Some Solutions

1. First, $k[X,Y]$ has a $k$-vector space basis $\{X^iY^j \mid i,j \geq 0\}$ and the maximal ideal $m = (X,Y)$ has basis $\{X^iY^j \mid i+j \geq 1\}$. It follows that $m^n$ has basis $\{X^iY^j \mid i+j \geq n\}$ and that $\dim_k (m^n/m^{n+1}) = n + 1$. Therefore

$$\dim_k \frac{k[X,Y]}{m^n} = |\{(i,j) \in \mathbb{N}^2 \mid i+j \leq n\}| = 1 + 2 + \cdots + n = \frac{1}{2} n(n+1).$$

The ideal $J$ in the question is $m^n$, so $\dim_k k[X,Y]/J = n(n+1)/2$.

Let $R$ be any $k$-algebra, $m$ a maximal ideal of $R$ such that $R/m \cong k$, and $I = a_1 R + \cdots + a_t R$ an ideal. Then $I/Im$ is an $R/m$-module generated by the images of $a_1, \ldots, a_t$, so is a $k$-vector space of dimension $\leq t$. Hence the minimal number of generators for the ideal $I$ is $\geq \dim_k I/Im$.

In this problem we see that $J/mJ$ has dimension $n+1$, so $J$ needs at least $n+1$ generators.

4. The kernel of the map $\psi : k[X,Y] \to k[T]$ defined by $\psi(X) = T^2$ and $\psi(Y) = T^3$ contains $J = (X^3 - Y^2)$. There is a vector space decomposition

$$k[X,Y] = k[X] \oplus Yk[X] \oplus J.$$

To see this first observe that each $X^iY^j$ belongs to the sum on the right: this is clear if $j = 0$ or $j = 1$ and, if $j \geq 2$, $X^iY^j = X^iY^{j-2}(Y^2 - X^3) + X^{i+3}Y^{j-2}$ so we may then argue by induction on $j$. The sum is direct because there is a direct sum $k[T^2] \oplus T^3 k[T^2]$ in $k[T]$. This also makes it clear that $\ker \psi = J$.

Thus $k[X,Y]/(X^3 - Y^2) \cong S = k[T^2,T^3] \subset k[T]$. There is an obvious bijection between $k$-algebra homomorphisms $\theta : S \to k$ and $k$-algebra homomorphisms $\theta : k[X,Y] \to k$ that vanish on $J$. A $k$-algebra homomorphism $\theta : k[X,Y] \to k$ is completely determined by $\alpha = \theta(X)$ and $\beta = \theta(Y)$, and $J \subset \ker \theta$ if and only if $\alpha^3 = \beta^2$, i.e., if and only if $(\alpha, \beta) \in C$ where $C$ is the curve $X^3 = Y^2$. So the $k$-algebra homomorphisms $\theta : k[T^2,T^3] \to k$ are in bijection with the points on $C$.

Now let $m$ be a maximal ideal of $S$. Write $x$ and $y$ for the images of $X$ and $Y$ in $k[X,Y]/J$ and $m = (x-\alpha, y-\beta)$ where $(\alpha, \beta) \in C$. Then $m$ is the image of the maximal ideal $n = (X-\alpha, Y-\beta)$ in $k[X,Y]$ so $m/m^2$ is a quotient of $n/n^2$ and $\dim_k m/m^2 \leq \dim_k n/n^2 = 2$. I showed in class (Lemma 5.7) that $m \neq m^2$.

If $m = (x,y) = kT^2 + kT^3 + \cdots$ (i.e., if $(\alpha, \beta) = (0,0)$) then $m^2 = kT^4 + kT^5 + \cdots$, so $\dim (m/m^2) = 2$.

We will now show that if $m \neq (x,y)$, then $\dim (m/m^2) = 1$. So assume $(0,0) \neq (\alpha, \beta) \in C$. It suffices to show that $x-\alpha$ and $y-\beta$ are linearly dependent modulo $m^2$. Certainly $m^2$ contains

$$(y-\beta)^2 = y^2 - 2\beta y + \beta^2 = x^3 - 2\beta y + \beta^2$$

and $(x-\alpha)^2 = x^2 - 2\alpha x + \alpha^2$, so contains $x^3 - 2\alpha x^2 + \alpha^2 x$ and $x^3 - 2\alpha(2\alpha x -$
\(\alpha^2 + \alpha^2 x = x^3 - 3\alpha^2 x + 2\alpha^3\). Hence \(m^2\) contains
\[
(x^3 - 2\beta y + \beta^2) - (x^3 - 3\alpha^2 x + 2\alpha^3) = -2\beta y + \beta^2 + 3\alpha^2 x - 2\alpha^3
\]
\[
= -2\beta y + 3\alpha^2 x - \alpha^3
\]
\[
= 3\alpha^2 (x - \alpha) - 2\beta(y - \beta).
\]

Hence \(x - \alpha\) and \(y - \beta\) are linearly dependent modulo \(m^2\) and \(\dim(m/m^2) = 1\).

