39. Should be 34!

**Lemma.** Let $f$ be a non-zero element in a domain $R$ and $R[f^{-1}]$ the subring of Fract $R$ generated by $R$ and $f^{-1}$. Then $R[X]/(Xf - 1) \cong R[f^{-1}]$.

**Proof:** It is clear that $R[f^{-1}]$ is the image of the map $\phi : R[X] \to \text{Fract } R$ defined by $\phi(r) = r$ for $r \in R$ and $\phi(X) = f^{-1}$.

It is clear that $\ker \phi \supset (Xf - 1)$. To prove the reverse inclusion, suppose that $a = \sum_{i=0}^{n} a_iX^i$ is in the kernel of $f$. We will argue by induction on $n = \deg a$. Multiplying $\phi(a)$ by $f^n$, we see that

$$0 = \sum_{i=0}^{n} a_i f^{n-i} = a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n,$$

whence $f$ divides $a_n$. Write $a_n = fb$. Now

$$a = bX^{n-1}(Xf - 1) + (a_{n-1} + b)X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1X + a_0.$$

The first term of the sum on the right-hand side belongs to $\ker \phi$, so the sum of the other terms is also in $\ker \phi$; but that sum has degree $\leq n - 1$ so, by the induction hypothesis, belongs to $(Xf - 1)$. Hence $a \in (Xf - 1)$. \hfill $\square$

It is clear that there is a bijection $p = (a_1, \ldots, a_n) \leftrightarrow (a_1, \ldots, a_n, f(p)^{-1})$ between the points of $Z_f := \{ p \in Z \mid f(p) \neq 0 \}$ and points of $W = V(I, X_{n+1}f - 1)$.

(c) The projection map $\pi : \mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$, $(p, q) \mapsto p$, is a morphism, so continuous with respect to the Zariski topologies. Hence the restriction of $\pi$ to $W = V(I, X_{n+1}f - 1)$ is a continuous map $\pi : W \to Z_f$. This is a bijection, so its inverse is also continuous. Thus $W$ is homeomorphic to $Z_f$.

87. Should be 81!

By definition, $(K : J)J \subset K$, so

$$V(K) \subset V((K : J)J) = V(K : J) \cup V(J),$$

whence $V(K) - V(J) \subset V(K : J)$ and, because $V(K : J)$ is closed,

$$\overline{V(K) - V(J)} \subset V(K : J).$$

To prove the reverse inclusion we need to assume that $K$ is radical, so assume that. This implies $(K : J)$ is radical too: if $a^2 \in (K : J)$ then $a^2J \subset K$ so $(aJ)^2 \subset K$, whence $aJ \subset K$.

Let $f \in I(V(K) - V(J))$ and $g \in J$. Let $p \in V(K)$. If $p \notin V(J)$, then $f(p) = 0$; if $p \in V(J)$, then $g(p) = 0$; in either case, $(fg)(p) = 0$. Hence $fg \in I(V(K)) = K$. But this is true for all $g \in J$, so $fJ \subset K$ and $f \in (K : J)$, so

$$I(V(K) - V(J)) \subset (K : J).$$

Now, using the fact established in class that $I(V(S)) = \overline{S}$ for any subset $S \subset \mathbb{A}^n$, we get

$$\overline{V(K) - V(J)} = V(I(V(K) - V(J))) \supset V(K : J).$$
92. Should be 85! We consider the map \( f : \text{Spec} \, S \to \text{Spec} \, R \), \( f(p) := R \cap p \). Write \( U := \{ q \in \text{Spec} \, R \mid q \cap S = \emptyset \} \). The image of \( f \) is contained in \( U \) because if \( f(p) \) were not in \( U \) then \( R \cap p \) would contain some \( s \in S \). But then \( p \) contains \( s \) and hence \( s^{-1}s = 1 \), contradicting the fact that \( p \) is prime (so not equal to \( S \! \)).

Claim: If \( I \) is an ideal of \( S = R[S^{-1}] \), then \( I = (R \cap I)S \). Proof: Certainly, \((R \cap I)S \subset I \) because \( I \) is an ideal of \( S \). On the other hand, every element of \( I \) is of the form \( xs^{-1} \) for some \( x \in R \) and \( s \in S \) and since \( x = xs^{-1}s, x \in R \cap I \); therefore \( x \in (R \cap I)S \).

It follows that \( f \) is injective.

Now let \( q \in \text{Spec} \, R \) be such that \( q \cap S = \emptyset \). We will show that \( f : \text{Spec} \, S \to U \) is surjective by showing that \( qS \) is prime and \( q = f(qS) \).

In order to show that \( qS \) is prime, suppose \( as^{-1}, bt^{-1} \in S \) are such that \( as^{-1}bt^{-1} = cu^{-1} \in qS \). We can assume \( a, b, c \in R \), \( c \in q \), and \( s, t, u \in S \). Then \( abu = cst \in q \). But \( q \) is prime so at least one of \( a, b, u \) belongs to \( q \). But \( S \cap q = \emptyset \) so either \( a \) or \( b \) is in \( q \), and therefore either \( as^{-1} \) or \( bt^{-1} \) in in \( qS \). This shows that \( qS \) is prime.

Now we show that \( f(qS) = q \). Certainly, \( q \subset R \cap qS \). Suppose that \( r \in R \cap qS \). Then \( r = as^{-1} \) where \( a \in q \) and \( s \in S \). Therefore \( sr \in q \); but \( s \notin q \), so \( r \in q \). Thus \( q = R \cap qS = f(qS) \). So \( f : \text{Spec} \, S \to U \) is a surjective map. This completes the proof of the bijection

\[ U \leftrightarrow \text{Spec} \, S. \]

Final Remark. If \( S \) is generated by \( s_1, \ldots, s_n \), then \( U \) is open because it is the union of the open sets \( \text{Spec} \, R - V(s_i) \), where \( V(s_i) = \{ q \mid s_i \notin q \} \).

93. Should be 86! To show that the map \( f : \text{Spec} \, S \to U \) in the previous exercise is a homeomorphism it remains to show it is continuous. But this is an immediate consequence of the fact that the map \( f : \text{Spec} \, S \to \text{Spec} \, R, p \mapsto \phi^{-1}(p) \) is continuous for any ring homomorphism \( \phi : R \to S \).

113. Should be 106! Write \( Y = Y_1 \cup Y_2 \cup \cdots \cup Y_n \) as a union of irreducible components. If \( Z \) is an irreducible component of \( X \subset Y \), then \( Z = (Z \cap Y_1) \cup \cdots \cup (Z \cap Y_n) \) expresses \( Z \) as a union of closed subsets so \( Z = Z \cap Y_i \) for some \( i \), whence \( Z \subset Y_i \).