Solutions HW1 Guangbin

1.20. Let $V \neq 0$ be an irreducible rep.
Let $S = \{0 \neq W \subset V : W \text{ is a sub-rep}\}$
Obviously $S \neq \emptyset$ since $V \in S$.
Pick $U \in S$ with smallest dimension. Then by the choice U must be irreducible. (For the second part, see next page).

1.21 Warning: k has to be algebraically closed !!

(a) For any $z \in Z(A)$, define
$$l_z : V \longrightarrow V$$
$$a \longmapsto za.$$
Since $z \in Z(A)$, it is easy to check $l_z \in \mathrm{End}_A(V)$.
Next we claim that $\mathrm{End}_A(V) = k$.
By Schur's lemma, $\mathrm{End}_A(V)$ is a division ring.
Since V is f.d, $\mathrm{End}_A(V)$ is also f.dim'l over k.
For any $\phi \in \mathrm{End}_A(V)$, consider $k(\phi) \subset \mathrm{End}_A(V)$.
By the fact $\mathrm{End}_A(V)$ is f.dim'l,
$k(\phi)$ is an algebraic field extension over k.
But k is algebraically closed, we must have
$\phi \in k$, i.e ϕ is a scalar multiplication.
Take $\phi = l_z$. the result follows.

(b) For any $v \in V$, $z_1, z_2 \in Z(A)$. $(z_1 z_2).v = \chi_V(z_1 z_2)\, v$.
On the other hand,
$$(z_1 z_2).v = z_1(z_2 v) = z_1(\chi_V(z_2) v) = \chi_V(z_1)\chi_V(z_2)\, v$$
Hence $\chi_V(z_1 z_2) = \chi_V(z_1)\chi_V(z_2)$
Also, easy to check χ_V is linear.
and $\chi_V(1) = 1$

1b) For any $\lambda \in k$, let
$$V_\lambda = \{ v \in V : (z-\lambda)^n . v = 0 \text{ for some } n \geq 1 \}.$$
Since $z \in Z(A)$, it is easy to check V_λ is a sub-rep of V.
Let $\Lambda = \{ \lambda \in k : V_\lambda \neq 0 \}$.
By basic linear algebra, $\Lambda \neq \emptyset$. We want to show $|\Lambda| = 1$.
If $|\Lambda| = n \geq 2$, and suppose $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$
then $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$. $(*)$

But we've shown each V_{λ_i} is a sub-rep.
Now $(*)$ contradicts the fact that V is indecomp.
Hence $|\Lambda| = 1$, the result follows.

1.20 (Second part)
Let $A = k[X]$. Then any irred rep W of $k[X]$ must be of the form $k[X]/(f)$ where f is an irred. polynomial. Hence any element w in W must be killed by a polynomial $f \neq 0$ (i.e. $fw = 0$).
Consider the rep $V = k[X]$
then V is torsion-free in the sense that any non-zero $v \in V$ cannot be killed by a non-zero element in $k[X]$. Hence V has no irred rep.

1.23 Pick a non-zero element $v \in V$

Since V is irred, every $\phi \in \mathrm{End}_A(V)$ is uniquely determined by $\phi(v)$ and therefore $\phi|_{kv}$ (the restriction of ϕ on the subspace kv)

Now we have an injective k-linear map

$$\begin{aligned} \mathrm{End}_A(V) &\longrightarrow \mathrm{Hom}_k(kv, V) \cong V. \\ \phi &\longmapsto \phi|_{kv} \end{aligned}$$

Hence $\mathrm{End}_A(V)$ is at most countably dimensional since V is.

Suppose $\phi \in \mathrm{End}_A(V)$ is not a scalar.

Consider the subfield $\mathbb{C}(\phi) \subset \mathrm{End}_A(V)$.

Since $\mathbb{C}$ is algebraically closed, $\mathbb{C}(\phi)$ must be a transcendental field extension $/ k$ (otherwise we would have $\phi \in \mathbb{C}$)

Now consider the set $\left\{ \dfrac{1}{\phi - \lambda} : \lambda \in \mathbb{C} \right\}$

It is easy to show this set consists of linearly independent elements.

But this contradicts the fact that $\mathrm{End}_A(V)$ is at most countably dimensional.