14. Let \(\phi : T = k[u, v, w] \rightarrow R = k[x, y]\) be the \(k\)-algebra homomorphism defined by \(\phi(u) = xy, \phi(v) = x^2\), and \(\phi(w) = y^2\). The image of \(\phi\) is the subring of \(k[x, y]\) consisting of all polynomials having no odd degree terms; i.e., \(\text{im}(\phi)\) is spanned by all \(x^i y^j\) such that \(i + j\) is even.

The kernel of \(\phi\) contains \(f = u^2 - vw\) and hence contains the ideal \(fT\). It is trickier to show this is exactly the kernel.

Here is one such argument based on the fact that \(R\) and \(T\) are graded \(k\)-algebras, i.e., there is a nice notion of “degree”.

Define \(T_n\) to be the linear span of all \(u^p v^q w^r\) such that \(p + q + r = n\). Define \(R_n\) to be the linear span of all \(x^i y^j\) such that \(i + j = n\). Then \(T = \bigoplus_{n \geq 0} T_n\) and \(R = \bigoplus_{n \geq 0} R_n\) and \(T_i T_j \subseteq T_{i+j}\) and \(R_i R_j \subseteq R_{i+j}\) for all \(i\) and \(j\).

Since \(\phi : T \rightarrow R\) maps \(T_n\) surjectively onto \(R_{2n}\),
\[
\ker \phi = \bigoplus_{n \geq 0} (T_n \cap \ker \phi).
\]

Thus, to show that \(\ker \phi = fT\) it suffices to show that \(T_n \cap \ker \phi = fT_{n-2}\) for all \(n\). Because \(T_n \rightarrow R_{2n}\) is surjective we know that
\[
\dim_k (T_n \cap \ker \phi) = \dim T_n - \dim R_{2n}.
\]

But \(\dim R_i = i + 1\) and \(\dim T_i = \binom{i+2}{2}\), so
\[
\dim_k (T_n \cap \ker \phi) = \frac{1}{2} n(n+1) - (2n+1) = \frac{1}{2} (n-1)(n-2) = \dim T_{n-2} = \dim fT_{n-2}.
\]

Hence \(T_n \cap \ker \phi = fT_{n-2}\) and \(\ker \phi = (u^2 - vw)T\).

Write \(S\) for \(\phi(T) \subset R\) and write \(S_i = S \cap R_i\). Thus \(S_i = 0\) if \(i\) is odd and \(S_i = R_i\) if \(i\) is even. Now we show that \(S\) is integrally closed in its field of fractions \(\text{Fract} S\). There are injections
\[
\begin{array}{ccc}
S & \rightarrow & R \\
\downarrow & & \downarrow \\
\text{Fract} S & \rightarrow & \text{Fract} R.
\end{array}
\]

Suppose that \(\xi \in \text{Fract} S\) is integral over \(S\). Then \(\xi \in \text{Fract} R\) is integral over \(R\) and hence is in \(R\) because \(k[x, y]\) in integrally closed, so it remains to show that \(R \cap \text{Fract} S = S\).
If \( R \cap \text{Fract} S \) is strictly larger than \( S \) we can find an \( r \in R \) of minimal degree (i.e., \( r \in R_0 \oplus R_1 \oplus \cdots \oplus R_n \) with \( n \) minimal) such that \( r = a/b \) for some \( a, b \in S \) with \( b \neq 0 \). Then \( rb = a \). Write
\[
\begin{align*}
r &= r_n + \text{lower degree terms}, \\
a &= a_m + \text{lower degree terms}, \\
b &= b_p + \text{lower degree terms},
\end{align*}
\]
where \( a_i, b_i, r_i \in R_i \). Looking at the highest degree term \( r_n b_p = a_m \); but \( a, b \in S \), so \( p \) and \( m \) are even. Hence \( n \) is also even and \( r_n \in S \). Thus \( r - r_n = ab^{-1} - r_n \in R \cap \text{Fract} S \); but \( \deg(r - r_n) < \deg r \), so \( r - r_n \in S \). Hence \( r \in S \).

15. The ideal \((x - 3)\) is prime in \( k[x, y] \) because \( k[x, y]/(x - 3) \cong k[y] \) which is a domain. To be precise, \((x - 3)\) is the kernel of the surjective \( k \)-algebra homomorphism \( \phi : k[x, y] \to k[y] \) defined by \( \phi(x) = 3 \) and \( \phi(y) = y \).

Notice that the ideal \( I = (x^2 - y^2, x + y) \) is equal to \((x + y)\), and this is a prime ideal in \( k[x, y] \) because \( k[x, y]/(x + y) \cong k[y] \).

Now the ideal \( J = (1 + x^2 - y^2, x + y) \) is equal to \((1, x + y)\) which equals the ring itself. But the ring itself is not considered to be a prime ideal.

The ideal \( K = (x^3 + y^3, x^6 + y^6) \) is not prime because it contains the product \((x + y)(x^2 - xy + y^2)\) and neither of these elements is in \( K \) because \( K \subset \sum_{i+j \geq 3} kx^iy^j \).