M$ lying between $L$ and $N$ corresponds to the submodule $M/L$ of $N/L$. Show that

$$N/M \cong \frac{N/L}{M/L}.$$ 

4. Let $M$ be a left $R$-module. If $I$ is an ideal of $R$ we write

$$IM := \{\sum_{i=1}^{n} a_i m_i \mid n \geq 0, \ a_i \in I, \ m_i \in M\}.$$
Show that $IM$ is a submodule of $M$. Show that if $IM = 0$, there is a natural way to make $M$ a left $R/I$-module.

5. Let $S$ be any subring of $R = k[x]$ that is strictly larger than $k$. Show that $R$ is a finitely generated $R$-module. What minimal information about $S$ would allow you to obtain an upper bound on the number of elements needed to generate $R$ as an $S$-module?

6. Hilbert showed that every ideal in a polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated. Find explicit generators for the ideal

$$I := \{ f \in \mathbb{R}[x, y, z] \mid f(a, b, c) = 0 \text{ for all } (a, b, c) \in S^2 \}.$$

where $S^2$ is the sphere

$$S^2 := \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}.$$
Math 504, Homework 6, November 9, 2001

In the exercises below $k$ denotes a field.

1. Show that $k[x]$ has infinitely many irreducible elements, and use that to show that $k(x)$ is not a finitely generated $k[x]$-module.

2. Let $R$ be a subring of the commutative ring $S$. Suppose that $S$ is a finitely generated $R$-module. Show that if $m$ is a maximal ideal of $R$, then $Sm \neq S$ (Hint: suppose to the contrary that $S = Sm$ and write $S = \sum_i R s_i$ and $s_i = \sum_j r_{ij} s_j$ with $r_{ij} \in m$; then prove that the determinant of the matrix $(r_{ij} - \delta_{ij})$ is zero, and conclude that $1 \in m$.)

Hence show that there is a maximal ideal $n$ of $S$ such that $n \cap R = m$.

What does this say in terms of the map $\text{Spec} S \to \text{Spec} R$ induced by the inclusion $R \to S$?

3. Let $p$ be a non-zero element in the commutative ring $R$. Show that $(p)$ is a prime ideal if and only if $p$ is a prime element of $R$.

4. Are the following ideals of $k[x, y]$ prime or not? Give reasons, and say whether your answer depends on the characteristic of $k$.

(a) $(x^3 - y^2)$
(b) $(1 + x^3 - y^3)$
(c) $(x^2 - y^2, x + y)$
(d) $(1 + x^2 - y^2, x + y)$
(e) $(x^2 + y^2, x + y)$

5. If $I$ is an ideal in a commutative ring $R$, a prime ideal $p$ containing $I$ is called a minimal prime over $I$ if there are no prime ideals $q$ such that $I \subset q \subset p$ other than $q = p$.

What are the minimal primes over the following ideals:

(a) $(1 + x^2 - y^2, x + y)$ in $k[x, y]$;
(b) $(xy, yz, zx)$ in $k[x, y, z]$;
(c) $(x^2 + y^2 + z^2, xy + yz + xz)$ in $k[x, y, z]$.

6. Find the radical of the ideal $(xy, z(x - y))$ in $k[x, y, z]$. 
Math 504, Homework 7, November 16, 2001
Due on November 28, 2001

In the exercises below $k$ denotes a field.

1. What are the closed points in $\text{Spec } R$?

2. If $p_1, \ldots, p_n$ are distinct prime ideals, what is the radical of $p_1^{e_1} \cdots p_n^{e_n}$?

3. Give an example to show that a sum of radical ideals need not be radical.

4. Describe the subvariety

$$x(y + z) = x(y - z) - 2y = 0$$

of $\mathbb{C}^3$ by describing its irreducible components and their intersections.

5. What are the irreducible components of the subvariety of $\mathbb{C}^2$ given by

$$x^4 - y^4 = y^4 - x^2y^2 + xy^2 - x^3 = 0?$$

6. Is $(x^2 - y^3, y^2 - z^3)$ a prime ideal of $k[x, y, z]$?

7. Let $C \subset \mathbb{R}^2$ be the curve $y^2 = x^3 - x^4$. Let $L_t, 0 \neq t \in \mathbb{R}$, be the line $y = tx$. Show that $L_t$ meets $C$ at $\{(0, 0), p_t\}$ where $p_t$ is a unique point of $C$. Define the map $f : \mathbb{R} \to C$ by $f(t) = p_t$. Is $f$ a morphism? Is it bijective? Sketch $C$.

8. If $f$ is an irreducible polynomial in $k[x_1, \ldots, x_n]$, is $V(f)$ irreducible? Explain. What if $f = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$?

9. Let $I = (x^2 + y^2, x^2 - y^2) \subset \mathbb{C}[x, y]$. Find $V(I)$ and $\dim_{\mathbb{C}} \mathbb{C}[x, y]/I$.

10. If $X \subset Y$ are closed subvarieties of $\mathbb{A}^n$, show that every irreducible component of $X$ is contained in some irreducible component of $Y$. 