42. Should be 36! Throughout $R$ denotes a commutative ring and $M$ and $N$ are $R$-modules.

**Lemma 0.1** $(F/E, \lambda)$ has the required universal property. That is, if $\theta : M \times N \to P$ is a bilinear map to an $R$-module $P$, there is a unique $R$-module map $\rho : F/E \to P$ such that $\theta = \rho \lambda$.

**Proof.** A diagram might help to keep track of things:

$$
\begin{array}{ccc}
F & \xleftarrow{i} & M \times N \\
\downarrow{\pi} & & \downarrow{\phi} \\
F/E & \xrightarrow{\rho} & P.
\end{array}
$$

Here $i : M \times N \to F$ is the inclusion map sending $M \times N$ to the natural basis for $F$. We are given the bilinear map $\theta$, and $\pi : F \to F/E$ is the quotient map. We will prove the existence of $\rho$ making this diagram commute. Since $\lambda = \pi \circ i : M \times N \to F/E$ we will then have shown that $\rho \lambda = \theta$.

Before establishing the existence of $\rho$ notice that there is an $R$-module map $\tilde{\rho} : F \to T$ such that $\tilde{\rho}(m,n) := \theta(m,n)$; here I should really write $i(m,n)$ and apply $\tilde{\rho}$ to $i(m,n)$. Because $\tilde{\rho}$ is an $R$-module homomorphism

$$
\tilde{\rho}(r(m,n)) = r\tilde{\rho}(m,n) = r\theta(m,n);
$$

since $\theta$ is $R$-bilinear $r\theta(m,n) = \theta(rm,n) = \theta(m,rn)$; by the definition of $\tilde{\rho}$ this gives $r\tilde{\rho}(m,n) = \tilde{\rho}(rm,n) = \tilde{\rho}(m,rn)$. Hence the kernel of $\tilde{\rho}$ contains $r(m,n) - (rm,n)$ and $r(m,n) - (m,rn)$. Similarly, $\theta(m_1 + m_2,n) = \theta(m_1,n) + \theta(m_2,n)$, so $\tilde{\rho}(m_1 + m_2,n) = \tilde{\rho}(m_1,n) + \tilde{\rho}(m_2,n)$, so the kernel of $\tilde{\rho}$ contains $(m_1 + m_2,n) - (m_1,n) - (m_2,n)$.

Continuing in this way we see that $E \subset \ker \tilde{\rho}$, so $\tilde{\rho}$ factors through $F/E$. That is, there is an $R$-module homomorphism $\rho : F/E \to P$ such that $\tilde{\rho} = \rho \pi$ where $\pi : F \to F/E$. Now $\tilde{\rho}$ was defined so that $\tilde{\rho} \circ i = \theta$, so $\theta = \rho \pi i$. But $\pi i = \lambda : M \times N \to F/E$, so we get $\theta = \rho \lambda$.

Finally the uniqueness of $\rho$ is clear because any other map $\rho' : F/E \to P$ such that $\rho' \lambda = \theta$ will agree with $\rho$ on the generators $\lambda(m,n)$ of $F/E$ so must equal $\rho$.

I will not give proofs of the standard properties of tensor product. It is pretty tedious establishing all the basic properties of $\otimes_R$ so I won’t do it here—look in the books.

Ohhh, OK, just one will do. I will prove there is an isomorphism of $R$-modules

$$
\Theta : \text{Hom}_R(M \otimes_R N, T) \to \text{Hom}_R(M, \text{Hom}_R(N, T))
$$

given by

$$
\Theta(f)(m)(n) = f(m \otimes n)
$$
for \( f \in \text{Hom}_R(M \otimes_R N, T) \), \( m \in M \), and \( n \in N \). I really only care that \( \Theta \) is an isomorphism of abelian groups, but it is no extra work to check that \( \Theta \) is an \( R \)-module map.

Let’s write \( \text{Bilin}(A \times B, C) \) for the set of \( R \)-bilinear maps \( f : A \times B \to C \) when \( A \), \( B \), and \( C \), are \( R \)-modules. This universal property of the pair \( (M \otimes_R N, \lambda) \) can then be stated as follows: there is an \( R \)-bilinear map \( \lambda : M \times N \to M \otimes_R N \) such that for every \( R \)-module \( D \) the map

\[
\text{Hom}_R(M \otimes_R N, D) \to \text{Bilin}(M \times N, D)
\]

\[
\rho \mapsto \rho \circ \lambda
\]

is bijective.

First we define \( \Psi : \text{Hom}_R(M, \text{Hom}_R(N, T)) \to \text{Bilin}(M \times N, T) \) by

\[
\Psi(f)(m, n) = f(m)(n).
\]

Since \( f \) is an \( R \)-module map \( \Psi(f)(-, n) : M \to T \) is \( R \)-linear. Since \( f(m) \) is an \( R \)-module map \( \Psi(f)(m, -) \) is \( R \)-linear. It follows that \( \Psi(f) \) is \( R \)-bilinear.

Define \( \Phi : \text{Bilin}(M \times N, T) \to \text{Hom}_R(M, \text{Hom}_R(N, T)) \) by

\[
\Phi(\alpha)(m) = \alpha(m, n).
\]

Since \( \alpha(m, -) \) is \( R \)-linear \( \Phi(\alpha)(m) \) is an \( R \)-module map. Since \( \alpha(-, n) \) is \( R \)-linear \( \Phi(\alpha) \) is an \( R \)-module map. Hence \( \Phi(\alpha) \in \text{Hom}_R(M, \text{Hom}_R(N, T)) \).

