Problem 13

(a) (i) If $\phi^{-1}(P) = R$ we are done, otherwise let $\pi : S \rightarrow S/P$ be the quotient map. The kernel of the composition $\pi \circ \phi$ is clearly $\phi^{-1}(P)$. Hence, by the First Isomorphism Theorem, $\phi^{-1}(P)$ is an ideal of $R$ and $R/\phi^{-1}(P)$ injects into $S/P$. We assumed $P$ was prime, so $S/P$ is a domain. Thus $R/\phi^{-1}(P)$ has no zero divisors since it is a subring of a domain. Clearly, since $\phi^{-1}(P) \neq R$, the image of $1 \in R$ in $R/\phi^{-1}(P)$ is a non-zero unit thus $R/\phi^{-1}(P)$ is a domain. It follows that $\phi^{-1}(P)$ is a prime ideal since $R/\phi^{-1}(P)$ is a domain if and only if $\phi^{-1}(P)$ is prime.

(ii) In the case that $R$ is a subring of $S$ and $\phi$ is the inclusion then $\phi^{-1}(P) = P \cap R$ so the result is clear.

(b)(i) As in part (a) let $\pi : S \rightarrow S/M$ be the quotient map. Again, by the First Isomorphism Theorem, $\phi^{-1}(P)$ is an ideal of $R$ and $R/\phi^{-1}(P)$ injects into $S/M$. Since both $\pi$ and $\phi$ are surjective, their composition is surjective and it follows $R/\phi^{-1}(P)$ is isomorphic to $S/M$. The fact that $M$ is maximal implies that $S/M$ is a field and hence $R/\phi^{-1}(P)$ is a field. But $R/\phi^{-1}(P)$ is a field if and only if $\phi^{-1}(M)$ is maximal.

(ii) Take $R = \mathbb{Z}$, $S = \mathbb{Q}$, $M = (0)$, and $\phi$ to be the natural inclusion.

Problem 37

(i) Suppose $R$ is a local ring with maximal ideal $M$ and $x \in R - M$. If $1 \in (x)$ then $x$ unit and we are done. Otherwise $(x)$ is a proper ideal and hence contained in the maximal ideal $M$. This gives a contradiction since $x \in (x)$ but $x \notin M$.

(ii) Suppose $R$ is a commutative ring with 1 in which the set of nonunits forms an ideal $M$. Since $1 \in R$ there is some maximal ideal containing $M$. If $N$ is any maximal ideal of $R$ then by proposition 9 pg 253 we have $N \subset M$. Since $N$ is maximal, we must have $N = M$. Hence there is precisely one maximal ideal $M$.

Problem 8

(a) Note that $x$ and $y = x^{2}y/x^2$ are in $\text{Fract}(R)$ thus $F[x, y] \subset \text{Fract}(R)$. But $\text{Fract}(R)$ is a field so if it contains $F[x, y]$ then it contains the inverses of every element of $F[x, y]$. Hence $\text{Fract}(F[x, y]) \subset \text{Fract}(R)$. Since $F[x, y]$ is a domain and $R \subset F[x, y]$, clearly $\text{Fract}(R) \subset \text{Fract}(F[x, y])$. It follows that $\text{Fract}(R) = \text{Fract}(F[x, y])$.

(b) If every ideal is finitely generated then $R$ is a noetherian ring and hence the ascending chain of ideals $I_{n} = \langle x, x^{2}y, x^{3}y^2, \ldots, x^{n}y^{n-1} \rangle$ is eventually constant. It follows that for some $n$ we have $x^{n+1}y^n \in (x, x^2y, x^3y^2, \ldots, x^{n}y^{n-1})$. Hence there exist $r_{i} \in R$ so that $x^{n+1}y^n = r_{1}x + r_{2}x^2y + r_{3}x^3y^2 + \ldots + r_{n}x^{n}y^{n-1}$. The monomials of $F[x, y]$ are a basis thus some $r_{i}x^{i}y^{i-1}$ contains an $x^{n+1}y^n$ term. This is impossible since it would require $x^{n+1}y^n = x^{n+1}y^{n-1}$ to be in $R$.

Problem 17

$I$ homogeneous $\implies I$ is generated by homogeneous polynomials.

For $p \in I$, define $P$ to be the set of homogeneous components of $p$. Then $\bigcup_{p \in I} P$ is a set of homogeneous polynomials which clearly generates $I$.

$I$ is generated by homogeneous polynomials $\implies I$ homogeneous.

Let $\{g_{i}\}$ be a set of homogenenous generators for $I$ and set $d_{i} = \text{deg}(g_{i})$. Let $f = \Sigma_{i}f_{i}g_{i}$ be an element of $I$. Let $f_{i,j}$ be the homogenous component of $f_{i}$ of degree $j - d_{i}$. Then clearly the homogenous component of $f$ of degree $j$ is $\Sigma f_{i,j}g_{i}$. But $g_{i} \subset I$ and $f_{i,j} \in F[x_{1}, \ldots, x_{n}]$ implies $\Sigma f_{i,j}g_{i} \in I$. So $I$ is homogeneous.

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