It is easy to check that \( \Phi \) and \( \Psi \) are mutually inverse so

\[
\text{Hom}_R(M, \text{Hom}_R(N, T)) \cong \text{Bilin}(M \times N, T) \cong \text{Hom}_R(M \otimes_R N, T).
\]

43. Should be 37! I will write \( M \otimes_R N \) rather than \( F/E \), so \( \lambda : M \times N \to M \otimes_R N \) is \( \lambda(m, n) = m \otimes n \).

**Lemma 0.2** Suppose that \( T \) is an \( R \)-module and \( \lambda' : M \times N \to T \) is an \( R \)-bilinear map with the property that for every \( R \)-module \( Q \) the map

\[
\text{Hom}_R(T, Q) \to \text{Bilin}(M \times N, Q)
\]

\[
\nu \mapsto \nu \circ \lambda'
\]

is bijective. Then there is a unique \( R \)-module homomorphism \( \rho : M \otimes_R N \to T \) such that \( \lambda' = \rho \circ \lambda \), and \( \rho \) is an isomorphism.

**Proof.** The existence of an \( R \)-module homomorphism \( \rho : M \otimes_R N \to T \) such that \( \lambda' = \rho \circ \lambda \) is guaranteed by the universal property for \( (M \otimes_R N, \lambda) \).

Because of the universal property satisfied by \((T, \lambda')\), if we take \( Q = M \otimes_R N \) there is an \( R \)-module homomorphism \( \nu : T \to M \otimes_R N \) such that \( \lambda = \nu \lambda' \).

The \( R \)-module homomorphism \( \nu \rho : M \otimes_R N \to M \otimes_R N \) has the property that \( \nu \rho \lambda = \nu \lambda' = \lambda \). However, the identity map \( \text{id}_{M \otimes_R N} \) also has the property
that \( \text{id}_{M \otimes_R N} \lambda = \lambda \) so, by the uniqueness clause in the universal property for \( M \otimes_R N \) (applied with \( T = M \otimes_R N \)) \( \nu \rho = \text{id}_{M \otimes_R N} \).

Repeating the argument in the previous paragraph for \( T \) in place of \( M \otimes_R N \), we see that \( \rho \nu = \text{id}_T \). Hence \( \nu \) is an isomorphism.

Finally, the uniqueness of the homomorphism \( \rho \) satisfying \( \lambda' = \rho \lambda \) is ensured by the uniqueness clause in the universal property for \( (M \otimes_R N, \lambda) \). \( \square \)

**45. Should be 39!** Let \( a \) and \( b \) be positive integers and \( d = \gcd(a, b) \). To show that

\[
\frac{\mathbb{Z}}{(a)} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{(b)} \cong \frac{\mathbb{Z}}{(d)}
\]

we will show that the bilinear map

\[
\lambda : \frac{\mathbb{Z}}{(a)} \times \frac{\mathbb{Z}}{(b)} \to \frac{\mathbb{Z}}{(d)}
\]

defined by

\[
\lambda(i, j) = \overline{ij}
\]

has the appropriate universal property. Notice that \( \lambda \) is well-defined because \( d \) divides \( a \) and \( b \). Let \( \theta : \frac{\mathbb{Z}}{(a)} \times \frac{\mathbb{Z}}{(b)} \to M \) be a \( \mathbb{Z} \)-bilinear map. Define the \( R \)-module map

\[
\rho : \frac{\mathbb{Z}}{(d)} \to M
\]

by \( \rho(1) = \theta(1, 1) \). To see that there really is an \( R \)-module map with this property we need only check that \( d \cdot \theta(1, 1) = 0 \). This is true because there are integers \( u, v \) such that \( d = au + bv \), whence

\[
d \cdot \theta(1, 1) = \theta(d, 1) = \theta(au + bv, 1) = \theta(bv, 1) = b \theta(v, 1) = \theta(v, b) = 0.
\]

Now, using the bilinearity of \( \theta \) twice,

\[
\rho \lambda(i, j) = \rho(\overline{ij}) = ij \rho(1) = ij \theta(1, 1) = i \theta(1, j) = \theta(i, j)
\]

whence \( \rho \lambda = \theta \). Moreover \( \rho \) is the only \( R \)-module map such that \( \rho \lambda = \theta \) because if \( \rho' \lambda = \theta \) also then \( \rho'(1) = \rho' \lambda(1, 1) = \theta(1, 1) \).

**90. Should be 83!** Label the rings

\[
R_1 = \mathbb{Z}/(4), \ R_2 = \mathbb{F}_2[x]/(x^2), \ R_3 = \mathbb{F}_2[t]/(t^2 - 1), \ R_4 = \mathbb{F}_2[y]/(y^2 + y + 1).
\]

The rings \( R_2 \) and \( R_3 \) are isomorphic and there are no other isomorphisms.

To see that \( R_2 \cong R_3 \) notice that \( (t^2 - 1) = (t - 1)^2 \), and \( x \mapsto t + 1 \) extends to an isomorphism \( R_2 \to R_3 \).

\( R_4 \) is a field because \( y^2 + y + 1 \) is irreducible over \( \mathbb{F}_2 \).

\( R_1 \) and \( R_2 \) have exactly 3 ideals each, namely 0, \( \sqrt{0} \), and the ring itself. Hence neither of these rings is isomorphic to \( R_4 \).

Finally \( R_1 \neq R_2 \) because 1 + 1 \( \neq 0 \) in \( R_1 \) but 1 + 1 = 0 in \( R_2 \).